Graded Betti numbers of the path ideals of cycles and lines

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1 Introduction

Path complexes are simplicial complexes whose facets encode paths of a fixed length in a graph. These simplicial complexes in turn correspond to monomial ideals called "path ideals". Path ideals of graphs were first introduced by Conca and De Negri [3] in a different algebraic context, but the study of algebraic invariants corresponding to their minimal free resolutions has become popular, with works of Bouchat, Hà and O'Keefe [2] and He and Van Tuyl [7], and the authors [1].

The papers cited above gives partial information on Betti numbers of path ideals. In this paper we use purely combinatorial arguments based on our results in [1] to give an explicit formula for all the graded Betti numbers of path ideals of line graphs and cycles. As a consequence we can give new and short proofs for the known formulas of regularity and projective dimensions of path ideals of line graphs.

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2 Preliminaries

A **simplicial complex** on vertex set $\mathcal{X} = \{x_1, \dots, x_n\}$ is a collection Δ of subsets of $\mathcal{X}$ such that $\{x_i\} \in \Delta$ for all i, and if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. The elements of Δ are called **faces** of Δ and the maximal faces under inclusion are called **facets** of Δ. We denote the simplicial complex Δ with facets $F_1, \dots, F_s$ by $\langle F_1, \dots, F_s \rangle$. We call $\{F_1, \dots, F_s\}$ the facet set of Δ and is denoted by $F(\Delta)$. The vertex set of Δ is denoted by Vert(Δ). A **subcollection** of Δ is a simplicial complex whose facet set is a subset of the facet set of Δ. For $\mathcal{Y} \subseteq \mathcal{X}$, an **induced subcollection** of Δ on $\mathcal{Y}$, denoted by $\Delta_{\mathcal{Y}}$, is the simplicial complex whose vertex set is a subset of $\mathcal{Y}$ and facet set is $\{F \in F(\Delta) \mid F \subseteq \mathcal{Y}\}$.

If F is a face of $\Delta = \langle F_1, \dots, F_s \rangle$, we define the **complement** of F in Δ to be

$$F^c_{\mathcal{X}} = \mathcal{X} \setminus F \quad \text{and} \quad \Delta^c_{\mathcal{X}} = \langle (F_1)^c_{\mathcal{X}}, \dots, (F_s)^c_{\mathcal{X}} \rangle.$$

Note that if $\mathcal{X} \subsetneqq$ Vert(Δ), then $\Delta^c_{\mathcal{X}} = (\Delta_{\mathcal{X}})^c_{\mathcal{X}}$.

From now on we assume that $R = K[x_1, \dots, x_n]$ is a polynomial ring over a field K. Suppose I an ideal in R minimally generated by square-free monomials $M_1, \dots, M_s$. The **facet complex** $\Delta(I)$ associated to I has vertex set $\{x_1, \dots, x_n\}$ and is defined as $\Delta(I) = \langle F_1, \dots, F_s \rangle$ where $F_i = \{x_j \mid x_j | M_i,\ 1 \leq j \leq n\},\ 1 \leq i \leq s$. Conversely if Δ is a simplicial complex with vertices labeled $x_1, \dots, x_n$, the **facet ideal** of Δ is defined as $I(\Delta) = (\prod_{x \in F} x \mid\ F \text{ is a facet of} \Delta)$.

Given a homogeneous ideal I of the polynomial ring R there exists a **graded minimal finite free resolution**

$$0 \to \bigoplus_d R(-d)^{\beta_{p,d}} \to \cdots \to \bigoplus_d R(-d)^{\beta_{1,d}} \to R \to R/I \to 0$$

of R/I in which $R(-d)$ denotes the graded free module obtained by shifting the degrees of elements in R by d. The numbers $\beta_{i,d}$ are the i-th $\mathbb{N}$-**graded Betti numbers** of degree d of R/I, and are independent of the choice of graded minimal finite free resolution.

The first step to our computations of Betti numbers is a form of Hochster's formula for Betti numbers that was proved in [1].

Theorem 2.1 ([1] Theorem 2.8)**.** *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, and I be a pure square-free monomial ideal in R. Then the $\mathbb{N}$-graded Betti numbers of R/I are given by*

$$\beta_{i,d}(R/I) = \sum_{\Gamma \subset \Delta(I), |\mathit{Vert}(\Gamma)|=d} \dim_K \widetilde{H}_{i-2}(\Gamma^c_{\mathit{Vert}(\Gamma)})$$

where the sum is taken over the induced subcollections Γ of $\Delta(I)$ which have d vertices.

Because of Theorem 2.1, to compute Betti numbers we only need to consider induced subcollections $\Gamma = \Delta_{\mathcal{Y}}$ of a simplicial complex Δ with $\mathcal{Y} = \mathrm{Vert}(\Gamma)$.

3 Path complexes and runs

Definition 3.1. Let $G = (\mathcal{X}, E)$ be a finite simple graph and t be an integer such that $t \geq 2$. If x and y are two vertices of G, a **path** of length $(t-1)$ from x to y is a sequence of vertices $x = x_{i_1}, \ldots, x_{i_t} = y$ of G such that $\{x_{i_j}, x_{i_{j+1}}\} \in E$ for all $j = 1, 2, \ldots, t-1$. We define the **path ideal** of G, denoted by $I_t(G)$ to be the ideal of $K[x_1, \ldots, x_n]$ generated by the monomials of the form $x_{i_1}x_{i_2}\ldots x_{i_t}$ where $x_{i_1}, x_{i_2}, \ldots, x_{i_t}$ is a path in G. The facet complex of $I_t(G)$, denoted by $\Delta_t(G)$, is called the **path complex** of the graph G.

Two special cases that we will be considering in this paper are when G is a **cycle** C_n, or a **line graph** L_n on vertices $\{x_1, \ldots, x_n\}$.

$$C_n = \langle x_1x_2, \ldots, x_{n-1}x_n, x_nx_1\rangle \text{ and } L_n = \langle x_1x_2, \ldots, x_{n-1}x_n\rangle.$$

Example 3.2. Consider the cycle C_5 with vertex set $\mathcal{X} = \{x_1, \ldots, x_5\}$ Then $I_4(C_5) = (x_1x_2x_3x_4, x_2x_3x_4x_5, x_3x_4x_5x_1, x_4x_5x_1x_2, x_5x_1x_2x_3)$.

Notation 3.3. Let i and n be two positive integers. For (a set of) labeled objects we use the notation $\bmod n$ to denote

$$x_i \mod n = \{x_j \mid 1 \leq j \leq n, i \equiv j \mod n\}$$

and

$$\{x_{u_1}, x_{u_2}, \ldots, x_{u_t}\} \mod n = \{x_{u_j} \mod n \mid j = 1, 2, \ldots, n\}.$$

Let C_n be a cycle on vertex set $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $t < n$. The **standard labeling** of the facets of $\Delta_t(C_n)$ is as follows. We let $\Delta_t(C_n) = \langle F_1, \ldots, F_n\rangle$ where $F_i = \{x_i, x_{i+1}, \ldots, x_{i+t-1}\}$ $\bmod n$ for all $1 \leq i \leq n$.

