Computational Algebra and Combinatorics of Toric Ideals

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Introduction

This book is based on six lectures and tutorials that were prepared for a workshop in computational commutative algebra at the Harish Chandra Research Institute (HRI) at Allahabad, India in December 2003. The workshop was aimed at graduate students and was conducted as part of the conference on Commutative Algebra and Combinatorics held at HRI from December 8–13, 2003. The material in the early chapters is based heavily on the research monograph *Gröbner Bases and Convex Polytopes* [Stu96] by Bernd Sturmfels. We have attempted to explain the key concepts in this monograph to students who are not familiar with either Gröbner bases or convex polytopes by building up the basics of these theories from scratch. The tutorials and examples are meant to help this development. There is a special emphasis on actual computations via various software packages.

Lectures 1, 3, and 5 were written and delivered by Rekha R. Thomas (University of Washington), and Lectures 2, 4, and 6 were written and delivered by Diane Maclagan (Rutgers University). The tutorials for the lectures were prepared and conducted by Tony Puthenpurakal of IIT Bombay and A.V. Jayanthan of HRI (Tutorial 1); Amit Khetan of the University of Massachusetts, Amherst (Tutorial 2); Leah Gold of Cleveland State University (Tutorials 3 and 5); and Sara Faridi of Dalhousie University (Tutorial 4). Amit Khetan also contributed to Lecture 6.

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Chapter 1

Gröbner Basics

1.1 Introduction

This first chapter aims to give a brief introduction to the basics of Gröbner basis theory. There are many excellent books on Gröbner bases and their applications such as [AL94], [CLO97], [CLO98], [GP02] and [KR00]. Our account here will be brief, biased and focused.

Throughout this book, we let $S := \mathbf{k}[x_1, \ldots, x_n] =: \mathbf{k}[\mathbf{x}]$ be a polynomial ring over a field $\mathbf{k}$. For the most part we may take $\mathbf{k}$ to be the field of complex numbers $\mathbb{C}$, the field of rational numbers $\mathbb{Q}$, or the field of real numbers $\mathbb{R}$. A nonempty subset $I \subset S$ is called an *ideal* of S if I is closed under

1. addition — i.e., for all $f, g \in I$, $f + g \in I$, and
2. multiplication by elements of S — i.e., $h \in S$, $f \in I$ implies $hf \in I$.

An ideal I is *finitely generated* if there exists a subset $\{f_1, \ldots, f_t\} \subset I$, called a *basis* of I, such that $I = \{\sum_{i=1}^{t} h_i f_i \,:\, h_i \in S\}$. We write $I = \langle f_1, \ldots, f_t \rangle$. In the 1890s David Hilbert proved that every polynomial ideal has a finite basis. This fact is known as the Hilbert basis theorem. In this lecture we define and construct special bases of I called Gröbner bases.

The simplest polynomials in S are *monomials*, which are the polynomials $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ where $\mathbf{a} := (a_1, \ldots, a_n) \in \mathbb{N}^n$. Here $\mathbb{N}$ stands for the set of nonnegative integers. We denote this monomial by $\mathbf{x}^{\mathbf{a}}$. If $c \in \mathbf{k}$, then $c\mathbf{x}^{\mathbf{a}}$ is called a *term* in S. Every polynomial is a finite sum of terms. The *support* of a polynomial $f \in S$ is the set $\mathrm{supp}(f) := \{\mathbf{a} \in \mathbb{N}^n \,:\, f = \sum c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}},\ c_{\mathbf{a}} \neq 0\}$.

Example 1.1.1. If $f = \frac{3}{2} x_1^6 x_5^8 - \sqrt{2} x_1 x_3^{16} - 12 \in \mathbb{R}[x_1, \ldots, x_6]$, then $\mathrm{supp}(f) = \{(6,0,0,0,8,0), (1,0,16,0,0,0), (0,0,0,0,0,0)\} \subset \mathbb{N}^6$.

The *variety* of $f_1, \ldots, f_t \in S$ is the set

$$\mathcal{V}(f_1, \ldots, f_t) := \{\mathbf{p} \in \mathbf{k}^n \,:\, f_1(\mathbf{p}) = f_2(\mathbf{p}) = \cdots = f_t(\mathbf{p}) = 0\}.$$

The variety of the ideal $I = \langle f_1, \ldots, f_t \rangle$ is

$$\mathcal{V}(I) = \{\mathbf{p} \in \mathbf{k}^n \,:\, f(\mathbf{p}) = 0, \forall\, f \in I\} = \mathcal{V}(f_1, \ldots, f_t).$$

1.2 Motivation

Given an ideal $I \subset S$ and a polynomial $f \in S$, a fundamental problem is to decide whether f belongs to I. This is known as the *ideal membership problem*. We will see shortly that Gröbner bases can be used to solve this. We begin by examining algorithms for ideal membership in the two well-known families of univariate and linear ideals. For ease of exposition we choose the field $\mathbf{k} = \mathbb{C}$ for the first family and $\mathbf{k} = \mathbb{R}$ for the second.

(i) **Univariate Ideals:** ([CLO97, §1.5]) Since $\mathbb{C}[x]$ is a principal ideal domain, every ideal in $\mathbb{C}[x]$ is of the form $I = \langle g \rangle$ where $g \in \mathbb{C}[x]$. The polynomial g is of the form

$$g = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0$$

where $c_i \in \mathbb{C}$. Without loss of generality, we may assume that the leading coefficient c_d is one. When this is the case, we say that g is *monic*. The degree of g, denoted as $\deg(g)$, is then d.

- **Finding the basis**: It is not hard to show theoretically that I is generated by any polynomial g in I of least degree. If I is given as $I = \langle f_1, \ldots, f_t \rangle \subset \mathbb{C}[x]$, then $g = \gcd(f_1, \ldots, f_t)$, the greatest common divisor of $f_1, \ldots, f_t$. The gcd can be computed by the Euclidean Algorithm.

- **Ideal membership**: If $f \in \mathbb{C}[x]$ then, by the usual division algorithm for univariate polynomials, there exist unique polynomials h and $r \in \mathbb{C}[x]$ such that $f = hg + r$ where r is the remainder and $\deg(r) < \deg(g)$. The polynomial f lies in $I = \langle g \rangle$ if and only if $r = 0$. Thus ideal membership can be determined by the division algorithm.

- **Solving** $\{f_1 = f_2 = \cdots = f_t = 0\}$: Let $g = \gcd(f_1, \ldots, f_t)$. Then the variety $\mathcal{V}(f_1, \ldots, f_t)$ equals $\mathcal{V}(g)$. The roots of a univariate polynomial can be found via radicals when its degree is small and by numerical methods otherwise. See [Stu02] for a recent survey of methods for solving polynomial equations.

(ii) **Linear ideals:** (See Example 1.5 in [Stu96]). Let $A \in \mathbb{Z}^{d\times n}$ be a matrix with ith row $A_i = (a_{i1}, \ldots, a_{in})$ and $\mathrm{rank}(A) = d$. Consider the *linear ideal*

$$I = \left\langle \sum_{j=1}^{n} a_{ij}x_j \, : \, i = 1, \ldots, d \right\rangle \subseteq \mathbb{R}[x_1, \ldots, x_n].$$

The variety $\mathcal{V}(I)$ is the $(n-d)$-dimensional vector space $\ker_{\mathbb{R}}(A) = \{\mathbf{p} \in \mathbb{R}^n : A\mathbf{p} = \mathbf{0}\}$.

A nonzero linear form f in I is a *circuit* of I if f has minimal support (with respect to inclusion) among all linear forms in I. The coefficient vector of a circuit of I is a vector in the row span of A of minimal support. For $J \subseteq [n] := \{1, \ldots, n\}$ with $|J| = d$, let $D[J] := \det(A_J)$ be the determinant of A_J, where A_J is the submatrix of A with column indices J. The following algorithm computes the circuits of I.

Algorithm 1.2.1. [Stu02, Chapter 8.3] Let B be an integer $(n-d)\times n$ matrix whose rows form a basis of $\ker_{\mathbb{R}}(A)$. Then every vector in the row span of A (which includes the rows of A) is a linear dependency of the columns of B since $BA^t = 0$. Thus the coefficient vectors of the circuits of I are the minimal dependencies of the columns of B which are also called circuits of B.

For any $(d+1)$-subset $\tau = \{\tau_1, \ldots, \tau_{d+1}\} \subseteq [n]$ form the vector

$$C_\tau := \sum_{i=1}^{d+1} (-1)^i \ \det(B_{\tau\setminus\{\tau_i\}}) \ \mathbf{e}_{\tau_i}$$

where $\mathbf{e}_j$ is the jth unit vector of $\mathbb{R}^n$. If C_τ is nonzero, then compute the primitive vector obtained by dividing through with the gcd of its components. The resulting vector is a circuit of B and all circuits of B are obtained this way. Can you prove this?

Example 1.2.2. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \end{pmatrix}$. Then

$$I = \langle x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5, \ 6x_1 + 7x_2 + 8x_3 + 9x_4 + 10x_5 \rangle.$$

The rows of

$$B = \begin{pmatrix} 3 & -4 & 0 & 0 & 1 \\ 2 & -3 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 \end{pmatrix}$$

form a basis for $\ker_{\mathbb{R}}(A)$. Check that $BA^t = 0$. Let us compute one of the circuits of B. For $\tau = \{1, 2, 3, 4\}$,

$$C_\tau = -\det\begin{pmatrix} -4 & 0 & 0 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}\mathbf{e}_1 + \det\begin{pmatrix} 3 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\mathbf{e}_2$$

$$-\det\begin{pmatrix} 3 & -4 & 0 \\ 2 & -3 & 1 \\ 1 & -2 & 0 \end{pmatrix}\mathbf{e}_3 + \det\begin{pmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 1 & -2 & 1 \end{pmatrix}\mathbf{e}_4$$

which equals $(-4, -3, -2, -1, 0)$. Hence $-4x_1 - 3x_2 - 2x_3 - x_4$ or equivalently, $4x_1 + 3x_2 + 2x_3 + x_4$ is a circuit of I.

The above algorithm provides a linear algebraic method for computing the circuits of a linear ideal. The software package `4ti2` [Hem] computes circuits via a different method. We now prove that Gaussian elimination computes circuits of I. Let I_d be the $d \times d$ identity matrix and $C = (c_{ij})$ be the Gauss-Jordan form of A. Since the rank of A is d, we can assume, after possibly permuting columns, that C has the form $[I_d|*]$.

Proposition 1.2.3. *Let $C = (c_{ij}) = [I_d|*] \in \mathbb{R}^{d\times n}$ be the Gauss-Jordan form (reduced row-echelon form) of A and $g_i = x_i + \sum_{j=d+1}^{n} c_{ij}x_j$, $i = 1, \ldots, d$ be the linear forms corresponding to the rows of C. Then*

1. *$\{g_1, \ldots, g_d\}$ is a minimal generating set for I, and*
2. *the linear forms $g_1, \ldots, g_d$ are circuits of I.*

Proof. 1. Since every row of C is a linear combination of rows of A and vice-versa, every g_i is a linear combination of the f_i's and every f_i a linear combination of the g_i's. Thus $I = \langle f_1, \ldots, f_d\rangle = \langle g_1, \ldots, g_d\rangle$.

2. Suppose g_1 is not a circuit. Then there exists a linear polynomial $g \in I$ such that $\mathrm{supp}(g) \subsetneq \mathrm{supp}(g_1)$. However, $g = t_1g_1 + \ldots + t_dg_d$ for scalars $t_1, \ldots, t_d \in \mathbb{R}$. Since $\mathrm{supp}(g) \subset \mathrm{supp}(g_1)$, $t_2 = t_3 = \cdots = t_d = 0$. This implies that $g = t_1g$, $t_1 \neq 0$ and hence $\mathrm{supp}(g) = \mathrm{supp}(g_1)$, a contradiction. The same argument can be repeated for $g_2, \ldots, g_d$.

$\square$

Proposition 1.2.4. *Assume the same setup as in Proposition 1.2.3.*

1. *A polynomial $f \in S$ is an element of I if and only if successively replacing every occurrence of x_i , $i = 1, \ldots, d$ in f with $-\sum_{j=d+1}^{n} c_{ij}x_j$ results in the zero polynomial.*

2. *The linear system $A\mathbf{x} = \mathbf{0}$ can be solved by back-solving the "triangularized" system $g_1 = g_2 = \cdots = g_d = 0$.*

Proof. 1. Let f' be obtained from f by successively replacing every occurrence of x_i, $i = 1, \ldots, d$ in f with $-\sum_{j=d+1}^{n} c_{ij}x_j$. Then $f' \in \mathbb{R}[x_{d+1}, \ldots, x_n]$. This implies that $f = \sum_{i=1}^{d} h_i g_i + f'$ where $h_i \in \mathbb{R}[\mathbf{x}]$. If $f' = 0$ then clearly $f \in I$. Conversely, if $f \in I$, then $f' = f - \sum_{i=1}^{d} h_i g_i \in I \cap \mathbb{R}[x_{d+1}, \ldots, x_n] = \{0\}$. The last equality follows from Proposition 1.2.3(1) which implies that no polynomial combination of $g_1, \ldots, g_d$ can lie in $\mathbb{R}[x_{d+1}, \ldots, x_n]$.

2. This is the familiar method of solving linear systems by Gaussian elimination from linear algebra.

□

The proof of Proposition 1.2.4(1) is employing a division algorithm for linear polynomials in many variables that succeeds in determining ideal membership for linear ideals. Note that when we perform Gauss-Jordan elimination on A to obtain $C = [I_d \,|\, *]$, we are implicitly ordering the variables in $\mathbb{R}[\mathbf{x}]$ such that $x_1 > x_2 > \cdots > x_n$. The division algorithm used in the proof of Proposition 1.2.4(1) replaces every occurrence of the "leading term" x_i in g_i with $x_i - g_i$, which is the sum of the "trailing terms" in g_i.

Thus the questions we started with have well-known algorithms and answers when I is either a univariate principal ideal or a multivariate linear ideal. We now seek a common generalization of these methods to multivariate polynomials of arbitrary degrees, and their ideals. This leads us to Gröbner bases of polynomial ideals.

1.3 Gröbner bases

In order to mimic the procedures from the last section, we first need to impose an ordering of the monomials in S so that the terms in a polynomial are always ordered. This is important if the generalized division algorithm is to replace the leading term of a divisor by the sum of its trailing terms.

Definition 1.3.1. A *term order* $\succ$ on S is a total order on the monomials of S such that

1. $\mathbf{x}^{\mathbf{a}} \succ \mathbf{x}^{\mathbf{b}}$ implies that $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{c}} \succ \mathbf{x}^{\mathbf{b}}\mathbf{x}^{\mathbf{c}}$ for all $\mathbf{c} \in \mathbb{N}^n$, and
2. $\mathbf{x}^{\mathbf{a}} \succ \mathbf{x}^{\mathbf{0}} = 1$ for all $\mathbf{a} \in \mathbb{N}^n \backslash \{\mathbf{0}\}$.

Part 2 of Definition 1.3.1 implies that every term order is a *well-ordering* on the monomials of S which will be important for the finite termination of the algorithms described in this chapter.

Example 1.3.2. The most common examples of term orders are the *lexicographic (lex) order* and the *graded reverse lexicographic order* on S with respect to a fixed ordering of the variables, such as $x_1 \succ x_2 \succ \cdots \succ x_n$.

In the lex ordering, $\mathbf{x}^{\mathbf{a}} \succ \mathbf{x}^{\mathbf{b}}$ if and only if the left-most nonzero term in $\mathbf{a} - \mathbf{b}$ is positive. For example, if $x \succ y \succ z$, then

$$x^3 \succ x^2y \succ x^2z \succ xy^2 \succ xyz \succ xz^2 \succ y^3 \succ y^2z \succ yz^2 \succ z^3.$$

In the (graded) reverse lexicographic order, $\mathbf{x}^{\mathbf{a}} \succ \mathbf{x}^{\mathbf{b}}$ if and only if either $\deg(\mathbf{x}^{\mathbf{a}}) = |\mathbf{a}| > \deg(\mathbf{x}^{\mathbf{b}}) = |\mathbf{b}|$, or $\deg(\mathbf{x}^{\mathbf{a}}) = \deg(\mathbf{x}^{\mathbf{b}})$ and the right-most nonzero term in $\mathbf{a} - \mathbf{b}$ is negative. Here $|\mathbf{a}|$ denotes the 1-norm of $\mathbf{a}$ which is the degree of $\mathbf{x}^{\mathbf{a}}$ under the total-degree grading of S. Again if $x \succ y \succ z$, then

$$x^3 \succ x^2y \succ xy^2 \succ y^3 \succ x^2z \succ xyz \succ y^2z \succ xz^2 \succ yz^2 \succ z^3.$$

The reverse lexicographic order defined here is *degree-compatible* or *graded* which means that it first compares two monomials by degree and then breaks ties using the rule described. We call it *grevlex*. Note that there are $n!$ lex and grevlex orders on S.

We will see in the next chapter that vectors in $\mathbb{R}^n_{\geq 0}$ can be used to define term orders. We now fix a term order $\succ$ on S. Given a polynomial $f = \sum c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}} \in S$, the *initial term* or *leading term* of f with respect to $\succ$ is the term $c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}}$ in f such that $\mathbf{x}^{\mathbf{a}} \succ \mathbf{x}^{\mathbf{a}'}$ for all $\mathbf{a}' \in \operatorname{supp}(f)$ different from $\mathbf{a}$. It is denoted by $\operatorname{in}_{\succ}(f)$. The *initial monomial* of f is the monic term $\mathbf{x}^{\mathbf{a}}$. We can write $f = \operatorname{in}_{\succ}(f) + f'$.

Example 1.3.3. Let $f = 3x_1x_3^2 + \sqrt{2}x_3^2 - x_1x_2^2 \in \mathbb{C}[x_1, x_2, x_3]$ and $\succ$ be the reverse lexicographic order with $x_1 \succ x_2 \succ x_3$. Then $x_1x_2^2$ is the initial monomial of f and $-x_1x_2^2 = \operatorname{in}_{\succ}(f)$ is the initial term of f.

We can now attempt to write down a division algorithm for multivariate polynomials. The following algorithm is from [CLO97, Theorem 3].

Algorithm 1.3.4. Division algorithm for multivariate polynomials [CLO97, Theorem 3]
INPUT: A dividend $f \in S$, an ordered set of divisors $\{f_1, \dots, f_s\}$, $f_i \in S$, and a term order $\succ$.
OUTPUT: Polynomials $a_1, \dots, a_s, r \in S$ such that $f = \sum_{i=1}^{s} a_i f_i + r$ where either $r = 0$ or no term in r is divisible by $\mathrm{in}_\succ(f_1), \dots, \mathrm{in}_\succ(f_s)$.

INITIALIZE: $a_1 := 0, \dots, a_s := 0, r := 0$; $p := f$
WHILE $p \neq 0$ DO
    $i := 1$
    divisionoccurred := false
    WHILE $i \leq s$ AND divisionoccurred = false DO
        IF $\mathrm{in}_\succ(f_i)$ divides $\mathrm{in}_\succ(p)$ THEN
            $a_i := a_i + \mathrm{in}_\succ(p)/\mathrm{in}_\succ(f_i)$
            $p := p - (\mathrm{in}_\succ(p)/\mathrm{in}_\succ(f_i))f_i$
            divisionoccurred := true
        ELSE
        $i := i + 1$
    IF divisionoccurred = false THEN
        $r := r + \mathrm{in}_\succ(p)$
        $p := p - \mathrm{in}_\succ(p)$

Example 1.3.5. (Taken from [CLO97, Chapter 3, §3])
Dividing $f = x^2y + xy^2 + y^2$ by the ordered list of polynomials $\{f_1 = xy - 1, f_2 = y^2 - 1\}$, we get $f = (x + y)f_1 + f_2 + x + y + 1$. Switching the order of the divisors and redoing the division gives $f = (x + 1)f_2 + x(f_1) + 2x + 1$. We list all polynomials in the two division algorithms according to lex order with $x \succ y$. Note that the remainders are different. This example shows that the above division algorithm for multivariate polynomials has several drawbacks one of which is that it does not produce unique remainders. This makes it impossible to check ideal membership of f in $\langle f_1, f_2 \rangle$ by dividing f by the generators f_1, f_2.

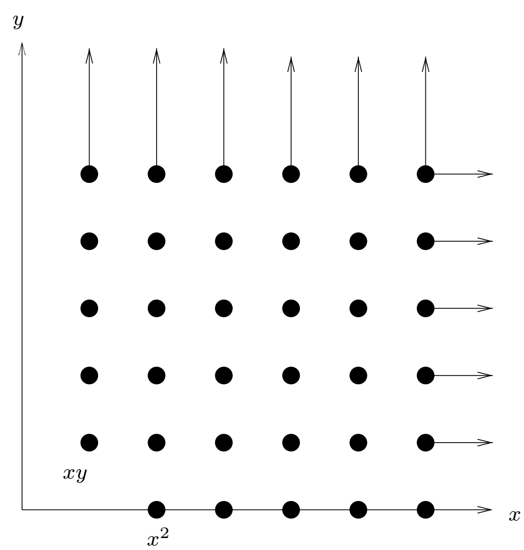
The above example shows that arbitrary generating sets of ideals and a naive extension of the usual division algorithm cannot be used for ideal membership. We will see that this difficulty disappears when the basis of the ideal is a Gröbner basis.

A *monomial ideal* is an ideal generated by monomials. The *initial ideal* of an ideal $I \subset S$ is the monomial ideal

$$\mathrm{in}_\succ(I) := \langle \mathrm{in}_\succ(f) \, : \, f \in I \rangle \subseteq S.$$

A monomial ideal in S can be depicted by its *staircase diagram* in $\mathbb{N}^n$ which is the collection of all exponent vectors of monomials in the ideal. Clearly, this set of "dots" is closed under the addition of $\mathbb{N}^n$ to any of the dots. Equivalently, the complement in $\mathbb{N}^n$ is a *down-set* or *order ideal* in $\mathbb{N}^n$.

Example 1.3.6. Let $I = \langle x^2 - y, x^3 - x\rangle \subset \mathbf{k}[x, y]$ and $\succ$ be the lexicographic order with $x \succ y$. The polynomial $x(x^2 - y) - (x^3 - x) = -xy + x \in I$ which shows that $\mathrm{in}_\succ(I) \supset \{x^2, xy\}$. We will see later that in fact, $\mathrm{in}_\succ(I) = \langle x^2, xy, y^2\rangle$. The monomial ideal $\langle x^2, xy\rangle$ has the following staircase diagram.



By the Gordan-Dickson Lemma [CLO97], all monomial ideals of S have a unique minimal finite generating set consisting of monomials. Hence there exists $g_1, \ldots, g_s \in I$ such that $\mathrm{in}_\succ(I) = \langle \mathrm{in}_\succ(g_1), \ldots, \mathrm{in}_\succ(g_s)\rangle$.

Definition 1.3.7.

1. A finite set of polynomials $\mathcal{G}_\succ(I) = \{g_1, \ldots, g_s\} \subset I$ is a *Gröbner basis* of I with respect to $\succ$ if $\mathrm{in}_\succ(I) = \langle \mathrm{in}_\succ(g_1), \ldots, \mathrm{in}_\succ(g_s)\rangle$. We may assume that each $\mathrm{in}_\succ(g_i)$ is a monomial.

2. If $\{\mathrm{in}_\succ(g_1), \ldots, \mathrm{in}_\succ(g_s)\}$ is the unique minimal generating set of $\mathrm{in}_\succ(I)$, we say that $\mathcal{G}_\succ(I)$ is a *minimal* Gröbner basis of I with respect to $\succ$.

3. A minimal Gröbner basis $\mathcal{G}_\succ(I)$ is *reduced* if no non-initial term of any g_i is divisible by any of $\mathrm{in}_\succ(g_1), \ldots, \mathrm{in}_\succ(g_s)$.

4. The monomials of S that do not lie in the initial ideal $\mathrm{in}_\succ(I)$ are called the *standard monomials* of $\mathrm{in}_\succ(I)$.

Every term order produces a unique reduced Gröbner basis of I. Check that a Gröbner basis of I is indeed a generating set (basis) of I.

Lemma 1.3.8. *If $\mathcal{G}_\succ(I)$ is a Gröbner basis of I with respect to the term order $\succ$, then the remainder of any polynomial after division by $\mathcal{G}_\succ(I)$ is unique.*

Proof. Suppose $\mathcal{G}_\succ(I) = \{g_1, \ldots, g_s\}$ and we divide a polynomial $f \in S$ by $\mathcal{G}_\succ(I)$ obtaining two remainders $r_1, r_2 \in S$. We get two expressions

$$f = \sum a_i g_i + r_1 = \sum a_i' g_i + r_2$$

which implies that $r_1 - r_2 \in I$ and that no term of $r_1 - r_2$ is divisible by $\mathrm{in}_\succ(g_i)$ for any $g_i \in \mathcal{G}_\succ(I)$. However this implies that $r_1 - r_2 = 0$ since otherwise the nonzero term $\mathrm{in}_\succ(r_1 - r_2) \in \mathrm{in}_\succ(I)$ and some $\mathrm{in}_\succ(g_i)$ would divide it. □

Corollary 1.3.9. *Gröbner bases solve the ideal membership problem.*

Definition 1.3.10.

1. The unique remainder of a polynomial $f \in S$ obtained after dividing f with the reduced Gröbner basis $\mathcal{G}_\succ(I)$ is called the *normal form* of f with respect to $\mathcal{G}_\succ(I)$.
2. The division of f by $\mathcal{G}_\succ(I)$ is called *reduction* by $\mathcal{G}_\succ(I)$.

The passage from an ideal to one of its initial ideals is a "flat" deformation that preserves several invariants such as Hilbert function, dimension and degree. This property has allowed Gröbner bases and initial ideals to become an important theoretical tool in algebra as it allows one to study a complicated ideal by passing to monomial ideals. The details of this deformation point of view can be found in Chapter 15 of [Eis94]. The connection was first worked out by Dave Bayer in his Ph.D. thesis [Bay82].

1.4 Buchberger's algorithm

In [Buc65] Buchberger developed an algorithm to compute the reduced Gröbner basis of an ideal $I = \langle f_1, \ldots, f_t \rangle$ with respect to any prescribed term order $\succ$ on S. The algorithm needs as a subroutine the calculation of the S-pair of two polynomials f and g, denoted by S-pair(f, g). Let $\mathrm{lcm}(\mathrm{in}_\succ(f), \mathrm{in}_\succ(g))$ be the least common multiple of $\mathrm{in}_\succ(f)$ and $\mathrm{in}_\succ(g)$. This lcm is the product of the coefficients in $\mathrm{in}_\succ(f)$ and $\mathrm{in}_\succ(g)$ and the lcm of the initial monomials in $\mathrm{in}_\succ(f)$ and $\mathrm{in}_\succ(g)$. Then

$$\text{S-pair}(f, g) = \frac{\mathrm{lcm}(\mathrm{in}_\succ(f), \mathrm{in}_\succ(g))}{\mathrm{in}_\succ(g)} f - \frac{\mathrm{lcm}(\mathrm{in}_\succ(f), \mathrm{in}_\succ(g))}{\mathrm{in}_\succ(f)} g.$$

We also let $\text{rem}_G(h)$ denote the remainder obtained by dividing the polynomial h by an ordered list of polynomials G.

We now describe Buchberger's algorithm. The algorithm hinges on the important fact that a set of polynomials G form a Gröbner basis with respect to $\succ$ if and only if for each pair $f, f' \in G$, $\text{rem}_G(\text{S-pair}(f, f')) = 0$. The proof can be found in any of the books mentioned in the introduction. We reproduce the algorithm from [CLO97, Theorem 2].

Algorithm 1.4.1. Buchberger's algorithm [CLO97, Theorem 2]
INPUT: $F = \{f_1, \ldots, f_t\}$ a basis of the ideal $I \subset S$ and a term order $\succ$.
OUTPUT: The reduced Gröbner basis $\mathcal{G}_\succ(I)$ of I with respect to $\succ$.

G := F
REPEAT
    $G' := G$
    For each pair $\{p, q\}$, $p \neq q$ in G' do
        $S := \text{rem}_{G'}(\text{S-pair}(p, q))$
        If $S \neq 0$ then $G := G \cup \{S\}$
UNTIL $G = G'$.
G is a Gröbner basis for I at this point.