Since for each $1 \leq i \leq n$ we have

$$F_{i+1} \setminus F_i = \{x_{t+i}\} \quad \text{and} \quad F_i \setminus F_{i+1} = \{x_i\} \qquad \bmod n,$$

it follows that $|F_i \setminus F_{i+1}| = 1$ and $|F_{i+1} \setminus F_i| = 1 \quad \bmod n$ for all $1 \leq i \leq n-1$.

Definition 3.4. Given an integer t, we define a **run** to be the path complex of a line graph. A run which has p facets is called a **run of length** p and corresponds to $\Delta_t(L_{p+t-1})$. Therefore a run of length p has $p+t-1$ vertices.

Example 3.5. Consider the cycle C_7 on vertex set $\mathcal{X} = \{x_1, \dots x_7\}$ and the simplicial complex $\Delta_4(C_7)$. The following induced subcollections are two runs in $\Delta_4(C_7)$

$$\begin{array}{rcl} \Delta_1 & = & \langle \{x_1, x_2, x_3, x_4\}, \{x_2, x_3, x_4, x_5\} \rangle \\ \Delta_2 & = & \langle \{x_1, x_2, x_6, x_7\}, \{x_1, x_2, x_3, x_7\}, \{x_1, x_2, x_3, x_4\} \rangle. \end{array}$$

In [1] we show that every induced subcollection of the path complex of a cycle is a disjoint union of runs ([1] Proposition 3.6), and that two induced subcollections of the path complex of a cycle composed of the same number of runs of the same lengths are homeomorphic ([1] Lemma 3.8). Therefore all the information we need to compute the homologies of induced subcollections of $\Delta_t(C_n)$ depends on the number and the lengths of the runs.

4 Graded Betti numbers of path ideals

We focus on Betti numbers of degree less than n, as those of degree n were computed in [1]. By Theorem 2.1 we need to count induced subcollections.

Definition 4.1. Let i and j be positive integers. We call an induced subcollection Γ of $\Delta_t(C_n)$ an (i, j)**-eligible subcollection** of $\Delta_t(C_n)$ if Γ is composed of disjoint runs of lengths

$$(t+1)p_1 + 1, \dots, (t+1)p_\alpha + 1, (t+1)q_1 + 2, \dots, (t+1)q_\beta + 2 \tag{4.1}$$

for nonnegative integers $\alpha, \beta, p_1, p_2, \dots, p_\alpha, q_1, q_2, \dots, q_\beta$, which satisfy the following conditions

$$\begin{array}{rcl} j & = & (t+1)(P+Q) + t(\alpha+\beta) + \beta \\ i & = & 2(P+Q) + 2\beta + \alpha, \end{array}$$

where $P = \sum_{i=1}^{\alpha} p_i$ and $Q = \sum_{i=1}^{\beta} q_i$.

Eligible subcollections count the graded Betti numbers.

Theorem 4.2 ([1] Theorem 5.3)**.** *Let $I = I(\Lambda)$ be the facet ideal of an induced subcollection Λ of $\Delta_t(C_n)$. Suppose i and j are integers with $i \leq j < n$. Then the $\mathbb{N}$-graded Betti number $\beta_{i,j}(R/I)$ is the number of (i, j)-eligible subcollections of Λ.*

The following corollary is a special case of Theorem 4.2.

Corollary 4.3. *Let $I = I(\Lambda)$ be the facet ideal of an induced subcollection Λ of $\Delta_t(C_n)$. Then for every i, $\beta_{i,ti}(R/I)$, is the number of induced subcollections of Λ which are composed of i runs of length 1.*

Proof. From Theorem 4.2 we have $\beta_{i,ti}(R/I)$ is the number of (i, ti)-eligible subcollections of Λ. With notation as in Definition 4.1 we have

$$\begin{cases} ti = (t+1)(P+Q) + t(\alpha+\beta) + \beta \\ i = 2(P+Q) + (\alpha+\beta) + \beta \end{cases} \Rightarrow ti = 2t(P+Q) + t(\alpha+\beta) + t\beta$$

Putting the two equations for ti together, we conclude that $(t-1)(P+Q+\beta) = 0$. But β, P, $Q \geq 0$ and $t \geq 2$, so we must have

$$\beta = P = Q = 0 \Rightarrow p_1 = p_2 = \cdots = p_\alpha = 0.$$

So $\alpha = i$ and Γ is composed of i runs of length one. $\square$

Theorem 4.2 holds in particular for $\Lambda = \Delta_t(L_m)$ and $\Lambda = \Delta_t(C_n)$ for any integers m, n. Our next statement is in a sense a converse to Theorem 4.2.

Proposition 4.4. *Let t and n be integers such that $2 \leq t \leq n$ and $I = I(\Lambda)$ be the facet ideal of Λ where Λ is an induced subcollection of $\Delta_t(C_n)$. Then for each $i, j \in \mathbb{N}$ with $i \leq d < n$, if $\beta_{i,j}(R/I) \neq 0$, there exist nonnegative integers ℓ, d such that*

$$\begin{cases} i &= \ell + d \\ j &= t\ell + d \end{cases}$$

Proof. From Theorem 4.2 we know $\beta_{i,j}$ is equal to the number of (i,j)-eligible subcollections of Λ, where with notation as in Definition 4.1 we have

$$\begin{cases} j = (t+1)(P+Q) + t(\alpha+\beta) + \beta \\ i = 2(P+Q) + (\alpha+\beta) + \beta. \end{cases}$$

It follows that

$$j - i = (t-1)(P+Q+\alpha+\beta) \quad \text{and} \quad ti - j = (t-1)(P+Q+\beta). \tag{4.2}$$

We now show that there exist positive integers ℓ, d such that $i = \ell + d$ and $j = t\ell + d$.

$$\begin{cases} i = \ell + d \\ j = t\ell + d \end{cases} \Rightarrow \ \ell = \frac{j-i}{t-1} \quad \text{and} \quad d = \frac{ti-j}{t-1} \ .$$

From (4.2) we can see that i and j as described above are nonnegative integers. $\square$

Theorem 4.2 tells us that to compute Betti numbers of induced subcollections of $\Delta_t(C_n)$ we need to count the number of its induced subcollections which consist of disjoint runs of lengths one and two. The next few pages are dedicated to counting such subcollections. We use some combinatorial methods to generalize a helpful formula which can be found in Stanley's book [9] on page 73.

Lemma 4.5. *Consider a collection of n points arranged on a line. The number of ways of coloring k points, when there are at least t uncolored points on the line between each colored point is*

$$\binom{n-(k-1)t}{k}.$$

Proof. First label the points from $1, 2, \ldots, n$ from left to right, and let $a_1 < a_2 < \cdots < a_k$ be the colored points. For $1 \leq i \leq k-1$, we define x_i to be the number of points, including a_i, which are between a_i and a_{i+1}, and x_0 to be the number of points which exist before a_1, and x_k the number of points, including a_k, which are after a_k.

$$\begin{array}{cccccc} \overbrace{\cdots}^{x_0} & \overbrace{\bullet\cdots}^{x_1} & \overbrace{\bullet\cdots}^{x_2} & \bullet\cdots & \overbrace{\bullet\cdots}^{x_{k-1}} & \overbrace{\bullet\cdots}^{x_k} \\ 1 & a_1 & a_2 & a_3 & a_{k-1} & a_k \ \ n \end{array}$$

If we consider the sequence $x_0, x_1, \ldots, x_k$ it is not difficult to see that there is a one to one correspondence between the positive integer solutions of the following equation and the ways of coloring k points of n points on a line with at least t uncolored points between each two colored points.

$$x_0 + x_1 + \cdots + x_k = n \quad x_0 \geq 0, x_i > t, \text{for } 1 \leq i \leq k-1, \text{and } x_k \geq 1.$$

So we only need to find the number of positive integer solutions of this equation. Consider the following equation

$$(x_0+1)+(x_1-t)+\cdots+(x_{k-1}-t)+x_k = n-(k-1)t+1$$

where $x_0+1 \geq 1$, $x_i - t \geq 1$, for $i = 0\ldots, k-1$ and $x_k \geq 1$. The number of positive integer solution of this equation is (see for example [5] page 29)

$$\binom{n-(k-1)t}{k}.$$

$\square$

Corollary 4.6. *Let C_n be a graph cycle and with the standard labeling let Γ be a proper subcollection of $\Delta_t(C_n)$ with k facets $F_a, \ldots, F_{a+k-1} \mod n$. The number of induced subcollections of Γ which are composed of m runs of length one is*

$$\binom{k-(m-1)t}{m}.$$

Proof. To compute the number of induced subcollections of Γ which are composed of m runs of length one, it is enough to consider the facets $F_a, \ldots, F_{a+k-1}$ as points arranged on a line and compute the number of ways which we can color m points of these k arranged points with at least t uncolored points between each two consecutive colored points. Therefore, by Lemma 4.5 we have the number of induced subcollections of Γ which are composed of m runs of length one is $\binom{k-(m-1)t}{m}$. $\square$

Proposition 4.7. *Let C_n be a graph cycle with vertex set $\mathcal{X} = \{x_1, \ldots, x_n\}$. The number of induced subcollections of $\Delta_t(C_n)$ which are composed of m runs of length one is*

$$\frac{n}{n-mt}\binom{n-mt}{m}.$$

Proof. Recall that $\Delta_t(C_n) = \langle F_1, \ldots, F_n\rangle$ with standard labeling. First we compute the number of induced subcollections of $\Delta_t(C_n)$ which consist of m runs of length one and do not contain the vertex x_n. There are t facets of $\Delta_t(C_n)$ which contain x_n, the remaining facets are $F_1, \ldots, F_{n-t}$, and so by Corollary 4.6 the number we are looking for is

$$\binom{n-t-(m-1)t}{m} = \binom{n-mt}{m}. \tag{4.3}$$

Now we are going to compute the number of induced subcollections Γ which consist of m runs of length one and include x_n. We have t facets which contain x_n, they are $F_{n-t+1}\ldots, F_n$. Each such Γ will contain one $F_i \in \{F_{n-t+1}\ldots, F_n\}$ as the run containing x_n, and $m-1$ other runs of length one which have to be chosen so that they are disjoint from F_i. So we are looking for $m-1$ runs of length one in the subcollection $\Gamma' = \langle F_{i+t}, \ldots, F_{i-t}\rangle \mod n$. The subcollection Γ' has $n-2t-1$ facets, so by Corollary 4.6 it has

$$\binom{n-2t-1-(m-2)t}{m-1} = \binom{n-mt-1}{m-1}$$

induced subcollections that consist of runs of length one. Putting this together with the number of ways to choose F_i and with (4.3) we conclude that the number of induced subcollections of $\Delta_t(C_n)$ which are composed of m runs of length one is

$$t\binom{n-mt-1}{m-1}+\binom{n-mt}{m}=\frac{n}{n-mt}\binom{n-mt}{m}.$$

□

We apply these counting facts to find Betti numbers in specific degrees; the formula in (iii) below (that of a line graph) was also computed by Bouchat, Ha and O'Keefe [2] using Eliahou-Kervaire techniques.

Corollary 4.8. *Let $n \geq 2$ and t be an integer such that $2 \leq t \leq n$. Then we have*

i. For the cycle C_n we have

$$\beta_{i,it}(R/I_t(C_n))=\frac{n}{n-it}\binom{n-it}{i}.$$

ii. For any proper induced subcollection Λ of $\Delta_t(C_n)$ with k facets we have

$$\beta_{i,ti}(R/I(\Lambda)=\binom{k-(i-1)t}{i}.$$

iii. For the line graph L_n, we have

$$\beta_{i,ti}(R/I_t(L_n)=\binom{n-it+1}{i}.$$

Proof. From Corollary 4.3 we have $\beta_{i,it}(R/I)$ in each of the three cases (i), (ii) and (iii) is the number of induced subcollections of $\Delta_t(C_n)$, Λ and $\Delta_t(L_n)$, respectively, which are composed of i runs of length 1. Case (i) now follows from Proposition 4.7, while (ii) and (iii) follow directly from Corollary 4.6. □

The following Lemma is the core of our counting later on in this section.

Lemma 4.9. *Let $\Delta_t(C_n) = \langle F_1, F_2, \ldots, F_n\rangle$, $2 \leq t \leq n$, be the standard labeling of the path complex of a cycle C_n on vertex set $\mathcal{X} = \{x_1, \ldots, x_n\}$. Let i be a positive integer and $\Gamma = \langle F_{c_1}, F_{c_2}, \ldots, F_{c_i}\rangle$ be an induced subcollection of $\Delta_t(C_n)$ consisting of i runs of length 1, with $1 \leq c_1 < c_2 < \cdots < c_i \leq n$. Suppose Σ is the induced subcollection on Vert$(\Gamma) \cup \{x_{c_u+t}\}$ for some $1 \leq u \leq i$. Then*

$$|\Sigma| = \begin{cases} |\Gamma|+t & u<i \text{ and } \quad c_{u+1}=c_u+t+1 \\ |\Gamma|+1 & u=i \text{ or } \quad c_{u+1}>c_u+t+1 \end{cases}$$

Proof. Since Γ consists of runs of length one and each $F_{c_u} = \{x_{c_u}, x_{c_u+1}, \ldots, x_{c_u+t-1}\}$ we must have $c_{u+1} > c_u + t \mod n$ for $u \in \{1, 2, \ldots, i-1\}$. There are two ways that x_{c_u+t} could add facets to Γ to obtain Σ.

1. If $c_{u+1} = c_u + t + 1$ then $F_{c_u}, F_{c_u+1}, \ldots, F_{c_u+t+1} = F_{c_{u+1}} \in \Sigma$ or in other words, we have added t new facets to Γ.

2. If $c_{u+1} > c_u + t + 1$ or $u = i$ then $F_{c_u+1} \in \Sigma$, and therefore one new facet is added to Γ.

$\square$

The following propositions, which generalize Lemma 7.4.22 in [8], will help us compute the remaining Betti numbers.