Producing a minimal Gröbner basis with respect to $\succ$.
Let G be a Gröbner basis of I. Make all elements of G monic by dividing each by its leading coefficient. For each $g \in G$, remove it from G if its leading term is divisible by the leading term of another element $g' \in G$.

Producing the reduced Gröbner basis with respect to $\succ$.
$G' := G$ where G is a minimal Gröbner basis of I, $\mathcal{G}_\succ(I) := \emptyset$
For each $g \in G$ do
    $g' = \text{rem}_{G' \setminus g}(g)$; $\mathcal{G}_\succ(I) = \mathcal{G}_\succ(I) \cup \{g'\}$; $G' = G' \setminus \{g\} \cup \{g'\}$.

Example 1.4.2. For the ideal $I = \langle f_1 := x^2 - y, f_2 := x^3 - x\rangle$ with the lex order $x \succ y$, we begin by setting $G = \{f_1, f_2\}$. The first step of the Buchberger algorithm computes $\text{S-pair}(f_1, f_2) = f_2 - x(f_1) = xy - x$. Note that $\text{rem}_G(xy - x) = xy - x$. Thus we define $f_3 := xy - x$ and update G to $G = \{f_1, f_2, f_3\}$. Next check that $\text{S-pair}(f_1, f_3) = y^2 - x^2$ which reduces modulo f_1 to $f_4 := y^2 - y$. Therefore, $G = \{f_1, f_2, f_3, f_4\}$. All other S-pairs reduce to zero modulo G. The reduced Gröbner basis of I with respect to $\succ$ is $\{f_1 = x^2 - y, f_3 = xy - x, f_4 = y^2 - y\}$.

Definition 1.4.3. A finite subset of I is a *universal* Gröbner basis of I if it is a Gröbner basis of I with respect to every term order.

In Chapter 2 we will see that every ideal has a universal Gröbner basis. We construct a universal Gröbner basis for linear ideals.

Proposition 1.4.4. (Linear ideals revisited)

1. *The set of linear forms $\{g_1, \dots, g_d\}$ computed from the Gauss-Jordan form C of the matrix A in Proposition 1.2.3 is the reduced Gröbner basis of the linear ideal $I = \langle g_1, \dots, g_d \rangle$ with respect to any term order such that $x_1 \succ \cdots \succ x_n$.*

2. *The set of all circuits of I is a minimal universal Gröbner basis of I. ([Stu96, Proposition 1.6]).*

Proof. 1. Note that the terms of each $g_i = x_i + \sum_{j=d+1}^{n} c_{ij} x_j$ are already ordered in decreasing order with respect to the above lex order and that $G = \{g_1, \dots, g_d\}$ is reduced in the sense that no term of any g_i other than x_i lies in $\mathrm{in}_\succ(I) = \langle x_1, \dots, x_d \rangle$. To show that G is a Gröbner basis it suffices to show that the remainder obtained by dividing S-pair(g_i, g_j) with respect to G is zero for all $i \neq j \in [d]$. This follows from the following general fact: If $f = \mathrm{in}_\succ(f) + f'$ and $g = \mathrm{in}_\succ(g) + g'$ are two monic polynomials such that $\mathrm{in}_\succ(f)$ and $\mathrm{in}_\succ(g)$ are relatively prime, then

$$\text{S-pair}(f, g) = \mathrm{in}_\succ(g)\, f - \mathrm{in}_\succ(f)\, g$$

which reduces to zero modulo $\{f, g\}$. This is an important criterion for avoiding S-pairs that will eventually reduce to zero, known as *Buchberger's first criterion.*

2. (proof from [Stu96]) The argument in part 1 shows that every reduced Gröbner basis of I arises from a Gauss-Jordan form. We argued earlier that all the elements of these Gröbner bases are circuits of I. Thus the circuits of I form a universal Gröbner basis of I.

   To prove minimality, we need to argue that each circuit l appears in some reduced Gröbner basis of I. Let $\succ$ be a term order such that $\{x_i : i \notin \mathrm{supp}(l)\} \succ \{x_i : i \in \mathrm{supp}(l)\}$ and $\mathcal{G} := \mathcal{G}_\succ(I)$. The term order makes any monomial not containing a variable in the first group of variables cheaper, regardless of its degree, than any monomial containing a variable in the first group. Such term orders are known as *elimination orders*. The lex order is an example of an elimination order with $\{x_1\} \succ \{x_2\} \succ \cdots \succ \{x_n\}$. Suppose l does not appear in $\mathcal{G}$. Then there exists $l' \in \mathcal{G}$ such that $\mathrm{in}_\succ(l) = \mathrm{in}_\succ(l')$. By the elimination property of $\succ$, $\mathrm{supp}(l') \subseteq \mathrm{supp}(l)$ and hence $\mathrm{supp}(l - l') \subsetneq \mathrm{supp}(l)$. However this contradicts that l is a circuit of I as $l - l'$ is a nonzero linear form with strictly smaller support. □

Here is another example of a universal Gröbner basis.

Proposition 1.4.5. *[Stu96, Example 1.4] Consider a polynomial ring in $2m$ indeterminates:*

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \end{pmatrix}$$

and the ideal I generated by the 2×2 minors $D_{ij} = x_{1i}x_{2j} - x_{1j}x_{2i}$, $1 \leq i < j \leq m$ of the above matrix. Then $\{D_{ij}\}$ is a universal Gröbner basis of I.

Finding universal Gröbner bases is a hard task in general. Chapter 2 gives a general algorithm for finding a universal Gröbner basis of an ideal. In special cases, a universal Gröbner basis can be described using intrinsic features of the ideal. In Chapter 3 we will construct a universal Gröbner basis for the special class of ideals called toric ideals.

1.5 Tutorial 1

Macaulay 2 [GS] is a mathematical software package, freely available from
http://www.math.uiuc.edu/Macaulay2/
Start Macaulay 2 with the command "M2" (without quotes), and you will be provided with an input prompt. We begin by illustrating some of the basic commands that we need in this tutorial. Most often the commands are self explanatory. We have provided an appropriate description whenever they are not.

```
i1 : 2+2
o1 = 4
i2 : 3/5 + 7/11
     68
o2 = --
     55
o2 : QQ
i3 : 2*3*4
o3 = 24
i4 : 2^8
o4 = 256
i5 : 6!
o5 = 720
```

Note that an input to Macaulay 2 is on a line that starts with an `i` and the outputs are on lines that start with `o`. For instance, `o2` is the answer to the input $3/5+7/11$ in `i2`. The second output line `o2` tells you what sort of object the output is. In this case, it is an element of the ring $\mathbb{Q}$ which in Macaulay 2 is denoted as `QQ`.

Rings and Fields in Macaulay 2 :

`ZZ` - The ring of integers.
`QQ` - The field of rationals.
`RR` - The field of reals.
`CC` - The field of complex numbers.
`ZZ/p` - Finite field of order `p`, where `p` is a prime number.

Warning : Macaulay 2 does not accept rings of order a composite integer. Polynomial rings are defined over `ZZ, ZZ/p` and `QQ`.

```
i6 : k = ZZ/32003
```

```
o6 = k
o6 : QuotientRing
i7 : R = k[x,y,z]
o7 = R
o7 : PolynomialRing
i8 : R=QQ[a_0..a_6]
o8 = R
o8 : PolynomialRing
```

Different term orders in Macaulay 2 :

The default order in Macaulay 2 is the graded reverse lexicographic order. Other orders one can use are :

GRevLex	Graded Reverse Lexicographic order (default)
GLex	Graded Lexicographic order
Lex	Lexicographic order
Eliminate n	Elimination order, eliminating first n variables
ProductOrder {n1, ..., nv}	Product order
Weight Order	Represent the term order using a vector (See Chapter 2 for details)

```
i9 : R = QQ[x,y,z, MonomialOrder => GLex]
o9 = R
o9 : PolynomialRing
i10 : R = QQ[a..e, Weights => {2,89,100,23,1}, MonomialSize => 16]
o10 = R
o10 : PolynomialRing
```

In line `i10`, the numbers in the curly brackets represent the weight for each variable. Setting `MonomialSize => n` specifies that monomial exponents may be as large as $2^{(n-1)} - 1$. The default value is 8, allowing for exponents up to 127. Currently the maximum value is 16, allowing for exponents up to 32767.

Ideals of a polynomial ring:

```
i11 : R = QQ[x,y,z]
o11 = R
o11 : PolynomialRing
i12 : I = ideal (x^2+y^2, x^3*y+y^3)
              2    2   3     3
o12 = ideal (x  + y , x y + y )
o12 : Ideal of R
```

The command `leadTerm(I)` gives the generators of the initial ideal of I with respect to the term order specified in the ring `R`.

```
i13 : leadTerm(I)
o13 = | x2 xy3 y5 |
               1       3
o13 : Matrix R  <--- R

i14 : gb I
o14 = | x2+y2 xy3-y3 y5+y3 |
o14 : GroebnerBasis
i15 : J = ideal(x^3*y+x*y^2,x^4,x^3*y^2)
               3       2   4   3 2
o15 = ideal (x y + x*y , x , x y )
o15 : Ideal of R
i16 : I + J
               2    2   3     3   3       2   4   3 2
o16 = ideal (x  + y , x y + y , x y + x*y , x , x y )
o16 : Ideal of R
i17 : I*J
               5     3 3    3 2      4   6    4 2   5 2    3 4
o17 = ideal (x y + x y  + x y  + x*y , x  + x y , x y  + x y ,
              6 2    4 3    3 4      5   7     4 3   6 3    3 5
             x y  + x y  + x y  + x*y , x y + x y , x y  + x y )
o17 : Ideal of R
```

To suppress the output, put a semicolon at the end of the command.

```
i18 : intersect(I, J);
o18 : Ideal of R

i19 : I:J
               2                2
o19 = ideal (y  + x, x*y - y, x  - x)
o19 : Ideal of R
```

1. Define the polynomial ring $\mathbb{Q}[x, y, z, w]$, ideals $I = \langle x^2 + 2xy^3, z^2 - w^3, xz - 3yw\rangle$ and $J = \langle y^3 - 2zw^2, z^2 - 3yw, x^2y - z^2w\rangle$. Compute initial ideal and Gröbner bases of I, J, $I + J$, IJ, $I : J$ and $I \cap J$ with respect to the monomial orderings lex and grevlex.

Do the following exercises without the help of Macaulay 2. They have been taken from [CLO97].

2. Use Buchberger's algorithm to compute Gröbner basis of the ideal $I = \langle y - z^2, z - x^3 \rangle \in \mathbb{Q}[x, y, z]$ with grevlex and lex orders.

3. Let $f, g \in S$ be such that $\text{in}_\succ(f)$ and $\text{in}_\succ(g)$ are relatively prime and the leading coefficients of f and g are 1. Show that $\text{S-pair}(f, g) = -(g - \text{in}_\succ(g))f + (f - \text{in}_\succ(f))g$ and hence reduce to zero modulo $\{f, g\}$. Deduce that the leading monomial of $\text{S-pair}(f, g)$ is a multiple of the leading monomial of either f or g in this case.

4. Show that the polynomials $f_1 = x - y^2 w$, $f_2 = y - zw$, $f_3 = z - w^3$, $f_4 = w^3 - w \in \mathbb{Q}[x, y, z, w]$ with lex ordering where $x \succ y \succ z \succ w$ form a Gröbner basis for the ideal they generate. Show that they do not form a lex Gröbner basis if $w \succ x \succ y \succ z$.

5. (Division algorithm)

   Divide $f = xy^2 + 1$ by $f_1 = xy + 1$ and $f_2 = y + 1$ using lex order with $x \succ y$.

6. Solve the linear equations

$$\begin{aligned} 3x + 4y - z + w &= 0 \\ x - 3y + 3z - 4w &= 0 \\ x - y + z - w &= 0 \end{aligned}$$

   by computing a Gröbner basis of the ideal generated by the polynomials

$$f_1 = 3x + 4y - z + w,\ f_2 = x - 3y + 3z - 4w,\ \text{and } f_3 = x - y + z - w.$$

   Note that the matrix corresponding to the Gröbner basis is a row reduced Gauss-Jordan form of the matrix of the original equations.

The next 3 exercises are from [Stu96, p. 6].

7. Compute all circuits in the following ideal of linear forms:

$$I = \langle 2x_1 + x_2 + x_3,\ x_2 + 2x_4 + x_5,\ x_3 + x_5 + 2x_6 \rangle.$$

8. Let $\mathcal{U}$ be a universal Gröbner basis for an ideal I in $\mathbf{k}[x_1, \ldots, x_n]$. Show that for every subset $Y \subseteq \{x_1, \ldots, x_n\}$ the elimination ideal $I \cap \mathbf{k}[Y]$ is generated by $\mathcal{U} \cap \mathbf{k}[Y]$.

9. Note that for a term order $\succ$, $\text{in}_\succ(f)\,\text{in}_\succ(g) = \text{in}_\succ(fg)$ for $f, g \in S$. Show that $\text{in}_\succ(I)\,\text{in}_\succ(J) \subseteq \text{in}_\succ(IJ)$ for any two ideals I and J. Find I and J such that this containment is proper.

10. **Saturation of ideals:** [CLO97, Ex. 8, p. 196], The following exercise illustrates an algorithm to compute the saturation of ideals. We need this exercise in Lecture 3. Let $I \subset S$ be an ideal, and fix $f \in S$. Then the *saturation* of I with respect to f is the set

$$(I : f^\infty) = \{g \in S \mid f^m g \in I \text{ for some } m > 0\}.$$

   (a) Prove that $(I : f^\infty)$ is an ideal.

   (b) Prove that we have the ascending chain of ideals

$$(I : f) \subseteq (I : f^2) \subseteq (I : f^3) \subseteq \cdots$$

   (c) By part (b) and the Ascending Chain Condition we have $(I : f^N) = (I : f^{N+1}) = \cdots$ for some integer N. Prove that $(I : f^\infty) = (I : f^N)$.

   (d) Prove that $(I : f^\infty) = (I : f^m)$ if and only if $(I : f^m) = (I : f^{m+1})$.

When the ideal I is homogeneous and f is one of the variables x_n then one can use the following strategy to compute the saturation [Stu96, Lemma 12.1].