Proposition 4.10. *Let $\Delta_t(C_n) = \langle F_1, F_2, \dots, F_n\rangle$, $2 \le t \le n$, be the standard labeling of the path complex of a cycle C_n on vertex set $\mathcal{X} = \{x_1, \dots, x_n\}$. Also let i, j be positive integers such that $j \le i$ and $\Gamma = \langle F_{c_1}, F_{c_2}, \dots, F_{c_i}\rangle$ be an induced subcollection of $\Delta_t(C_n)$ consisting of i runs of length 1, with $1 \le c_1 < c_2 < \cdots < c_i \le n$. Suppose $W = Vert(\Gamma) \cup A \subsetneq \mathcal{X}$ for some subset A of $\{x_{c_1+t}, \dots, x_{c_i+t}\} \mod n$ with $|A| = j$. Then the induced subcollection Σ of $\Delta_t(C_n)$ on W is an $(i+j, ti+j)$-eligible subcollection.*

Proof. Since Γ consists of runs of length one and each $F_{c_u} = \{x_{c_u}, x_{c_u+1}, \dots, x_{c_u+t-1}\}$ we must have $c_{u+1} > c_u + t \mod n$ for $u \in \{1, 2, \dots, i-1\}$. The runs (or connected components) of Σ are of the form $\Sigma' = \Sigma_U$ where $U \subseteq W$, and can have one of the following possible forms.

a. For some $a \le i$:
$$U = F_{c_a},$$
and therefore $\Sigma' = \langle F_{c_a}\rangle$ is a run of length 1.

b. For some $a \le i$:
$$U = F_{c_a} \cup \{x_{c_a+t}\},$$
and therefore $c_{a+1} > c_a + t + 1$, so from Lemma 4.9 we have $\Sigma' = \langle F_{c_a}, F_{c_a+1}\rangle$ is a run of length 2.

c. For some $a \le i$:
$$U = F_{c_a} \cup F_{c_{a+1}} \cup \cdots \cup F_{c_{a+r}} \cup \{x_{c_a+t}, x_{c_{a+1}+t}, \dots, x_{c_{a+r-1}+t}\} \mod n$$
and $F_{c_{a+j}} = F_{c_a+j(t+1)}$ for $j = 0, 1, \dots, r$ and $r \ge 1$. Then from Lemma 4.9 above we know Σ' is a run of length $r + 1 + tr = (t+1)r + 1$.

d. For some $a \le i$:
$$U = F_{c_a} \cup F_{c_{a+1}} \cup \cdots \cup F_{c_{a+r}} \cup \{x_{c_a+t}, x_{c_{a+1}+t}, \dots, x_{c_{a+r}+t}\} \mod n$$
and $F_{c_{a+j}} = F_{c_a+j(t+1)}$ for $j = 0, 1, \dots, r$ and $r \ge 1$, and $c_{a+r+1} > c_{a+r} + t + 1$ or $a + r = i$. Then from Lemma 4.9 we have Σ' is a run of length $r+1+tr+1 = (t+1)r+2$.

So we have shown that Σ consists of runs of length 1 and 2 $\mod t+1$.

Suppose the runs in Σ are of the form described in (4.1). By Definition 3.4 we have

$$\begin{aligned}
|\mathrm{Vert}(\Sigma)| &= (t+1)p_1 + t + \cdots + (t+1)p_\alpha + t + (t+1)q_1 + t + 1 + \cdots + (t+1)q_\beta + t + 1 \\
&= (t+1)P + t\alpha + (t+1)Q + t\beta + \beta \\
&= (t+1)(P+Q) + t(\alpha+\beta) + \beta.
\end{aligned}$$

On the other hand by the definition of Σ we know that, Σ has $ti + j$ vertices and therefore

$$ti + j = (t+1)(P+Q) + t(\alpha+\beta) + \beta.$$

It remains to show that $i+j = 2(P+Q)+(\alpha+\beta)+\beta$. Note that if $j=0$ then $\beta = P = Q = 0$ and hence

$$j = 0 \implies P+Q+\beta = 0. \tag{4.4}$$

Moreover each vertex $x_{c_v+t} \in A$ either increases the length of a run in Γ by one and hence increases β (the number of runs of length 2 in Γ) by one, or increases the length of a run by $t+1$, in which case $P+Q$ increases by 1. We can conclude that if we add j vertices to Γ, $P+Q+\beta$ increases by j. From this and (4.4) we have $j = P+Q+\beta$. Now we solve the following system

$$\left\{ \begin{array}{rcl} ti+j & = & (t+1)(P+Q)+t(\alpha+\beta)+\beta \\ j & = & P+Q+\beta \end{array} \right. \begin{array}{l} \implies ti = t(P+Q)+t(\alpha+\beta) \\ \implies i = P+Q+\alpha+\beta \end{array}$$

$$\implies \left\{ \begin{array}{rcl} i & = & P+Q+\alpha+\beta \\ j & = & P+Q+\beta \end{array} \right. \implies i+j = 2(P+Q)+(\alpha+\beta)+\beta.$$

□

Proposition 4.11. *Let C_n be a cycle, $2 \le t \le n$, and i and j be positive integers. Suppose Σ is an $(i+j, ti+j)$-eligible subcollection of $\Delta_t(C_n)$, $2 \le t \le n$. Then with notation as in Definition 4.1, there exists a unique induced subcollection Γ of $\Delta_t(C_n)$ of the form $\langle F_{c_1}, F_{c_2}, \ldots, F_{c_i} \rangle$ with $1 \le c_1 < c_2 < \cdots < c_i \le n$ consisting of i runs of length 1, and a subset A of $\{x_{c_1+t}, \ldots, x_{c_i+t}\} \mod n$, with $|A| = j$ such that $\Sigma = \Delta_t(C_n)_W$ where $W =$ Vert$(\Gamma) \cup A$.*

Moreover if $\mathcal{R} = \langle F_h, F_{h+1}, \ldots, F_{h+m} \rangle \mod n$ is a run in Σ with $|\mathcal{R}| = 2 \mod (t+1)$, then $F_{h+m} \notin \Gamma \mod n$.

Proof. Suppose Σ consists of runs $R'_1, R'_2, \ldots, R'_{\alpha+\beta}$ where for $k = 1, 2, \ldots, \alpha+\beta$

$$\begin{array}{ll} R'_k = \langle F_{h_k}, F_{h_k+1}, \ldots, F_{h_k+m_k-1} \rangle & \mod n \\ \text{Vert}(R'_k) = \{x_{h_k}, x_{h_k+1}, \ldots, x_{h_k+m_k+t-2}\} & \mod n \\ h_{k+1} \ge t + h_k + m_k & \mod n \end{array}$$

and

$$m_k = \left\{ \begin{array}{lll} (t+1)p_k + 1 & \text{for} & k = 1, 2, \ldots, \alpha \\ (t+1)q_{k-\alpha} + 2 & \text{for} & k = \alpha+1, \alpha+2, \ldots, \alpha+\beta. \end{array} \right. \tag{4.5}$$

For each k, we remove the following vertices from Vert(R'_k)

$$\begin{array}{lll} x_{h_k+t}, x_{h_k+2t+1}, \ldots, x_{h_k+p_kt+(p_k-1)} & \mod n & \text{if } 1 \le k \le \alpha \text{ and } p_k \neq 0 \\ x_{h_k+t}, x_{h_k+2t+1}, \ldots, x_{h_k+(q_{k-\alpha}+1)t+q_{k-\alpha}} & \mod n & \text{if } \alpha+1 \le k \le \alpha+\beta \end{array} \tag{4.6}$$

Let $\Gamma = \langle R_1, R_2, \ldots R_{\alpha+\beta} \rangle$ be the induced subcollection on the remaining vertices of Σ, where

$$R_k = \left\{ \begin{array}{lll} \langle F_{h_k}, F_{h_k+t+1}, \ldots, F_{h_k+(t+1)p_k} \rangle & \mod n & \text{for } 1 \le k \le \alpha \\ \langle F_{h_k}, F_{h_k+t+1}, \ldots, F_{h_k+(t+1)q_{k-\alpha}} \rangle & \mod n & \text{for } \alpha+1 \le k \le \alpha+\beta. \end{array} \right. \tag{4.7}$$

In other words, $\mod n$, Γ has facets

$$F_{h_1}, F_{h_1+t+1}, \ldots, F_{h_1+(t+1)p_1}, F_{h_2}, F_{h_2+t+1}, \ldots, F_{h_2+(t+1)p_2}, \ldots, F_{h_{\alpha+\beta}}, \ldots, F_{h_{\alpha+\beta}+(t+1)q_\beta}.$$

It is clear that each R_k consists of runs of length one. Since Γ is a subcollection of Σ, no runs of R_k and $R_{k'}$ are connected to one another if $k \neq k'$, and hence we can conclude Γ is an induced subcollection of $\Delta_t(C_n)$ which is composed of runs of length one. From (4.7) we have the number of runs of length 1 in Γ (or the number of facets of Γ) is equal to

$$(p_1+1)+(p_2+1)+\cdots+(p_\alpha+1)+(q_1+1)+\cdots+(q_\beta+1) = P+Q+\alpha+\beta = i.$$

Therefore, Γ is an induced subcollection of $\Delta_t(C_n)$ which is composed of i runs of length 1. We relabel the facets of Γ as $\Gamma = \langle F_{c_1}, \ldots, F_{c_i}\rangle$. Now consider the following subset of $\{x_{c_1+t}, \ldots, x_{c_i+t}\}$ as A

$$\bigcup_{k=1, p_k\neq 0}^{\alpha} \{x_{h_k+t}, x_{h_k+2t+1}, \ldots, x_{h_k+p_k t+(p_k-1)}\} \cup \bigcup_{k=\alpha+1}^{\alpha+\beta} \{x_{h_k+t}, x_{h_k+2t+1}, \ldots, x_{h_k+(q_{k-\alpha}+1)t+q_{k-\alpha}}\}$$

by (4.6) we have:

$$|A| = (p_1+p_2+\cdots+p_\alpha)+(q_1+1\cdots+q_\beta+1) = P+Q+\beta = j.$$

Then if we set

$$W = (\bigcup_{h=1}^{i} F_{c_h}) \cup A$$

we clearly have $\Sigma = (\Delta_t(C_n))_W$. This proves the existence of Γ, we now prove its uniqueness. Let $\Lambda = \langle F_{s_1}, F_{s_2}, \ldots, F_{s_i}\rangle$ be an induced subcollection of $\Delta_t(C_n)$ which is composed of i runs of length 1 such that $1 \leq s_1 < s_2 < \cdots < s_i \leq n$. Also let B be a j- subset of the set $\{x_{s_1+t}, x_{s_2+t}, \ldots, x_{s_i+t}\} \mod n$ such that

$$\Sigma = (\Delta_t(C_n))_{\text{Vert}(\Lambda)\cup B}. \tag{4.8}$$

Suppose $\Lambda = \langle S_1, S_2, \ldots, S_{\alpha+\beta}\rangle$, such that for $k = 1, 2, \ldots, \alpha+\beta$, S_k is an induced subcollection of R'_k which consists of y_k runs of length one. By (4.8) we have $y_k \neq 0$ for all k. Now we prove the following claims for each $k \in \{1, 2, \ldots, \alpha+\beta\}$.

a. $F_{h_k} \in \Lambda$. Suppose $1 \leq k \leq \alpha+\beta$. If $p_k = 0$ we are clearly done, so consider the case $p_k \neq 0$.

   Assume $F_{h_k} \notin \Lambda$. Since F_{h_k} is the only facet of Σ which contains x_{h_k} we can conclude $x_{h_k} \notin$ Vert(Λ). From (4.8), it follows that $x_{h_k} \in \{x_{s_1+t}, x_{s_2+t}, \ldots, x_{s_i+t}\}$, so

$$x_{h_k} = x_{s_a+t} \mod n \text{ for some } a. \tag{4.9}$$

   On the other hand we know

$$\begin{aligned} F_{s_a} &= \{x_{s_a}, x_{s_a+1}, \ldots, x_{s_a+t-1}\} \mod n \\ F_{s_a+1} &= \{x_{s_a+1}, x_{s_a+2}, \ldots, x_{s_a+t}\} \mod n. \end{aligned}$$

   Since R'_k is an induced connected component of Σ, by (4.9) we can conclude $x_{h_k} \in F_{s_a+1}$ and $F_{s_a}, F_{s_a+1} \in R'_k$. However, we know F_{h_k} is the only facet of R'_k which contains x_{h_k} and so $F_{s_a+1} = F_{h_k}$ and then $s_a+1 = h_k \mod n$. This and (4.9) imply that $t = 1 \mod n$, which contradicts our assumption $2 \leq t \leq n$.

b. *If $F_u \in S_k$ for some u and $F_{u+t+1} \in R'_k$, then $F_{u+t+1} \in S_k$.* Assume $F_{u+t+1} \notin S_k$ and $F_{u+t+1} \in R'_k$. Let
$$r_0 = \min\{r : r > u, F_r \in S_k \mod n\}.$$

Since S_k consists of runs of length one we can conclude $r_0 \geq u+t+1$. Since $r_0 \neq u+t+1$ we have $r_0 \geq u+t+2$. But then

$$x_{u+t+1} \notin \text{Vert}(\Lambda) \cup \{x_{s_1+t}, x_{s_2+t}, \dots, x_{s_i+t}\}$$

and therefore $x_{u+t+1} \notin \text{Vert}(\Sigma)$ which is a contradiction.

Now for each k, by (a) we have $F_{h_k} \in \Lambda$ and from repeated applications of (b) we find that

$$F_{h_k+f(t+1)} \in S_k \quad \text{for } f = \begin{cases} 1, 2, \dots, p_k & 1 \leq k \leq \alpha \\ 1, 2, \dots, q_{k-\alpha} & \alpha+1 \leq k \leq \alpha+\beta. \end{cases}$$

So $R_k \subseteq S_k$. On the other hand S_k consists of runs of length one, so no other facet of R'_k can be added to it, and therefore $S_k = R_k$ for all k. We conclude that $\Lambda = \Gamma$ and we are therefore done. The last claim of the proposition is also apparent from this proof. $\square$

We are now ready to compute the remaining Betti numbers.