Fix the grevlex order induced by $x_1 \succ x_2 \succ \ldots \succ x_n$ and let G be a reduced Gröbner basis of a homogeneous ideal $I \subset S$.

   (e) Show that the set $G' =$

$$\{f \in G \;:\; x_n \text{ does not divide } f\} \cup \{f/x_n \;:\; f \in G \text{ and } x_n \text{ divides } f\}$$

   is a Gröbner basis of $(I : x_n)$.

   (f) Show that a Gröbner basis of $(I : x_n^\infty)$ is obtained by dividing each element $f \in G$ by the highest power of x_n that divides f.

1.6 Solutions to Tutorial 1

1. We provide Macaulay 2 code for the solution of Exercise 1. The calculation shown here is only for the lex order and only for some of the ideals. The commands can be repeated on the other ideals. The grevlex order can be done similarly. We show how to change the monomial order in the ring at the end.

```
i1 : S = QQ[x,y,z,w, MonomialOrder=>Lex, MonomialSize=>16]
o1 = S
o1 : PolynomialRing
i2 : I = ideal(x^2+2*x*y^3, z^2-w^3, x*z-3*y*w)
             2       3   2    3
o2 = ideal (x  + 2x*y , z  - w , x*z - 3y*w)
o2 : Ideal of S
i3 : J = ideal(y^3-2*z*w^2, z^2-3*y*w, x^2*y-z^2*w)
             3       2            2   2     2
o3 = ideal (y  - 2z*w , - 3y*w + z , x y - z w)
o3 : Ideal of S
i4 : IplusJ = I + J;
i5 : toString IplusJ
o5 = ideal(x^2+2*x*y^3,z^2-w^3,x*z-3*y*w,y^3-2*z*w^2,
-3*y*w+z^2,x^2*y-z^2*w)
i6 : ItimesJ = I*J;
o6 : Ideal of S
i7 : IcolonJ = I:J;
o7 : Ideal of S
i8 : IintersectJ = intersect(I,J);
Abort (y/n)? y
returning to top level
```

The command `intersect(I,J)` took more than several seconds and so it was aborted by the user by typing `CTRL C`. The reader can try it. Sometimes Macaulay 2 will bring you back to the session, but other times it will quit which is what happened here. So we restart the program and redo our calculations to continue. We next calculate the reduced Gröbner basis and initial ideal with respect to lex order for the ideal I. The commands can be repeated on the other ideals.

```
i7 : GI = gens gb I
```

```
o7 = | z2-w3 xz-3yw x2+2xy3 xw3-3yzw xyw+2y4w
       y4zw+3/2y2w2 y4w3+3/2y2zw |
             1       7
o7 : Matrix S  <--- S
i8 : leadTerm GI
o8 = | z2 xz x2 xw3 xyw y4zw y4w3 |
             1       7
o8 : Matrix S  <--- S
i9 : leadTerm I
o9 = | z2 xz x2 xw3 xyw y4zw y4w3 |
             1       7
o9 : Matrix S  <--- S
```

Note that the commands in `i8` and `i9` produced the same answer. We now change the monomial order to grevlex and import ideals into this new ring `R`. Note that since grevlex is the default order, no monomial order needs to be specified.

```
i14 : R = QQ[gens S, MonomialSize => 16]
o14 = R
o14 : PolynomialRing
i15 : I = substitute(I,R)
                 3    2     3    2
o15 = ideal (2x*y  + x , - w  + z , x*z - 3y*w)
o15 : Ideal of R
i16 : GI = gens gb I
o16 = | xz-3yw w3-z2 xy3+1/2x2 y4w+1/2xyw y4z2+3/2y2zw |
              1       5
o16 : Matrix R  <--- R
i17 : leadTerm I
o17 = | xz w3 xy3 y4w y4z2 |
              1       5
o17 : Matrix R  <--- R
```

We now retry the computation of $I \cap J$ in the ring `R` with grevlex order and see that it does compute the intersection. We show just the initial ideal of $I \cap J$ with respect to grevlex.

```
i18 : J = substitute(J,R);
```

```
i19 : IintersectJ = intersect(I,J);
i20 : numgens IintersectJ
o20 = 23
i21 : leadTerm IintersectJ
o21 = | xz3 z2w3 xz2w2 x2zw2 xy2z2 x3z2 xy3z x2y2z x3yz
y3w3 x2yw3 y4w2 x2yz2w y4zw y5w xy4w y4z2 xy5 x2y4 |
             1       19
o21 : Matrix R  <--- R
```

Now that we have the ideal $I \cap J$ in Macaulay 2, the reader can try to change the term order in the ring back to lex and compute the required initial ideal and Gröbner basis.

2. With respect to the lex ordering $x \succ y \succ z$, the leading terms of $f = y - z^2$ and $g = z - x^3$ are y and $-x^3$ respectively. By the division algorithm, we obtain that S-pair$(f,g) = x^3z^2 - yz$ and $x^3z^2 - yz = -z^2(z - x^3) - z(y - z^2)$. Hence $\{f, g\}$ is a Gröbner basis for I.

   Similarly, one can see that $\{f, g\}$ is a Gröbner basis with respect to the grevlex order as well.

3. Since $\text{in}_\succ(f)$ and $\text{in}_\succ(g)$ are relatively prime, S-pair$(f,g) = \text{in}_\succ(g)f - \text{in}_\succ(f)g$. Thus, every monomial of S-pair(f,g) is divisible by either $\text{in}_\succ(f)$ or $\text{in}_\succ(g)$. In particular, this is true for the leading monomial of S-pair(f,g).

4. The leading terms of f_i for $1 \le i \le 4$ are x, y,z and w^3 respectively. We tabulate the S-pairs below.
   S-pair$(f_1, f_2) = -y^3w + xzw = -y^2w(f_2) + zw(f_1)$.
   S-pair$(f_1, f_3) = -y^2wz + xw^3 = w^3(f_1) - y^2w(f_3)$.
   S-pair$(f_1, f_4) = -y^2w^4 + xw = w(f_1) - y^2w(f_4)$.
   S-pair$(f_2, f_3) = -z^2w + yw^3 = w^3(f_2) - zw(f_3)$.
   S-pair$(f_2, f_4) = -zw^4 + yw = w(f_2) - zw(f_4)$.
   S-pair$(f_3, f_4) = -w^6 + zw = w(f_3) - w^3(f_4)$.
   The second equalities above are obtained by applying the division algorithm. Hence, the f_i's form a Gröbner basis for I.

   If the term order is $w \succ x \succ y \succ z$, then the leading terms of the f_i's are $-wy^2$, $-zw$, $-w^3$ and w^3 respectively. Note that the S-pair$(f_1, f_2) = -zx + y^3$ is not divisible by the leading term of any of the f_i's.

5. Check that $f = y(f_1) - f_2 + 2$.

6. Fix any term order with $x \succ y \succ z \succ w$. Recall that $f_1 = 3x+4y-z+w$, $f_2 = x - 3y + 3z - 4w$ and $f_3 = x - y + z - w$. It can be checked that S-pair$(f_1, f_2) = m_1 = 13y - 10z + 13w$, S-pair$(f_2, f_3) = m_2 = -2y + 2z - 3w$, and S-pair$(f_1, f_3) = m_3 = 7y - 4z + 4w$. None of these S-pairs are divisible by f_1, f_2, f_3 and hence the current partial Gröbner basis consists of f_1, f_2, f_3, m_1, m_2 and m_3. Computing all S-pairs and their normal forms we see that the reduced Gröbner basis of the ideal is $x + 1/2w, y - 2/3w, z - 13/6w$. Check that the matrix of coefficient vectors of these three polynomials is a reduced row echelon form of the original matrix of coefficient vectors.

7. Let
$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}.$$
Then one can see that the rows of
$$B = \begin{pmatrix} 0 & 2 & -2 & -1 & 0 & 1 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{pmatrix}$$
form a basis for $\ker(A)$. Let $\tau = \{3, 4, 5, 6\}$. We compute C_τ, circuit corresponding to τ.
$$\begin{aligned} C_\tau &= -\det\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{e}_3 + \det\begin{pmatrix} -2 & 0 & 1 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\mathbf{e}_4 \\ &\quad -\det\begin{pmatrix} -2 & -1 & 1 \\ -1 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix}\mathbf{e}_5 + \det\begin{pmatrix} -2 & -1 & 0 \\ -1 & 0 & 1 \\ -2 & 0 & 0 \end{pmatrix}\mathbf{e}_6 \\ &= (0, 0, 1, 0, 1, 2). \end{aligned}$$
Therefore, $x_3+x_5+2x_6$ is a circuit of I. By varying the set τ over all four element subsets of $\{1, \ldots, 6\}$, one can compute all circuits of I. They are $\{x_3+x_5+2x_6, x_2-x_3+2x_4-2x_6,\ 2x_1+x_2-x_5-2x_6,\ x_1+x_3-x_4+x_6,\ 2x_1+x_3-2x_4-x_5,\ x_1-x_4-x_5-x_6,\ x_2+2x_4+x_5,\ 2x_1+x_2+x_3,\ x_1+x_2+x_4-x_6\}$.

8. Without loss of generality assume $Y = \{x_1, \ldots, x_r\}$ for some $r \leq n$. Consider an elimination term order $\succ$ on S with the property that

$\{x_{r+1}, \ldots, x_n\} \succ \{x_1, \ldots, x_r\}$. Since $\mathcal{U}$ is a universal Gröbner basis for I, $\mathcal{U}$ is a Gröbner basis of I with respect to the term order $\succ$. To show that $\mathcal{U} \cap \mathbf{k}[Y]$ generates $I \cap \mathbf{k}[Y]$ it suffices to show that $\mathcal{U} \cap \mathbf{k}[Y]$ contains a Gröbner basis for $I \cap \mathbf{k}[Y]$. Let $f \in I \cap \mathbf{k}[Y]$. Since $\mathcal{U}$ is a Gröbner basis for I with respect to $\succ$, there exists an element $u \in \mathcal{U}$ such that $\mathrm{in}_\succ(u)$ divides $\mathrm{in}_\succ(f) \subset \mathbf{k}[Y]$. This implies that $\mathrm{in}_\succ(u)$ and hence all of u lies in $\mathbf{k}[Y]$ since $\succ$ is the elimination order defined above. Therefore, we conclude that $\mathrm{in}_\succ(I \cap \mathbf{k}[Y]) \subseteq \langle \mathrm{in}_\succ(u) : u \in \mathcal{U} \cap \mathbf{k}[Y] \rangle$. Since the reverse containment is obviously true, we conclude that $\mathrm{in}_\succ(I \cap \mathbf{k}[Y]) = \langle \mathrm{in}_\succ(u) : u \in \mathcal{U} \cap \mathbf{k}[Y] \rangle$. By the definition of a Gröbner basis, $\mathcal{U} \cap \mathbf{k}[Y]$ is a Gröbner basis for $I \cap \mathbf{k}[Y]$ with respect to $\succ$ and hence generates $I \cap \mathbf{k}[Y]$.

9. We prove that $\mathrm{in}_\succ(I)\,\mathrm{in}_\succ(J) \subseteq \mathrm{in}_\succ(IJ)$. Let $f \in \mathrm{in}_\succ(I)$ and $g \in \mathrm{in}_\succ(J)$. Since $\mathrm{in}_\succ(I)$ and $\mathrm{in}_\succ(J)$ are generated by initial forms of elements in I and J respectively, we may assume that $f = \mathrm{in}_\succ(f_1)$ and $g = \mathrm{in}_\succ(g_1)$ for some polynomials $f_1, g_1 \in \mathbf{k}[x_1, \ldots, x_n]$. Then $f_1 g_1 \in IJ$ and by the first part $\mathrm{in}_\succ(f_1 g_1) = fg$. Therefore $fg \in \mathrm{in}_\succ(IJ)$. Hence $\mathrm{in}_\succ(I)\,\mathrm{in}_\succ(J) \subseteq \mathrm{in}_\succ(IJ)$.

   Let $S = \mathbf{k}[x, y]$ with grevlex monomial order and $I = (x^5, x^4y^2, x^2y^5(x + y), xy^8, y^{10})$. Then it can be seen that

$$\begin{aligned}
\mathrm{in}(I) &= (x^5, x^4y^2, x^3y^5, x^2y^7, xy^8, y^{10}), \\
\mathrm{in}(I)^2 &= (x^{10}, x^9y^2, x^8y^4, x^7y^7, x^6y^8, x^5y^{10}, x^4y^{12}, x^2y^{16}, x^3y^{15}, xy^{18}, y^{20}), \\
\mathrm{in}(I^2) &= (x^{10}, x^9y^2, x^8y^4, x^7y^6, x^6y^8, x^5y^{10}, x^4y^{12}, x^3y^{14}, x^2y^{16}, xy^{18}, y^{20}).
\end{aligned}$$

   Therefore $x^7y^6 \in \mathrm{in}(I^2)$ but $x^7y^6 \notin \mathrm{in}(I)^2$.

10. (a) If $g, h \in (I : f^\infty)$, then $f^m g, f^n h \in I$. Thus $f^{\max(m,n)}(g + h) \in I$. This implies $(g + h) \in (I : f^\infty)$. Now if $g \in (I : f^\infty)$, then $f^m g \in I$ for some $m > 0$. This implies that $h(f^m g) \in I$. But $h(f^m g) = f^m(hg)$. Thus $hg \in (I : f^\infty)$. Therefore $(I : f^\infty)$ is an ideal.

    (b) If $f^m g \in I$, then $f^{m+1} g \in I$. Therefore the chain of inequalities follows.

    (c) Let $g \in (I : f^\infty)$. Thus $f^m g \in I$ for some $m > 0$. If $m \leq N$, then $g \in (I : f^m) \subseteq (I : f^N)$. If $m > N$, then since $(I : f^N) = \cdots = (I : f^m)$, $g \in (I : f^N)$.

    (d) Clearly, if $(I : f^\infty) = (I : f^m)$, then $(I : f^m) = (I : f^{m+1})$. Now suppose that $(I : f^m) = (I : f^{m+1})$. Assume by induction that for any $0 \le r < \ell$, $(I : f^{m+r}) = (I : f^{m+r+1})$. Then $(I : f^{m+l+1}) = ((I : f^{m+l}) : f) = ((I : f^{m+l-1}) : f) = (I : f^{m+l})$. Therefore, $(I : f^{\infty}) = (I : f^m)$.