Theorem 4.12. *Let n, i, j and t be integers such that $n \geq 2$, $2 \leq t \leq n$, and $ti + j < n$. Then*

i. For the cycle C_n
$$\beta_{i+j,ti+j}(R/I_t(C_n)) = \frac{n}{n-it}\binom{i}{j}\binom{n-it}{i}$$

ii. For the line graph L_n

$$\beta_{i+j,ti+j}(R/I_t(L_n)) = \binom{i}{j}\binom{n-it}{i} + \binom{i-1}{j}\binom{n-it}{i-1}$$

Proof. If $I = I_t(C_n)$ (or $I = I_t(L_n)$), from Theorem 4.2, $\beta_{i+j,ti+j}(R/I)$ is the number of $(i+j, ti+j)$-eligible subcollections of $\Delta_t(C_n)$ (or $\Delta_t(L_n)$). We consider two separate cases for C_n and for L_n.

i. For the cycle C_n, suppose $\mathcal{R}_{(i)}$ denotes the set of all induced subcollections of $\Delta_t(C_n)$ which are composed of i runs of length one. By propositions 4.10 and 4.11 there exists a one to one correspondence between the set of all $(i+j, ti+j)$-eligible subcollections of $\Delta_t(C_n)$ and the set
$$\mathcal{R}_{(i)} \times \binom{[i]}{j}$$

where $\binom{[i]}{j}$ is the set of all j-subsets of a set with i elements. By Corollary 4.3 we have $|\mathcal{R}_{(i)}| = \beta_{i,ti}$ and since $|\binom{[i]}{j}| = \binom{i}{j}$ and so we apply Corollary 4.8 to observe that

$$\beta_{i+j,ti+j}(R/I_t(C_n)) = \binom{i}{j}\beta_{i,ti}(R/I_t(C_n)) = \frac{n}{n-it}\binom{i}{j}\binom{n-it}{i}.$$

ii. For the line graph L_n, recall that

$$\Delta_t(L_n) = \langle F_1, \ldots, F_{n-t+1} \rangle.$$

Let $\Lambda = \langle F_{c_1}, F_{c_2}, \ldots, F_{c_i} \rangle$ be the induced subcollection of $\Delta_t(L_n)$ which is composed of i runs of length 1 and $\text{Vert}(\Lambda) \subset \mathcal{X} \setminus \{x_n\}$, so that it is also an induced subcollection of $\Delta_t(L_{n-1})$. Also let A be a j - subset of $\{x_{c_1+t}, x_{c_2+t}, \ldots, x_{c_i+t}\} \mod n$. So by Propositions 4.10 and 4.11 the induced subcollections on $\text{Vert}(\Lambda) \cup A$ are $(i+j, ti+j)$-eligible and if one denotes these induced subcollections by $\mathcal{B}$ we have the following bijection

$$\mathcal{B} \rightleftharpoons \binom{[i]}{j} \times \{\Gamma \subset \Delta_t(L_{n-1}) : \Gamma \quad \text{is composed of } i \text{ runs of length } 1\}. \tag{4.10}$$

We make the following claim:

Claim: *Let Γ be an $(i+j, ti+j)$-eligible subcollection of $\Delta_t(L_n)$ which contains a run $\mathcal{R}$ with $F_{n-t+1} \in \mathcal{R}$. Then $\Gamma \in \mathcal{B}$ if and only if $|\mathcal{R}| = 2 \mod t+1$.*

Proof of Claim. Let $\Gamma \in \mathcal{B}$ and assume that $\Lambda = \langle F_{c_1}, F_{c_2}, \ldots, F_{c_i} \rangle$ is the subcollection of $\Delta_t(L_{n-1})$ used to build Γ as described above. Then we must have $c_i = n - t$. Now, the run $\mathcal{R}$ contains F_{n-t+1} and F_{n-t}.

If $|\mathcal{R}| > 2$, then $c_{i-1} = n - 2t - 1$ and $x_{c_{i-1}+t} = x_{n-t-1} \in A$ and from Lemma 4.9 we can see that another $t+1$ facets $F_{n-2t-1}, \ldots, F_{n-t-1}$ are in $\mathcal{R}$. If we have all elements of $\mathcal{R}$, we stop, and otherwise, we continue the same way. At each stage $t+1$ new facets are added to $\mathcal{R}$ and therefore in the end $|\mathcal{R}| = 2 \mod t+1$.

Conversely, if $|\mathcal{R}| = (t+1)q+2$ then let $\Lambda = \langle F_{c_1}, F_{c_2}, \ldots, F_{c_i} \rangle$ be the unique subcollection of $\Delta_t(L_n)$ consisting of i runs of length one from which we can build Γ. Since $F_{n-t-1} \in \mathcal{R}$, we must have $c_i = n - t$ or $c_i = n - t + 1$.

If $c_i = n - t$, then we are done, since Λ will be subcollection of $\Delta_t(L_{n-1})$ and so $\Gamma \in \mathcal{B}$. If $c_i = n - t + 1$, then R has one facet F_{n-t+1} and if $x_{c_{i-1}+t} \in A$, then by Lemma 4.9 $\mathcal{R}$ gets an additional $t+1$ facets. And so on: for each c_u either 0 or $t+1$ facets are contributed to $\mathcal{R}$. Therefore, for some p, $|\mathcal{R}| = (t+1)p+1$ which is a contradiction. This settles our claim. $\square$

We now denote the set of remaining $(i+j, ti+j)$-eligible induced subcollections of $\Delta_t(L_n)$ by $\mathcal{C}$. First we note that $\mathcal{C}$ consists of those induced subcollections which contain F_{n-t+1} and are not in $\mathcal{B}$. Also, if $j = i$, then a $(2i, (t+1)j)$-eligible subcollection Γ of $\Delta_t(L_n)$ would have no runs of length 1, as the equations in Definition 4.1 would give $\alpha = 0$. So $\Gamma \in \mathcal{C}$ and we can assume from now on that $j < i$.

We consider $\Lambda = \langle F_{c_1}, F_{c_2}, \ldots, F_{c_{i-1}} \rangle \subset \Delta_t(L_n)$ which is composed of $i-1$ runs of length 1 with $\text{Vert}(\Lambda) \subset \mathcal{X} \setminus F_{n-t+1} \cup \{x_{n-t}\}$. If A is a j-subset of the set $\{x_{c_1+t}, x_{c_2+t}, \ldots, x_{c_{i-1}+t}\}$, we claim that the induced subcollection Γ on $\text{Vert}(\Lambda) \cup A \cup F_{n-t+1}$ belongs to $\mathcal{C}$.

Suppose $\mathcal{R}$ is the run in Γ which includes F_{n-t+1}. If $|\mathcal{R}| \neq 1$ then $c_{i-1} + t = n - t$ which implies that $c_{i-1} = n - 2t$. By Lemma 4.9 we see that $t+1$ facets $F_{n-2t}, F_{n-2t+1}, \ldots, F_{n-t}$ are added to $\mathcal{R}$. If these facets are not all the facets of $\mathcal{R}$ then with the same method we can see that in each step $t+1$ new facets will be added to $\mathcal{R}$ and since $F_{n-t+1} \in \mathcal{R}$ we can conclude $|\mathcal{R}| = 1 \mod t+1$. Therefore $\Gamma \notin \mathcal{B}$.

Now we only need to show that Γ is an $(i+j, ti+j)$-eligible induced subcollection. By Proposition 4.10 the induced subcollection Γ' on Vert$(\Lambda) \cup A$ is an $(i-1+j, t(i-1)+j)$-eligible induced subcollection. Suppose Γ' is composed of runs $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_{\alpha'+\beta'}$ and then we have

$$\begin{cases} t(i-1)+j & = (t+1)(P'+Q')+t(\alpha'+\beta')+\beta' \\ i-1+j & = 2(P'+Q')+2\beta'+\alpha' \end{cases} \Longrightarrow \begin{cases} i-1 & = P'+Q'+\alpha'+\beta' \\ j & = P'+Q'+\beta' \end{cases} \tag{4.11}$$

So Γ consists of all or all but one of the runs $\mathcal{R}_1, \mathcal{R}_2 \dots, \mathcal{R}_{\alpha'+\beta'}$ as well as $\mathcal{R}$ where $\mathcal{R}$ is the run which includes F_{n-t+1}.

As we have seen $|\mathcal{R}| = 1 \mod t+1$. If we suppose $|\mathcal{R}| = 1$ then we can claim that Γ is composed of $\alpha = \alpha'+1$ runs of length 1 and $\beta = \beta'$ runs of length 2 $\mod t+1$, and with $P = P'$ and $Q = Q'$, by (4.11) we have Γ is an $(i+j, ti+j)$-eligible induced subcollection. Now assume $|\mathcal{R}| = (t+1)p+1$, so clearly we have $F_{n-2t} \in \Lambda$ and $x_{n-t} \in A$. Let $\mathcal{R}'$ be the induced subcollection on Vert$(\mathcal{R}) \setminus F_{n-t+1}$. Then clearly we have $\mathcal{R}'$ is a run in Γ' and since the only facets which belong to $\mathcal{R}$ but not to $\mathcal{R}'$ are the t facets $F_{n-2t+2}, \dots, F_{n-t+1}$ we have

$$|\mathcal{R}'| = (t+1)p+1-t = (t+1)(p-1)+2 \tag{4.12}$$

Therefore we have shown the run in Γ which includes F_{n-t+1} has been generated by a run of length 2 $\mod (t+1)$ in Γ'. Using (4.11) we can conclude Γ consists of $\alpha = \alpha'+1$ runs of length 1 and $\beta = \beta'-1$ runs of length 2 $\mod t+1$. We set $P = P'+p$ and $Q = Q'-(p-1)$, and use (4.12) to conclude that

$$\begin{cases} P+Q+\alpha+\beta = (P'+p)+(Q'-p+1)+(\alpha'+1)+(\beta'-1) = i \\ P+Q+\beta = (P'+p)+(Q'-p+1)+(\beta'-1) = j. \end{cases}$$

Therefore $\Gamma \in \mathcal{C}$ as we had claimed.

Conversely, let $\Gamma \in \mathcal{C}$ then one can consider the induced subcollection Γ' on Vert$(\Gamma) \setminus F_{n-t+1}$. Assume Γ is composed of runs $\mathcal{R}_1, \mathcal{R}_2 \dots, \mathcal{R}_{\alpha+\beta}$, so that $\mod t+1$, $\mathcal{R}_h$ is a run of length 1 if $h \leq \alpha$ and length 2 otherwise.

Suppose $\mathcal{R}_h$ is the run which includes F_{n-t+1}. By our assumption we have $|\mathcal{R}_h| = 1 \mod t+1$, so $h \leq \alpha$. If $|\mathcal{R}_h| = 1$ then $\mathcal{R}_h \notin \Gamma'$ and therefore we delete one run of length one from Γ to obtain Γ', in which case Γ' is $(i-1+j, t(i-1)+j)$-eligible.

If $|\mathcal{R}_h| = (t+1)p_h+1 > 1$ then the t facets $F_{n-2t+2}, \dots, F_{n-t+1} \in \mathcal{R}_h$ do not belong to Γ'. So Γ' consists of $\alpha+\beta$ runs $\mathcal{R}_1, \dots, \widehat{\mathcal{R}_h}, \dots, \mathcal{R}_{\alpha+\beta}, \mathcal{R}'_h$ where

$$|\mathcal{R}'_h| = (t+1)p_h+1-t = (t+1)(p_h-1)+2.$$

Setting $\alpha' = \alpha-1$, $\beta' = \beta+1$, $P' = P-p_h$ and $Q' = Q+p_h-1$ it follows that Γ' is $(i-1+j, t(i-1)+j)$-eligible. By Proposition 4.11 there exists a unique induced subcollection $\Lambda = \langle F_{c_1}, F_{c_2}, \dots, F_{c_{i-1}} \rangle$ of $\Delta_t(L_{n-t-1})$ which is composed of $i-1$ runs of length one and a j subset A of $\{x_{c_1+t}, \dots, x_{c_{i-1}+t}\}$ such that Γ' equals to induced subcollection on Vert$(\Lambda) \cup A$. So Γ is the induced subcollection on Vert$(\Lambda) \cup A \cup F_{n-t+1}$. Therefore there is a one to one correspondence between elements of $\mathcal{C}$ and

$$\binom{[i-1]}{j} \times \{\Gamma \subset \Delta_t(L_{n-t-1}) : \Gamma \quad \text{is composed of } i-1 \text{ runs of length } 1\} \tag{4.13}$$

By (4.10), (4.13) and Corollary 4.8 (iii) we have

$$\begin{aligned}
\beta_{i+j,ti+j}(R/I) &= |\mathcal{B}| + |\mathcal{C}| \\
&= \tbinom{i}{j}\beta_{i,ti}(R/I_t(L_{n-1})) + \tbinom{i-1}{j}\beta_{i-1,t(i-1)}(R/I_t(L_{n-t-1}) \\
&= \tbinom{i}{j}\tbinom{n-it}{i} + \tbinom{i-1}{j}\tbinom{n-it}{i-1}.
\end{aligned}$$

□

Finally, we put together Theorem 4.2, Proposition 4.4, Theorem 5.1 of [1] and Theorem 2.1. Note that the case $t = 2$ is the case of graphs which appears in Jacques [8]. Also note that $\beta_{i,j}(R/I_t(C_n)) = 0$ for all $i \geq 1$ and $j > ti$, see for example see for example [8] 3.3.4.

Theorem 4.13 (**Betti numbers of path ideals of lines and cycles).** *Let n, t, p and d be integers such that $n \geq 2$, $2 \leq t \leq n$, $n = (t+1)p + d$, where $p \geq 0$, $0 \leq d \leq t$. Then*

i. The $\mathbb{N}$-graded Betti numbers of the path ideal of the graph cycle C_n are given by

$$\beta_{i,j}(R/I_t(C_n)) = \begin{cases} t & j = n,\ d = 0,\ i = 2\left(\dfrac{n}{t+1}\right) \\[2ex] 1 & j = n,\ d \neq 0,\ i = 2\left(\dfrac{n-d}{t+1}\right) + 1 \\[2ex] \dfrac{n}{n - t\left(\frac{j-i}{t-1}\right)} \dbinom{\frac{j-i}{t-1}}{\frac{ti-j}{t-1}} \dbinom{n - t\left(\frac{j-i}{t-1}\right)}{\frac{j-i}{t-1}} & \begin{cases} j < n,\ i \leq j \leq ti, \text{ and} \\[1ex] 2p \geq \dfrac{2(j-i)}{t-1} \geq i \end{cases} \\[2ex] 0 & \text{otherwise.} \end{cases}$$

ii. The $\mathbb{N}$-graded Betti numbers of the path ideal of the line graph L_n are nonzero and equal to

$$\beta_{i,j}(R/I_t(L_n)) = \dbinom{\frac{j-i}{t-1}}{\frac{ti-j}{t-1}} \dbinom{n - t\left(\frac{j-i}{t-1}\right)}{\frac{j-i}{t-1}} + \dbinom{\frac{j-i}{t-1} - 1}{\frac{ti-j}{t-1}} \dbinom{n - t\left(\frac{j-i}{t-1}\right)}{\frac{j-i}{t-1} - 1}$$

if and only if

(a) $j \leq n$ and $i \leq j \leq ti$;

(b) If $d < t$ then $p \geq \dfrac{j-i}{t-1} \geq i/2$ where both inequalities cannot be $=$ at the same time;

(c) If $d = t$ then $(p+1) \geq \dfrac{j-i}{t-1} \geq i/2$ where both inequalities cannot be $=$ at the same time.