(e) Let $J = (I : x_n)$. To prove that G' is a Gröbner basis of J with respect to the given reverse lexicographic order, it suffices to show that $\{\mathrm{in}(g) : g \in G'\}$ generates $\mathrm{in}(J)$. Let $f \in J$ be a homogeneous element. Since G is a Gröbner basis for I and $x_n f \in I$, there exists homogeneous $r_i \in S$ such that $x_n f = \sum_i r_i g_i$ where $g_i \in G$. Since $x_n f$ and all the elements of G are homogeneous, in fact, $\mathrm{in}(x_n f) = \sum_i r_i \,\mathrm{in}(g_i)$. Since the term order is reverse lex with x_n cheapest, $\mathrm{in}(x_n f) = x_n \,\mathrm{in}(f)$. Therefore $\mathrm{in}(f) = \sum_{x_n \nmid \mathrm{in}(g_i)} (r_i/x_n) \,\mathrm{in}(g_i) + \sum_{x_n | \mathrm{in}(g_i)} r_i (\mathrm{in}(g_i)/x_n)$. Again due to the choice of term order, it is clear that if $x_n \mid \mathrm{in}(g)$, then $x_n | g$, for any homogeneous element g. Therefore $\mathrm{in}(g_i)/x_n = \mathrm{in}(g_i/x_n)$ and hence we obtain that $\mathrm{in}(J)$ is generated by $\{\mathrm{in}(g) : g \in G'\}$.

(f) We prove by induction that the set $G_m = \bigcup_{i=0}^{m-1} \{(f/x_n^i) : f \in G \text{ and } x_n^i \text{ is the largest power of } x_n \text{ dividing } f\} \bigcup \{f/x_n^m : f \in G \text{ and } x_n^m \text{ divides } f\}$ is a Gröbner basis for $(I : x_n^m)$, $m \ge 1$. The assertion for $m = 1$ is proved in (e). Assume by induction that $m > 1$ and the assertion holds for all $\ell < m$. Let $J = (I : x_n^{m-1})$. By induction, G_{m-1} is a Gröbner basis for J. Now using the fact $(I : x_n^m) = (J : x_n)$ and proceeding as in (e), we one proves the assertion.

Chapter 2

The Gröbner Fan

2.1 Introduction

The main goal of this chapter is to introduce the Gröbner fan of an ideal in a polynomial ring, which is a polyhedral fan associated to the given ideal, with one top-dimensional cone for each initial ideal of the ideal. No familiarity with polyhedral theory is assumed. Throughout this chapter, the polynomial ring is written as $S = \mathbf{k}[x_1, \ldots, x_n]$, with no assumptions on the field $\mathbf{k}$. We denote by $\mathbb{R}^n_{\geq 0}$ the set of vectors $\mathbf{v} \in \mathbb{R}^n$ with $v_i \geq 0$ for $1 \leq i \leq n$.

2.2 Some more Gröbner facts

We begin by introducing some more Gröbner facts. Recall that a Gröbner basis, and hence the corresponding initial ideal, for an ideal is determined by a term order on the monomials in the polynomial ring. Note that there are an uncountable number of different term orders. One way to see this is to observe that if $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$ is a vector with algebraically independent transcendental entries, then the order defined by setting $\mathbf{x}^{\mathbf{u}} \prec \mathbf{x}^{\mathbf{v}}$ if $\mathbf{w} \cdot \mathbf{u} < \mathbf{w} \cdot \mathbf{v}$ is a total order that obeys the term order axioms. Different transcendental vectors with norm one give different term orders, so we conclude that there are an uncountable number of term orders. We will use these term orders extensively in this chapter. The vector $\mathbf{w}$ is called a *weight vector*.

In light of this infinite number of term orders, it is somewhat surprising that a fixed ideal has only a finite number of different initial ideals. The intuitive explanation is that most of the term orders constructed above only differ in very high degree, and so the Buchberger algorithm does not see the difference between them. We now present two proofs of this finiteness result.

Proposition 2.2.1. *Let I be an ideal in S. Then there are only finitely many distinct initial ideals of I.*

Both proofs of this proposition rely on the following lemma.

Lemma 2.2.2. *Let M be an initial ideal of an ideal $I \subseteq S$ with respect to a term order $\prec$. Then the monomials of S not in M form a vector space basis for S/I.*

Proof. To see that the monomials not in M are independent modulo I, note that any dependency relation would give a nonzero polynomial f in I none of whose terms lie in M. But this contradicts M being the initial ideal of I, as we must have $\text{in}_\prec(f) \in M$. To see that they span, note that any polynomial in S has a normal form g for which $f - g \in I$, and g is a linear combination of monomials not in M. □

Corollary 2.2.3. *If $J = \text{in}_\prec(I)$ and $K = \text{in}_{\prec'}(I)$ are two initial ideals of an ideal $I \subseteq S$, with $J \subseteq K$, then $J = K$.*

Proof. Suppose that $J \subsetneq K$, and let $\mathbf{x}^\mathbf{u} \in K \setminus J$. By Lemma 2.2.2 we know that the monomials not in K form a basis for S/I, so there is some polynomial g none of whose terms lie in K for which $\mathbf{x}^\mathbf{u} - g \in I$. But none of the terms of $\mathbf{x}^\mathbf{u} - g$ lies in J, so in particular its leading term with respect to $\prec$ does not lie in J, contradicting the fact that J is the initial ideal of I with respect to $\prec$. From this contradiction we conclude that $J = K$. □

First proof of Proposition 2.2.1. Suppose I has an infinite number of initial ideals. Let Σ_0 be the set of all initial ideals of I. Since Σ_0 is infinite, I is not the zero ideal, so we can choose an element $f_1 \in I$. Since f_1 has only a finite number of terms, and each initial ideal $M \in \Sigma_0$ contains a term of f_1, there must be one term m_1 of f_1 that is contained in an infinite number of ideals in Σ_0. Let $\Sigma_1 = \{M \in \Sigma_0 : m_1 \in M\}$. Let $J_1 = \langle m_1 \rangle$. Since infinitely many initial ideals contain J_1, there is some initial ideal that properly contains J_1. Thus Lemma 2.2.2 implies that the monomials of S outside J_1 are linearly dependent modulo I, so there is some polynomial f_2 in I with no term lying in J_1. Again, there is a term m_2 of f_2 that is contained in an infinite number of ideals in Σ_1. Let $\Sigma_2 = \{M \in \Sigma_1 : m_2 \in M\}$, and let $J_2 = J_1 + \langle m_2 \rangle$. This procedure can now be iterated, at each stage finding a polynomial f_k none of whose terms are contained in J_{k-1}, and one of which, m_k, lies in infinitely many initial ideals in Σ_{k-1}. The new ideal $J_k = J_{k-1} + \langle m_k \rangle$ will be properly contained in some initial ideal, so the new f_{k+1} can be created.

In this fashion we get a properly increasing sequence of ideals

$$J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq \dots .$$

Since S is Noetherian, this is impossible. We thus conclude that I has only finitely many initial ideals. $\square$

The other proof relies on the following theorem in addition to Lemma 2.2.2.

Theorem 2.2.4. *[Mac01, Theorem 1.1] Let $\mathcal{I}$ be an infinite collection of monomial ideals in S. Then there exist $I, J \in \mathcal{I}$ such that $I \subseteq J$.*

Second proof of Proposition 2.2.1. Suppose there are an infinite number of initial ideals of I. Then by Theorem 2.2.4 there are two distinct initial ideals J and K with $J \subseteq K$. This is impossible, by Corollary 2.2.3. $\square$

Corollary 2.2.5. *Let I be an ideal in S. Then there is a finite generating set for I that is a Gröbner basis for I with respect to any term order.*

Proof. Note that if $\mathrm{in}_{\prec_1}(I) = \mathrm{in}_{\prec_2}(I) = J$, then the reduced Gröbner bases for I with respect to the term orders $\prec_1$ and $\prec_2$ are identical. To see this note that for each minimal generator $\mathbf{x}^{\mathbf{u}}$ of J, there is a unique polynomial $p_{\mathbf{u}}(\mathbf{x}) = \mathbf{x}^{\mathbf{u}} - q(\mathbf{x})$ in I where every monomial occurring in $q(\mathbf{x})$ does not lie in J. The existence of $p_{\mathbf{u}}$ follows from Lemma 2.2.2, since $\mathbf{x}^{\mathbf{u}} \in J$, while if $p_{\mathbf{u}}$ were not unique the difference of two such polynomials would yield a polynomial in I whose leading term did not lie in J. The polynomial $q(\mathbf{x})$ is the normal form of $\mathbf{x}^{\mathbf{u}}$ with respect to $\prec_1$ and $\prec_2$. The union of all $p_{\mathbf{u}}$ as $\mathbf{x}^{\mathbf{u}}$ varies over all minimal generators of J is the reduced Gröbner basis of I for any term order $\prec$ for which $\mathrm{in}_{\prec}(I) = J$. We can now take the union over the finitely many initial ideals of I of the corresponding reduced Gröbner bases to get the desired finite generating set. $\square$

Remark 2.2.6. Corollary 2.2.5 shows that a universal Gröbner basis (see Definition 1.4.3) for an ideal I always exists. When we refer to the universal Gröbner basis of I, we will usually mean the union of the reduced Gröbner bases corresponding to each initial ideal of I.

2.3 Polytopal basics

In this section we review the basics of polytope theory.

Definition 2.3.1. A set $\mathcal{U} \subseteq \mathbb{R}^d$ is *convex* if $\lambda\mathbf{v} + (1-\lambda)\mathbf{w} \in \mathcal{U}$ for any $\mathbf{v}, \mathbf{w} \in \mathcal{U}$, and $0 \leq \lambda \leq 1$. The *convex hull* of a set $\mathcal{V} \subseteq \mathbb{R}^d$ is the intersection of all convex sets containing $\mathcal{V}$, and is itself convex. A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^d$. We write $P = \operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \{\sum_{i=1}^n \lambda_i \mathbf{v}_i : \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1\}$.

Example 2.3.2. Classic examples of polytopes in $\mathbb{R}^3$ include the cube, the tetrahedron and the octahedron.

Definition 2.3.3. A *face* of a polytope $P \subseteq \mathbb{R}^d$ is a set of the form $\operatorname{face}_{\mathbf{c}}(P) := \{\mathbf{x} \in P : \mathbf{c} \cdot \mathbf{x} \geq \mathbf{c} \cdot \mathbf{y} \ \forall \mathbf{y} \in P\}$, where $\mathbf{c}$ is any vector in $\mathbb{R}^d$. The *affine span* of a set $V \subseteq \mathbb{R}^d$ is the affine subspace $\mathbf{v} + H$, where $\mathbf{v} \in V$, and H is the subspace of $\mathbb{R}^d$ spanned by $\{\mathbf{w} - \mathbf{v} : \mathbf{w} \in V\}$. The *dimension* of a face F of P is the dimension of its affine span.

Example 2.3.4. If P is the square $\operatorname{conv}((0,0),(1,0),(0,1),(1,1))$, then P has one two-dimensional face, four one-dimensional faces (edges), and four zero-dimensional faces (vertices). The whole square is the unique two-dimensional face $\operatorname{face}_{\mathbf{0}}(P)$. Note that the vertex $(1,0)$ is $\operatorname{face}_{(1,-1)}(P)$. Similarly $\operatorname{face}_{(0,1)}(P)$ is the the edge $\operatorname{conv}((0,1),(1,1))$. It is straightforward to find vectors $\mathbf{c}$ for the other three vertices and three edges. Definition 2.3.10 will make it easier to check that we have not omitted any faces from the above list.

The $(d-1)$-dimensional faces of a d-dimensional polytope are called *facets*. A facet F is of the form $F = \{\mathbf{x} : \mathbf{c} \cdot \mathbf{x} = \mathbf{b}\} \cap P$ for some $\mathbf{c} \in \mathbb{R}^d, \mathbf{b} \in \mathbb{R}$. The vector $\mathbf{c} =: \mathbf{c}_F$ is called the *facet normal*, and the corresponding hyperplane is a *defining hyperplane*. A normal vector to a facet is only defined up to sign. We shall choose the sign that makes $F = \operatorname{face}_{\mathbf{c}}(P)$. The facets determine the polytope in the following way.

Proposition 2.3.5. *Let P be a polytope with facet normals $\{\mathbf{c}_F : F$ facet of $P\}$. Then $P = \{\mathbf{x} : \mathbf{c}_F \cdot \mathbf{x} \leq \mathbf{b}_F\}$, where $\mathbf{b}_F = \max_{\mathbf{y} \in P}\{\mathbf{c}_F \cdot \mathbf{y}\}$.*

A proof of this proposition can be found in [Zie95, Lecture 1].

Example 2.3.6. The square P of Example 2.3.4 has facets $\{\mathbf{x} \cdot \mathbf{c}_F = \mathbf{b}_F\} \cap P$ for $\mathbf{c}_F, \mathbf{b}_F = \{((0,-1),0),((-1,0),0),((1,0),1),((0,1),1)\}$. Thus $P = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

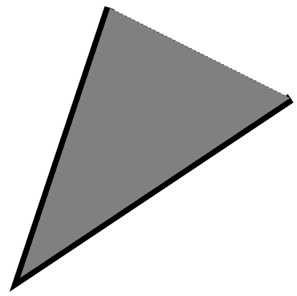


Figure 2.1: A polyhedral cone.

Definition 2.3.7. A *cone* is a set $C \subseteq \mathbb{R}^d$ that is closed under addition and positive scalar multiplication, so if $\mathbf{v}, \mathbf{w} \in C$, then $\mathbf{v} + \mathbf{w} \in C$, and if $\mathbf{v} \in C$ and $\lambda > 0$, then $\lambda \mathbf{v} \in C$. If $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq \mathbb{R}^d$ then its *positive hull* is $\mathrm{pos}(\mathcal{V}) = \{\sum_i \lambda_i \mathbf{v}_i : \lambda_i \geq 0\}$. We call $\mathrm{pos}(\mathcal{V})$ the cone spanned by $\mathcal{V}$. A cone C is *polyhedral* if $C = \mathrm{pos}(\mathcal{V})$ for a finite subset $\mathcal{V}$ of $\mathbb{R}^d$. As for polytopes, we define a face of a cone C to be $\mathrm{face}_{\mathbf{c}}(C) = \{\mathbf{x} \in C : \mathbf{c} \cdot \mathbf{x} \geq \mathbf{c} \cdot \mathbf{y} \ \forall \mathbf{y} \in C\}$.