Proof. We only need to make sure we have the correct conditions for the Betti numbers to be nonzero.

i. When $j < n$, $\beta_{i,j}(R/I_t(C_n)) \neq 0 \Longleftrightarrow$

$$\begin{aligned}
&\Longleftrightarrow \begin{cases} \dfrac{j-i}{t-1} \geq \dfrac{ti-j}{t-1} \\ n - \dfrac{t(j-i)}{t-1} \geq \dfrac{j-i}{t-1} \end{cases} \\
&\Longleftrightarrow \begin{cases} 2j \geq (t+1)i \\ n \geq \left(\dfrac{t+1}{t-1}\right)(j-i) \end{cases} \\
&\Longleftrightarrow \begin{cases} 2j \geq (t+1)i \\ (t+1)p + d \geq \left(\dfrac{t+1}{t-1}\right)(j-i) \end{cases} \\
&\Longleftrightarrow \begin{cases} 2j \geq (t+1)i \\ p + \dfrac{d}{t+1} \geq \dfrac{j-i}{t-1} \end{cases} \\
&\Longleftrightarrow 2p \geq \frac{2(j-i)}{t-1} \geq i \qquad\qquad \text{as } d < t+1
\end{aligned} \tag{4.14}$$

ii. $\beta_{i,j}(R/I_t(L_n)) \neq 0 \Longleftrightarrow$

$$\begin{aligned}
&\Longleftrightarrow \begin{cases} \dfrac{j-i}{t-1} \geq \dfrac{ti-j}{t-1} \\ n - \dfrac{t(j-i)}{t-1} \geq \dfrac{j-i}{t-1} - 1 \end{cases} && \text{both } \geq \text{ cannot be } = \text{ at the same time} \\
&\Longleftrightarrow \begin{cases} 2j \geq (t+1)i \\ n+1 \geq \left(\dfrac{t+1}{t-1}\right)(j-i) \end{cases} && \text{both } \geq \text{ cannot be } = \text{ at the same time} \\
&\Longleftrightarrow \begin{cases} 2j \geq (t+1)i \\ (t+1)p + d + 1 \geq \left(\dfrac{t+1}{t-1}\right)(j-i) \end{cases} && \text{both } \geq \text{ cannot be } = \text{ at the same time} \\
&\Longleftrightarrow \begin{cases} 2j \geq (t+1)i \\ p + \dfrac{d+1}{t+1} \geq \dfrac{j-i}{t-1} \end{cases} && \text{both } \geq \text{ cannot be } = \text{ at the same time} \\
&\Longleftrightarrow \begin{cases} 2p \geq \dfrac{2(j-i)}{t-1} \geq i & \text{if } d < t \\ 2(p+1) \geq \dfrac{2(j-i)}{t-1} \geq i & \text{if } d = t \end{cases} && \text{both } \geq \text{ cannot be } = \text{ at the same time in the second line.}
\end{aligned} \tag{4.15}$$

□

We can now easily derive the projective dimension and regularity of path ideals of lines, which were known before. The projective dimension of lines (Part i below) was computed by He and Van Tuyl in [7] using different methods. The case $t = 2$ is the case of graphs which appears in Jacques [8]. Part ii of the following Corollary reproves Theorem 5.3 in [2] which computes the Castelnuovo-Mumford regularity of path ideal of a line. The case of cycles was done in [1].

Corollary 4.14 (Projective dimension and regularity of path ideals of lines). *Let n, t, p and d be integers such that $n \geq 2$, $2 \leq t \leq n$, $n = (t+1)p + d$, where $p \geq 0$, $0 \leq d \leq t$. Then*

i. The projective dimension of the path ideal of a line L_n is given by

$$pd(R/I_t(L_n)) = \begin{cases} 2p & d \neq t \\ 2p+1 & d = t \end{cases}$$

ii. The regularity of the path ideal of a line L_n is given by

$$reg(R/I_t(L_n)) = \begin{cases} p(t-1) & d < t \\ (p+1)(t-1) & d = t \end{cases}$$

Proof. i. From (4.15) we see that the largest possible i where $\beta_{i,j}(R/I_t(L_n)) \neq 0$ is $2p+1$ when $d = t$ and $2p$ when $d \neq t$. By applying Theorem 4.13 if $d = t$ we have

$$\beta_{2p+1,n}(R/I_t(L_n)) = \binom{p+1}{p}\binom{p}{p+1} + \binom{p}{p}\binom{p}{p} = 1 \neq 0.$$

and if $d \neq t$

$$\beta_{2p,p(t+1)}(R/I_t(L_n)) = \binom{p}{p}\binom{p+d}{p} + \binom{p-1}{p}\binom{p}{p} = \binom{p+d}{p} \neq 0.$$

The formula now follows.

ii. By definition, the regularity of a module M is $\max\{j - i \mid \beta_{i,j}(M) \neq 0\}$. By Theorem 4.13, we know exactly when the graded Betti numbers of $R/I_t(L_n)$ are nonzero, and the formula follows directly from (4.15).

□

References

[1] A. Alilooee, S. Faridi. On the resolution of path ideals of cycles. *J. Comm. Algebra*, to appear, 2012.

[2] R. Bouchat, H. T. Ha, and A. O' Keefe. Path ideals of rooted trees and their graded betti numbers. *Journal of Combinatorial Theory, Series A*, 118 (2011) 2411-2425.

[3] A. Conca and E. De Negri. M-sequences, graph ideals and ladder ideals of linear types. *J. Algebra*, 211:599–624, 1999.

[4] CoCoA Team. Cocoa: a system for doing computations in commutative algebra. Technical report, Available at http://cocoa.dima.unige.it.

[5] R. P. Grimaldi. *Discrete and Combinatorial Mathematics, An applied Introduction*. Pearson, 2004.

[6] D. R. Grayson and M. E. Stillman. Macaulay 2, a software system for research in algebraic geometry. Technical report, http://www.math.uiuc.edu/Macaulay2/.

[7] J. He and A. Van Tuyl. Algebraic properties of path ideal of a tree. *Comm. Algebra*, 38:1725–1742, 2010.

[8] S. Jacques. Betti numbers of graph ideals. PhD thesis available on line *http://arxive.org/math/0410107*, 2004.

[9] R. P. Stanley. *Enumerative Combinatorics*. Cambridge University Press, 1997.