Example 2.3.8. An example of a polyhedral cone in $\mathbb{R}^2$ is shown in Figure 2.1.

Polytopes and cones are special cases of *polyhedra*, which are the intersection of finitely many halfspaces. Polytopes are bounded polyhedra, while cones are polyhedra for which the origin lies on every defining hyperplane. We are interested in particular families of polyhedral cones that fit together nicely.

For a subset V of $\mathbb{R}^n$, the *relative interior* is the interior of V inside the affine span of V. A set is *relatively open* if it is its own relative interior.

Definition 2.3.9. A *polyhedral fan* is a collection of polyhedral cones in $\mathbb{R}^n$ such that the intersection of any two cones is a face of each. The fan is called *complete* if in addition the union of the cones covers $\mathbb{R}^n$. A fan $\mathcal{F}$ is *simplicial* if every i-dimensional cone in $\mathcal{F}$ is the positive hull of i vectors for all i. It suffices to check this condition for the maximal cones in $\mathcal{F}$ (those not contained in any larger cone in $\mathcal{F}$).

We denote by $\overline{V}$ the closure (in the standard topology) of a set $V \subseteq \mathbb{R}^n$.

Definition 2.3.10. The *outer normal fan* of a polytope P is a polyhedral fan whose cones are indexed by the faces F of P. The cone corresponding to the face F of P is the closure of the relatively open cone $C[F] = \{\mathbf{c} \in \mathbb{R}^n$ such that $\mathrm{face}_{\mathbf{c}}(P) = F\}$.

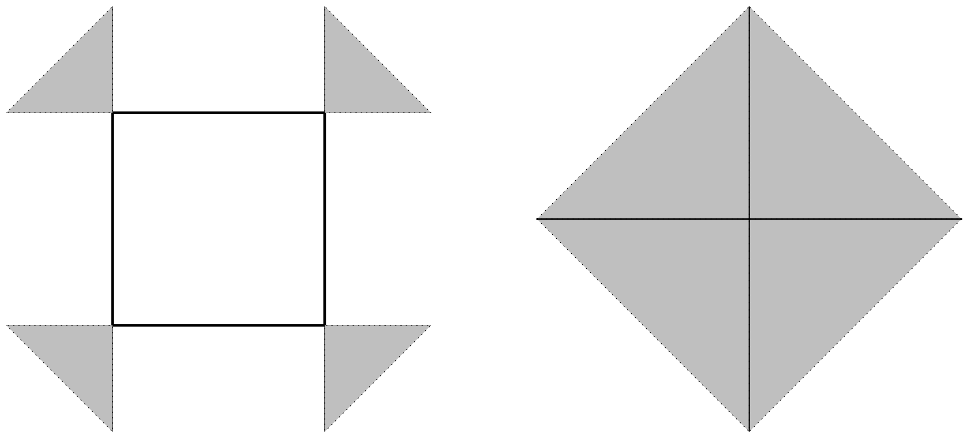


Figure 2.2: The outer normal fan for a square

Example 2.3.11. For the face $F = \{(1,0)\}$ of the square from Example 2.3.4, we have $C[F] = \{\mathbf{v} \in \mathbb{R}^2 : v_1 > 0, v_2 < 0\}$. Its closure is the polyhedral cone $\overline{C[F]} = \{\mathbf{v} \in \mathbb{R}^2 : v_1 \geq 0, v_2 \leq 0\}$. The face $F = \operatorname{conv}((0,1),(1,1))$ has $C[F] = \{\lambda(0,1) : \lambda > 0\}$. The closure $\overline{C[F]}$ is the polyhedral cone $\operatorname{pos}(\{(0,1)\})$. Check that in both cases we have that $C[F]$ is the relative interior of $\overline{C[F]}$.

Figure 2.2 shows the outer normal fan for the square. The first picture shows the cones $C[\mathbf{p}]$ attached to each vertex $\mathbf{p}$, while the second is the standard picture of the fan.

Definition 2.3.12. A polyhedral fan $\mathcal{F}$ is *polytopal* if there is a polytope P for which $\mathcal{F}$ is the outer normal fan of P.

All complete fans in $\mathbb{R}^2$ are polytopal, but this is false in $\mathbb{R}^3$. See [Ful93, §1.5] for an example. We next describe a notion of sum for polytopes.

Definition 2.3.13. The *Minkowski sum* of two polytopes P and Q in $\mathbb{R}^n$ is the set $\{\mathbf{x} + \mathbf{y} : \mathbf{x} \in P, \mathbf{y} \in Q\}$. This is again a polytope.

Example 2.3.14. Figure 2.3 illustrates the Minkowski sum of a square and a triangle in $\mathbb{R}^2$. Note that the picture of the sum can be obtained by marking the points covered by the triangle as we slide it so its bottom vertex lies in the square.

We finish this section with a fundamental theorem in linear programming which we will use later.

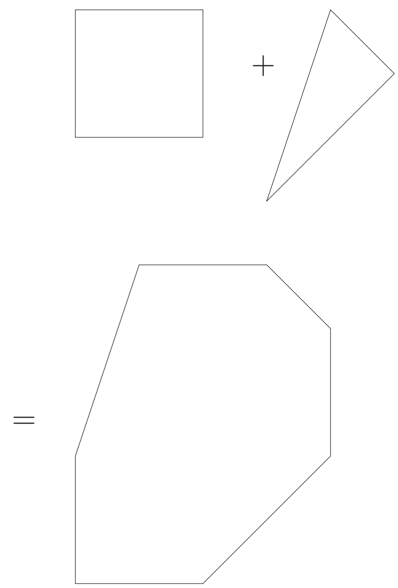


Figure 2.3: The Minkowski sum of a square and a triangle.

Theorem 2.3.15. *Let* $A \in \mathbb{R}^{d\times n}$ *and* $\mathbf{z} \in \mathbb{R}^d$*. Either there exists* $\mathbf{x} \in \mathbb{R}^n$ *with* $A\mathbf{x} \leq \mathbf{z}$*, where the inequality is term-wise, or there exists a vector* $\mathbf{c} \in \mathbb{R}^d$ *with* $\mathbf{c} \geq 0$ *such that* $\mathbf{c}^T A = 0$ *and* $\mathbf{c} \cdot \mathbf{z} < 0$*, but not both.*

This theorem is known as the Farkas Lemma. A proof can be found in [Zie95, Section 1.4].

2.4 The Gröbner fan

In this section we associate a polyhedral fan to an ideal, each top-dimensional cone of which corresponds to a different initial ideal of the given ideal. We begin by noting that the term orders determined by weight vectors mentioned at the beginning of this chapter are the only ones that need to be considered.

Definition 2.4.1. For $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$, and $f = \sum c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}} \in S$, we define the *leading form* $\mathrm{in}_{\mathbf{w}}(f)$ to be the sum of terms $c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}$ in f with $\mathbf{w}\cdot\mathbf{u}$ maximized. The initial ideal $\mathrm{in}_{\mathbf{w}}(I) := \langle \mathrm{in}_{\mathbf{w}}(f) : f \in I\rangle$. Note that $\mathrm{in}_{\mathbf{w}}(I)$ need not be a monomial ideal. Given a term order $\prec$, we also define the term order $\prec_{\mathbf{w}}$, for which $\mathbf{x}^{\mathbf{u}} \prec_{\mathbf{w}} \mathbf{x}^{\mathbf{v}}$ if $\mathbf{w}\cdot\mathbf{u} < \mathbf{w}\cdot\mathbf{v}$ or if $\mathbf{w}\cdot\mathbf{u} = \mathbf{w}\cdot\mathbf{v}$ and $\mathbf{x}^{\mathbf{u}} \prec \mathbf{x}^{\mathbf{v}}$. If no specific term order is specified when referring to $\prec_{\mathbf{w}}$, we take $\prec$ to be the lexicographic term order with $x_1 \succ x_2 \succ \cdots \succ x_n$.

We note that the above definitions still make sense for a general $\mathbf{w} \in \mathbb{R}^n$, possibly with negative coordinates. However for general $\mathbf{w}$ there may be no relation between any invariants of $\text{in}_{\mathbf{w}}(I)$ and those of I (such as the Hilbert function, or dimension). This is because if $\mathbf{w}$ has negative coordinates $\prec_{\mathbf{w}}$ is not a term order, as 1 is no longer the smallest monomial.

Lemma 2.4.2. *Let $\prec$ be a term order. If $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$ and $\mathcal{G}$ is a Gröbner basis for I with respect to $\prec_{\mathbf{w}}$, then $\{\text{in}_{\mathbf{w}}(g) : g \in \mathcal{G}\}$ is a Gröbner basis for $\text{in}_{\mathbf{w}}(I)$ with respect to $\prec$.*

Proof. For every polynomial $f \in I$ we have $\text{in}_{\prec}(\text{in}_{\mathbf{w}}(f)) = \text{in}_{\prec_{\mathbf{w}}}(f)$ by the definition of $\prec_{\mathbf{w}}$. Since every monomial in the ideal $\text{in}_{\prec_{\mathbf{w}}}(I)$ is of the form $\text{in}_{\prec_{\mathbf{w}}}(f)$ for some $f \in I$, this means that $\text{in}_{\prec_{\mathbf{w}}}(I) \subseteq \text{in}_{\prec}(\text{in}_{\mathbf{w}}(I))$.

To see the reverse inclusion, we first note that the vector $\mathbf{w}$ gives a $\mathbb{R}$-grading of S by $\deg(x_i) = w_i$. Since the group generated by the w_i is isomorphic to $\mathbb{Z}^k$ for some $k \leq n$, this is a grading by a finitely-generated abelian group. The ideal $\text{in}_{\mathbf{w}}(I)$ is homogeneous with respect to this grading, and thus so is the reduced Gröbner basis for $\text{in}_{\mathbf{w}}(I)$ with respect to $\prec$. Let $\mathbf{x}^{\mathbf{u}}$ be a minimal generator of $\text{in}_{\prec}(\text{in}_{\mathbf{w}}(I))$, so $\mathbf{x}^{\mathbf{u}} = \text{in}_{\prec}(g)$ for some $\mathbf{w}$-homogeneous $g \in \text{in}_{\mathbf{w}}(I)$. We first show that $g = \text{in}_{\mathbf{w}}(f)$ for some $f \in I$. Indeed, we can write $g = \sum h_i \,\text{in}_{\mathbf{w}}(g_i)$, where $g_i \in I$, h_i is a monomial for all i, and the sum has as few terms as possible. Since multiplying by a monomial multiplies the leading form by that monomial, $g = \sum \text{in}_{\mathbf{w}}(h_i g_i)$. Since we have as few terms as possible, the polynomials $\text{in}_{\mathbf{w}}(h_i g_i)$ must be $\mathbf{w}$-homogeneous of the same degree, since g is $\mathbf{w}$-homogeneous, so $g = \text{in}_{\mathbf{w}}(\sum h_i g_i)$. This proves the claim, since $f = \sum h_i g_i \in I$.

Now this means that $\text{in}_{\prec}(g) = \text{in}_{\prec}(\text{in}_{\mathbf{w}}(f)) = \text{in}_{\prec_{\mathbf{w}}}(f)$, and therefore $\text{in}_{\prec}(\text{in}_{\mathbf{w}}(I)) \subseteq \text{in}_{\prec_{\mathbf{w}}}(I)$. Thus $\text{in}_{\prec}(\text{in}_{\mathbf{w}}(I)) = \text{in}_{\prec_{\mathbf{w}}}(I)$, and so $\langle \text{in}_{\prec}(\text{in}_{\mathbf{w}}(g)) : g \in \mathcal{G} \rangle = \langle \text{in}_{\prec_{\mathbf{w}}}(g) : g \in \mathcal{G} \rangle = \text{in}_{\prec}(\text{in}_{\mathbf{w}}(I))$, which completes the proof. $\square$

Example 2.4.3. For $\mathbf{w} = (10, 1, 11, 12)$, $\text{in}_{\mathbf{w}}(a^2c - b^2) = a^2c$, and $\text{in}_{\mathbf{w}}(a^2b^2 - c^2) = a^2b^2 - c^2$. Let $I = \langle ad - bc, ac - b^2, bd - c^2 \rangle \subseteq \mathbf{k}[a, b, c, d]$. The initial ideal $\text{in}_{\mathbf{w}}(I) = \langle ad, ac, c^2 \rangle$. The ideal $\text{in}_{\mathbf{w}}(I)$ can computed using Lemma 2.4.2 and Macaulay 2.

Proposition 2.4.4. *Let I be a fixed ideal contained in S. For every term order $\prec$ there is a weight vector $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$ for which $\text{in}_{\prec}(I) = \text{in}_{\mathbf{w}}(I)$.*

Proof. Let $\mathcal{G} = \{g_1, \dots, g_r\}$ be the reduced Gröbner basis of I with respect to $\prec$. Write $g_i = \sum_j c_{ij} \mathbf{x}^{\mathbf{u}_{ij}}$, where $\text{in}_{\prec}(g_i) = c_{i1} \mathbf{x}^{\mathbf{u}_{i1}}$. Let $C_{\prec} = \{\mathbf{w} \in \mathbb{R}^n_{\geq 0} : \mathbf{w} \cdot \mathbf{u}_{i1} > \mathbf{w} \cdot \mathbf{u}_{ij} \text{ for all } j \geq 2, 1 \leq i \leq r\}$. For any weight vector $\mathbf{w} \in C_{\prec}$ we have $\mathrm{in}_{\prec}(I) \subseteq \mathrm{in}_{\prec_{\mathbf{w}}}(I)$, so since we cannot have a proper inclusion of initial ideals by Corollary 2.2.3, we conclude that $\mathrm{in}_{\prec}(I) = \mathrm{in}_{\mathbf{w}}(I)$. It thus remains to show that $C_{\prec}$ is nonempty.

Suppose that $C_{\prec} = \emptyset$. Form the matrix U with n columns whose s rows are the vectors $\mathbf{u_{i1}} - \mathbf{u_{ij}}$ for $1 \leq i \leq r$, and $j \geq 2$. The fact that $C_{\prec} = \emptyset$ means that there is no vector $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$ with $U\mathbf{w} > 0$, where the inequality is term-wise. It follows that there is no $\mathbf{w'} \in \mathbb{R}^n$ with $U'\mathbf{w'} \leq (-\mathbf{1}, \mathbf{0})^T$, where U' is the $(s+n) \times n$ matrix with first s rows $-U$, and last n rows $-I$, $\mathbf{1}$ is the vector of length s all of whose components are one, and $\mathbf{0}$ is the zero vector of length n. The Farkas Lemma (Theorem 2.3.15) now implies that there is some vector $\mathbf{c} \in \mathbb{R}^{s+n}$ with $\mathbf{c} \geq 0$, $\mathbf{c} \neq 0$, and $\mathbf{c}^T U' = 0$. Since U' has integral entries, we can choose $\mathbf{c} \in \mathbb{N}^{s+n}$. Let $c_{i,m} \geq 0$ be the component of $\mathbf{c}$ corresponding to the row $\mathbf{u}_{im} - \mathbf{u}_{i1}$ of U'. Then we have $\sum_{i,m} c_{im}(\mathbf{u_{im}} - \mathbf{u_{i1}}) \geq 0$, because when this sum is subtracted from $\mathbf{c}U' = 0$ the result has all nonpositive coordinates. Thus

$$\prod_{i,m} (\mathbf{x}^{\mathbf{u_{i1}}})^{c_{im}} \text{ divides } \prod_{i,m} (\mathbf{x}^{\mathbf{u_{im}}})^{c_{im}},$$

so

$$\prod_{i,m} (\mathbf{x}^{\mathbf{u_{i1}}})^{c_{im}} \preceq \prod_{i,m} (\mathbf{x}^{\mathbf{u_{im}}})^{c_{im}}.$$

But for all i, m we have $\mathbf{x}^{\mathbf{u}_{im}} \prec \mathbf{x}^{\mathbf{u}_{i1}}$, so

$$\prod_{i,m} (\mathbf{x}^{\mathbf{u_{im}}})^{c_{im}} \prec \prod_{i,m} (\mathbf{x}^{\mathbf{u_{i1}}})^{c_{im}}.$$

From this contradiction we conclude that $C_{\prec}$ is nonempty, so the proposition follows. □

Remark 2.4.5. A *rational cone* is the positive hull of vectors in $\mathbb{Q}^n$, or equivalently one whose facet normals all lie in $\mathbb{Q}^n$. Note that $C_{\prec}$ is the interior of an n-dimensional rational cone in $\mathbb{R}^n_{\geq 0}$, so it contains a nonnegative rational vector, and thus a vector in $\mathbb{N}^n$. Hence we can take $\mathbf{w} \in \mathbb{N}^n$. By interior we mean here in the induced topology on $\mathbb{R}^n_{\geq 0}$.

Also, we emphasize that the weight vector $\mathbf{w}$ assigned to $\prec$ in Proposition 2.4.4 depends on I. For example there is no weight vector $\mathbf{w}$ for which $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathrm{lex}}(I)$ for every ideal $I \subseteq S$, where lex denotes the lexicographic term order.

Proposition 2.4.6. *Let I be a fixed ideal, let $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$ and let $C[\mathbf{w}]$ be the set $\{\mathbf{w'} \in \mathbb{R}^n_{\geq 0} : \mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathbf{w'}}(I)\}$. Then $C[\mathbf{w}]$ is the relative interior of a polyhedral cone inside $\mathbb{R}^n_{\geq 0}$.*

Proof. For a given $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$, let $J = \mathrm{in}_{\mathbf{w}}(I)$, and let $\mathcal{G} = \{g_1, \dots, g_r\}$ be the reduced Gröbner basis for I with respect to $\prec_{\mathbf{w}}$. We note that J need not be a monomial ideal. For $g_i \in \mathcal{G}$, write $g_i = \sum_j c_{ij}\mathbf{x}^{\mathbf{a}_{ij}} + \sum_j c'_{ij}\mathbf{x}^{\mathbf{b}_{ij}}$ where $\mathrm{in}_{\mathbf{w}}(g_i) = \sum_j c_{ij}\mathbf{x}^{\mathbf{a}_{ij}}$. It suffices to show that

$$C[\mathbf{w}] = \{\mathbf{w}' \in \mathbb{R}^n_{\geq 0} : \mathrm{in}_{\mathbf{w}'}(g) = \mathrm{in}_{\mathbf{w}}(g) \text{ for all } g \in \mathcal{G}\} \tag{2.1}$$

The proposition then follows because the set on the right in this equation is

$$\{\mathbf{w}' \in \mathbb{R}^n_{\geq 0} : \mathbf{w}' \cdot \mathbf{a}_{ij} = \mathbf{w}' \cdot \mathbf{a}_{ik}, \mathbf{w}' \cdot \mathbf{a}_{ij} > \mathbf{w}' \cdot \mathbf{b}_{ik} \text{ for } i = 1, \dots, r \text{ and all } j, k\}.$$

This is the relative interior of a polyhedral cone by definition.

Let $\mathbf{w}'$ lie in the set on the right-hand side of Equation 2.1. Then $\mathrm{in}_{\mathbf{w}}(I) \subseteq \mathrm{in}_{\mathbf{w}'}(I)$. These ideals may not be monomial ideals, but the containment (and whether it is proper) must be preserved by taking any initial ideal with respect to an arbitrary order $\prec$. So we have $\mathrm{in}_{\prec}(\mathrm{in}_{\mathbf{w}}(I)) \subseteq \mathrm{in}_{\prec}(\mathrm{in}_{\mathbf{w}'}(I))$. By Corollary 2.2.3 we know that this inclusion cannot be proper, so we conclude that $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathbf{w}'}(I)$. This shows that the set on the right is contained in $C[\mathbf{w}]$.

Now consider $\mathbf{w}' \in C[\mathbf{w}]$. By Lemma 2.4.2 we know that $\mathrm{in}_{\mathbf{w}}(\mathcal{G}) = \{\mathrm{in}_{\mathbf{w}}(g) : g \in \mathcal{G}\}$ is a Gröbner basis for $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathbf{w}'}(I)$ with respect to $\prec$. Fix $g \in \mathcal{G}$. Then $\mathrm{in}_{\mathbf{w}'}(g)$ reduces to zero with respect to $\mathrm{in}_{\mathbf{w}}(\mathcal{G})$ using $\prec$. Now $m := \mathrm{in}_{\prec_{\mathbf{w}}}(g)$ is the only monomial occurring in g which is divisible by the leading term with respect to $\prec$ of a polynomial in $\mathrm{in}_{\mathbf{w}}(\mathcal{G})$, so it must occur in $\mathrm{in}_{\mathbf{w}'}(g)$ for the reduction to be possible. Write $\mathrm{in}_{\mathbf{w}}(g) = m + h$, and $\mathrm{in}_{\mathbf{w}'}(g) = m + h'$. By the choice of m we know that h and h' both are sums of terms not in $\mathrm{in}_{\prec_{\mathbf{w}}}(I)$. However $\mathrm{in}_{\mathbf{w}}(g) - \mathrm{in}_{\mathbf{w}'}(g) = h - h' \in \mathrm{in}_{\mathbf{w}}(I)$, so $\mathrm{in}_{\prec}(h - h') \in \mathrm{in}_{\mathbf{w}}(\mathrm{in}_{\prec_{\mathbf{w}}}(I)) = \mathrm{in}_{\prec_{\mathbf{w}}}(I)$. This is only possible if $h - h' = 0$, so $\mathrm{in}_{\mathbf{w}}(g) = \mathrm{in}_{\mathbf{w}'}(g)$, and thus $\mathbf{w}'$ lies in the right-hand side of Equation 2.1. From this we conclude that Equation 2.1 holds, and so the proposition follows. □

Remark 2.4.7. If I is homogeneous with respect to some positive grading $\deg(x_i) = p_i > 0$, then we can define $\prec_{\mathbf{w}}$ for all $\mathbf{w} \in \mathbb{R}^n$ by setting $\mathbf{x}^{\mathbf{u}} \prec_{\mathbf{w}} \mathbf{x}^{\mathbf{v}}$ if $\deg(\mathbf{x}^{\mathbf{u}}) < \deg(\mathbf{x}^{\mathbf{v}})$ or if $\deg(\mathbf{x}^{\mathbf{u}}) = \deg(\mathbf{x}^{\mathbf{v}})$ and $\mathbf{w} \cdot \mathbf{u} < \mathbf{w} \cdot \mathbf{v}$, or finally if $\deg(\mathbf{x}^{\mathbf{u}}) = \deg(\mathbf{x}^{\mathbf{v}})$, $\mathbf{w} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{v}$ and $\mathbf{x}^{\mathbf{u}} \prec \mathbf{x}^{\mathbf{v}}$, where deg is with respect to the grading by p_i. While this definition differs from the previous one, the term $\mathrm{in}_{\prec_{\mathbf{w}}}(f)$ is unchanged for homogeneous f. The same leading terms will also be achieved if we use $\prec_{\mathbf{w}'}$, where $\mathbf{w}' = \mathbf{w} + N\mathbf{p}$, where $\mathbf{p} = (p_i)$, and $N > 0$ has been chosen to be sufficiently large so $\mathbf{w}' \in \mathbb{R}^n_{\geq 0}$.

With this new definition the result of Lemma 2.4.2 still holds, and thus we can define $\mathrm{in}_{\mathbf{w}}(I)$ for any $\mathbf{w} \in \mathbb{R}^n$. This means that the conclusion of Proposition 2.4.6 holds for the cone $C_h[\mathbf{w}] = \{\mathbf{w}' \in \mathbb{R}^n : \mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathbf{w}'}(I)\}$.

Note that $C_h[\mathbf{w}] \supseteq C[\mathbf{w}] + \mathrm{span}(\mathbf{p})$. In fact the lineality space of $C_h[\mathbf{w}]$ (the largest subspace contained in $C_h[\mathbf{w}]$) consists of all degree vectors $\mathbf{p}$ for which I is homogeneous. We leave this fact as an exercise to the reader.

Example 2.4.8. Let $S = \mathbf{k}[a,b,c,d,e]$ and let I be the ideal $\langle ac-b^2, ae-bd, be-cd\rangle$. Let $\mathbf{w}$ be the weight-vector $(0,2,0,1,0)$. The initial ideal $\mathrm{in}_{\prec_{\mathbf{w}}}(I)$ is then $\langle b^2, bd, be, cd^2\rangle$. The reduced Gröbner basis for $\prec_{\mathbf{w}}$ is $\{b^2-ac, bd-ae, be-cd, cd^2-ae^2\}$, so by Lemma 2.4.2 we see that $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\prec_{\mathbf{w}}}(I)$. Thus $C[\mathbf{w}] = \{\mathbf{w}' \in \mathbb{R}^5_{\geq 0} : 2w_2' > w_1' + w_3', w_2' + w_4' > w_1' + w_5', w_2' + w_5' > w_3' + w_4', w_3' + 2w_4' > w_1' + 2w_5'\}$. To obtain the closure $\overline{C[\mathbf{w}]}$ we turn the strict inequalities in the above set into non-strict inequalities.

Note that I is homogeneous with respect to the standard degree $\deg(a) = \deg(b) = \deg(c) = \deg(d) = \deg(e) = 1$, so we can also consider $C_h[\mathbf{w}]$. The vectors $\{(1,1,1,1,1),(0,0,0,1,1),(0,1,2,0,1)\}$ are all contained in the lineality space of $C_h[\mathbf{w}]$, so we can project onto the orthogonal subspace spanned by $(1,-1,0,-1,1)$ and $(1,-2,1,0,0)$. In these coordinates we have $C_h[\mathbf{w}] = \{(s,t) : s > 3t, 0 > s\}$. To see this, note that only the last two inequalities above are facet-defining for $\overline{C_h[\mathbf{w}]}$. The first of these then turns into $(-s-2t)+s > t + (-s)$, which simplifies to $s > 3t$, and similarly for the second inequality.

We next note that these cones fit together to form a polyhedral fan.

Proposition 2.4.9. *The set* $\{\overline{C[\mathbf{w}]} : \mathbf{w} \in \mathbb{R}^n_{\geq 0}\}$ *forms a polyhedral fan.*

Proof. We first show that if $\mathbf{w}'$ lies in a face of $\overline{C[\mathbf{w}]}$ with $\mathbf{w}' \notin C[\mathbf{w}]$, then $\overline{C[\mathbf{w}']}$ is a face of $\overline{C[\mathbf{w}]}$. Fix a monomial term order $\prec$ and let $\mathcal{G}$ be the reduced Gröbner basis for I with respect to $\prec_{\mathbf{w}}$. Let $J = \langle \mathrm{in}_{\mathbf{w}'}(g) : g \in \mathcal{G}\rangle$. Since $\mathbf{w}'$ lies in a face of $C[\mathbf{w}]$, we know that $\mathrm{in}_{\mathbf{w}}(g) = \mathrm{in}_{\mathbf{w}}(\mathrm{in}_{\mathbf{w}'}(g))$ for all $g \in \mathcal{G}$. This means $\mathrm{in}_{\prec}(\mathrm{in}_{\mathbf{w}}(I)) \subseteq \mathrm{in}_{\prec}(\mathrm{in}_{\mathbf{w}}(J))$ By Lemma 2.4.2 we have $\mathrm{in}_{\prec_{\mathbf{w}}}(I) = \mathrm{in}_{\prec}(\mathrm{in}_{\mathbf{w}}(I))$. Suppose that $J \subsetneq \mathrm{in}_{\mathbf{w}'}(I)$. Then $\mathrm{in}_{\prec_{\mathbf{w}}}(J) \subsetneq \mathrm{in}_{\prec_{\mathbf{w}}}(\mathrm{in}_{\mathbf{w}'}(I)) = \mathrm{in}_{\prec_{\mathbf{w},\mathbf{w}'}}(I)$, where $\prec_{\mathbf{w},\mathbf{w}'}$ is the term order that first compares monomials using $\mathbf{w}'$, and then breaks ties with $\prec_{\mathbf{w}}$. But this means $\mathrm{in}_{\prec_{\mathbf{w}}}(I) \subsetneq \mathrm{in}_{\prec_{\mathbf{w},\mathbf{w}'}}(I)$ is a proper inclusion of initial ideals, which is impossible by Corollary 2.2.3. Thus $J = \mathrm{in}_{\mathbf{w}'}(I)$. Since $\mathrm{in}_{\prec_{\mathbf{w}}}(I) \subseteq \mathrm{in}_{\prec_{\mathbf{w}}}(J) = \mathrm{in}_{\prec_{\mathbf{w},\mathbf{w}'}}(I)$, we conclude, again by Corollary 2.2.3, that they are equal, and so $\mathcal{G}$ is the reduced Gröbner basis for $\prec_{\mathbf{w},\mathbf{w}'}$. This implies that $C[\mathbf{w}'] = \{\mathbf{w}'' \in \mathbb{R}^n_{\geq 0} : \mathrm{in}_{\mathbf{w}''}(g) = \mathrm{in}_{\mathbf{w}'}(g) \text{ for all } g \in \mathcal{G}\}$ is a face of $C[\mathbf{w}]$.

Now suppose that $\overline{C[\mathbf{w}_1]}$ and $\overline{C[\mathbf{w}_2]}$ are two cones with neither closure contained in the other. Then by the above argument we know that the intersection, which at least contains 0, is a union of common faces of these two closed cones. Since this intersection is convex, it must in fact be one face, so $\overline{C[\mathbf{w}_1]} \cap \overline{C[\mathbf{w}_2]}$ is a face of each. □

Definition 2.4.10. The polyhedral fan $\{\overline{C[\mathbf{w}]} : \mathbf{w} \in \mathbb{R}^n_{\geq 0}\}$ is called the *Gröbner fan* of the ideal I. If I is homogeneous with respect to some positive grading, as in Remark 2.4.7, then the homogeneous Gröbner fan is the fan $\{\overline{C_h[\mathbf{w}]} : \mathbf{w} \in \mathbb{R}^n_{\geq 0}\}$. This is a complete fan.

From now until the end of this section we assume that I is homogeneous with respect to a positive grading, which allows us to use the forms of the definitions in Remark 2.4.7.

Proposition 2.4.11. *Suppose I is homogeneous with respect to some positive grading* $\deg(x_i) = p_i > 0$. *Then the homogeneous Gröbner fan of I is polytopal.*

Proof. For a monomial ideal M, we define $(\sum M)_d$ to be the vector $\sum \mathbf{u}$ where the sum is over all vectors $\mathbf{u}$ with $\mathbf{x}^{\mathbf{u}} \in M$ with $\sum_i p_i u_i = d$. The hypothesis that the grading is positive is used here to guarantee that this sum is finite. Define

$$\mathrm{State}_d(I) = \mathrm{conv}(\{(\textstyle\sum J)_d : J \text{ is a monomial initial ideal of } I\}).$$

Set D to be the largest degree (using the weights p_i) of any element of a universal Gröbner basis for I. We then set

$$\mathrm{State}(I) = \sum_{d=1}^{D} \mathrm{State}_d(I),$$

where the sum is the Minkowski sum of polytopes.

To prove the proposition it suffices to prove that the outer normal fan to State(I) is the Gröbner fan of I. To show that two polyhedral fans are the same it suffices to show that their maximal open cones are the same. So we need only prove the claim that for *generic* weight vectors $\mathbf{w}, \mathbf{w}' \in \mathbb{R}^n$ we have $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathbf{w}'}(I)$ if and only if the vertices of State(I) maximizing $\mathbf{w}$ and $\mathbf{w}'$ are the same. The notion of genericity we are using here is that $\mathbf{w}$ is generic if both $\mathrm{in}_{\mathbf{w}}(I)$ is a monomial ideal, and $\mathrm{face}_{\mathbf{w}}(\mathrm{State}(I))$ is a vertex of State(I).

For the "only-if" direction, suppose that $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathbf{w}'}(I)$. This means that $(\mathrm{in}_{\mathbf{w}}(I))_d = (\mathrm{in}_{\mathbf{w}'}(I))_d$ for $d = 1 \dots D$. Lemma 2.4.12, which follows this proof, thus says that the faces of $\mathrm{State}_d(I)$ maximizing $\mathbf{w}$ and $\mathbf{w}'$ are the same. This direction now follows from the fact that the faces of State(I) maximizing $\mathbf{w}$ and $\mathbf{w}'$ are the Minkowski sums of the faces of $\mathrm{State}_d(I)$ maximizing $\mathbf{w}$

and $\mathbf{w}'$ respectively. Indeed, if $\mathbf{v}_d \in \mathrm{State}_d(I)$ satisfies $\mathbf{w} \cdot \mathbf{v}_d \geq \mathbf{w} \cdot \mathbf{y}_d$ for all $\mathbf{y}_d \in \mathrm{State}_d(I)$, for $1 \leq d \leq D$, then $\mathbf{w} \cdot \sum_{d=1}^{D} \mathbf{v}_d \geq \mathbf{w} \cdot \mathbf{y}$, where $\mathbf{y} = \sum_{d=1}^{D} \mathbf{y}_d$ for $\mathbf{y}_d \in \mathrm{State}_d(I)$ is an arbitrary point of $\mathrm{State}(I)$.

For the "if" direction, suppose that the vertices of $\mathrm{State}(I)$ maximizing $\mathbf{w}$ and $\mathbf{w}'$ coincide. Properties of Minkowski sums now imply that the vertices of each $\mathrm{State}_d(I)$ maximizing $\mathbf{w}$ must coincide. Since $\mathbf{w}, \mathbf{w}'$ were chosen to be generic, $\mathrm{in}_{\mathbf{w}}(I)$ and $\mathrm{in}_{\mathbf{w}'}(I)$ are honest monomial initial ideals of I. Corollary 2.4.13, which follows this proof, now implies that $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathbf{w}'}(I)$, which completes the proof. $\square$

Lemma 2.4.12. *For all $\mathbf{w} \in \mathbb{R}^n$ we have $face_{\mathbf{w}}(\mathrm{State}_d(I)) = \mathrm{State}_d(\mathrm{in}_{\mathbf{w}}(I))$.*

Proof. We first prove the lemma in the case where $\mathbf{w}$ is generic, so $\mathrm{in}_{\mathbf{w}}(I)$ is a monomial ideal, and the face of $\mathrm{State}_d(I)$ is a vertex. We denote by I_d the vector space of homogeneous polynomials of degree d in I. Let $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}$ be the monomials of degree d in S, and let $r = \dim_{\mathbf{k}}(I_d) \leq m$. Let $\prec$ be a term order for which $\mathrm{in}_{\prec}(I) = \mathrm{in}_{\mathbf{w}}(I)$. We may assume that $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r}$ are the monomials in $\mathrm{in}_{\prec}(I)$. Since every initial ideal of a monomial ideal is itself, we have $\mathrm{State}_d(\mathrm{in}_{\mathbf{w}}(I)) = \mathbf{a}_1 + \cdots + \mathbf{a}_r$. Lemma 2.2.2 implies the existence of polynomials $\mathbf{x}^{\mathbf{a}_i} - \sum_{j=r+1}^{m} c_{ij}\mathbf{x}^{\mathbf{a}_j} \in I_d$ for $1 \leq i \leq r$. In each of these polynomials $\mathbf{x}^{\mathbf{a}_i}$ is the leading term with respect to the weight vector $\mathbf{w}$.

By the construction of $\mathrm{State}_d(I)$, we know that there is some term order $\prec'$ for which the face of $\mathrm{State}_d(I)$ maximized by $\mathbf{w}$ is $(\sum \mathrm{in}_{\prec'}(I))_d$. Let $\mathbf{x}^{\mathbf{a}_{i_1}}, \ldots, \mathbf{x}^{\mathbf{a}_{i_r}}$ be the monomials of degree d in $\mathrm{in}_{\prec'}(I)$.

If the lemma were not true we would have $\mathbf{w} \cdot (\mathbf{a}_{i_1} + \cdots + \mathbf{a}_{i_r}) > \mathbf{w} \cdot (\mathbf{a}_1 + \cdots + \mathbf{a}_r)$. This would mean that $\mathbf{w} \cdot (\sum_{k \neq i_j} \mathbf{a}_k) < \mathbf{w} \cdot (\sum_{k>r} \mathbf{a}_k)$ for the two bases $\mathcal{B}_1 = \{\mathbf{x}^{\mathbf{a}_k} : \mathbf{x}^{\mathbf{a}_k} \notin \mathrm{in}_{\prec'}(I)\}$ and $\mathcal{B}_2 = \{\mathbf{x}^{\mathbf{a}_k} : k > r\}$ of $(S/I)_d$. We will obtain a contradiction to this statement by constructing a path of bases for this vector space from $\mathcal{B}_1$ to $\mathcal{B}_2$ each of which differs from the previous one by one element, and consists of elements from $\mathcal{B}_1 \cup \mathcal{B}_2$. These bases will have the property that the dot product with $\mathbf{w}$ of their sum decreases at each step, which will give the desired contradiction.

Suppose we have constructed a path from $\mathcal{B}_1$ to a basis $\mathcal{B}$. Let $\mathbf{x}^{\mathbf{a}_k}$ be an element of $\mathcal{B} \setminus \mathcal{B}_2$ (so $k \leq r$), and let $f = \mathbf{x}^{\mathbf{a}_k} - \sum_{j=r+1}^{m} c_{kj}\mathbf{x}^{\mathbf{a}_j}$ be the corresponding element of I_d. Let $\mathcal{C} = \{\mathbf{x}^{\mathbf{a}_j} : c_{kj} \neq 0\}$. The collection $\mathcal{C}$ is linearly independent modulo I, since it is a subset of $\mathcal{B}_2$. Each element $\mathbf{x}^{\mathbf{a}_j}$ of $\mathcal{C}$ can be written as a linear combination of elements of $\mathcal{B}$. If none of these linear combinations involved the element $\mathbf{x}^{\mathbf{a}_k}$, then $\mathcal{C} \cup \{\mathbf{x}^{\mathbf{a}_k}\}$ would be a linearly independent set, which is not the case since f is a linear dependency relation. Thus there is some $\mathbf{x}^{\mathbf{a}_i}$ that equals $c\mathbf{x}^{\mathbf{a}_k} + \sum_{\mathbf{x}^j \in \mathcal{B}, j \neq k} b_j \mathbf{x}^j$ where $c \neq 0$. This means that $\mathcal{B}' = \mathcal{B} \setminus \{\mathbf{x}^{\mathbf{a}_k}\} \cup \{\mathbf{x}^{\mathbf{a}_i}\}$ has the same span as $\mathcal{B}$, and thus is also a basis. Note that $\mathbf{w} \cdot (\sum_{\mathbf{x}^{\mathbf{a}_j} \in \mathcal{B}'} \mathbf{a}_j) < \mathbf{w} \cdot (\sum_{\mathbf{x}^{\mathbf{a}_j} \in \mathcal{B}} \mathbf{a}_j)$. Thus comparing along the constructed path from $\mathcal{B}_1$ to $\mathcal{B}_2$ we see that $\mathbf{w} \cdot (\sum_{\mathbf{x}^{\mathbf{a}_k} \in \mathcal{B}_1} \mathbf{a}_k) < \mathbf{w} \cdot (\sum_{\mathbf{x}^{\mathbf{a}_k} \in \mathcal{B}_2} \mathbf{a}_k)$. From this contradiction we conclude that the two bases are the same, and so the face of $\mathrm{State}_d(I)$ maximizing $\mathbf{w}$ is $\mathrm{State}_d(\mathrm{in}_{\mathbf{w}}(I))$ for generic $\mathbf{w}$.

To complete the proof for nongeneric $\mathbf{w}$ it suffices to show that the face of $\mathrm{State}_d(I)$ maximizing $\mathbf{w}$ has the same vertices as $\mathrm{State}_d(\mathrm{in}_{\mathbf{w}}(I))$. This follows from the generic case by considering the face of each polytope with respect to a generic weight vector $\mathbf{w}'$. $\square$

Corollary 2.4.13. *If $\prec$, $\prec'$ are two distinct term orders then* $\mathrm{in}_{\prec}(I)_d \neq \mathrm{in}_{\prec'}(I)_d$ *implies that* $(\sum \mathrm{in}_{\prec}(I))_d \neq (\sum \mathrm{in}_{\prec'}(I))_d$.

Proof. Let the monomials of degree d in $\mathrm{in}_{\prec}(I)$ be $\{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_r}\}$ and the monomials of degree d in $\mathrm{in}_{\prec'}(I)$ be $\{\mathbf{x}^{\mathbf{a}_{i_1}}, \dots, \mathbf{x}^{\mathbf{a}_{i_r}}\}$, and pick $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$ with $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\prec}(I)$. In the proof of Lemma 2.4.12 we showed that $\mathbf{w} \cdot (\mathbf{a}_1 + \cdots + \mathbf{a}_r) < \mathbf{w} \cdot (\mathbf{a}_{i_1} + \cdots + \mathbf{a}_{i_r})$. Since $(\sum \mathrm{in}_{\prec}(I))_d = \mathbf{a}_1 + \cdots + \mathbf{a}_r$ and $(\sum \mathrm{in}_{\prec'}(I))_d = \mathbf{a}_{i_1} + \cdots + \mathbf{a}_{i_r}$, this shows that they are not the same. $\square$

Definition 2.4.14. A polytope P whose normal fan is the Gröbner fan of an ideal $I \subseteq S$ is called a *state polytope* of I.

Example 2.4.15. Let $S = \mathbf{k}[a, b, c, d, e]$, and let I be the ideal $\langle ac - b^2, ae - bd, be - cd \rangle$.

The ideal I has seven initial ideals:

1. $\langle ac, ae, cd \rangle$
2. $\langle ac, bd, cd \rangle$
3. $\langle b^2, bd, cd \rangle$
4. $\langle b^2, bd, be, cd^2 \rangle$
5. $\langle ae^2, b^2, bd, be \rangle$
6. $\langle ae, b^2, be \rangle$
7. $\langle ac, ae, be \rangle$

A state polytope for I is shown in Figure 2.4. As in Example 2.4.8 the Gröbner fan has a three-dimensional lineality space, so we draw a two-dimensional polytope.

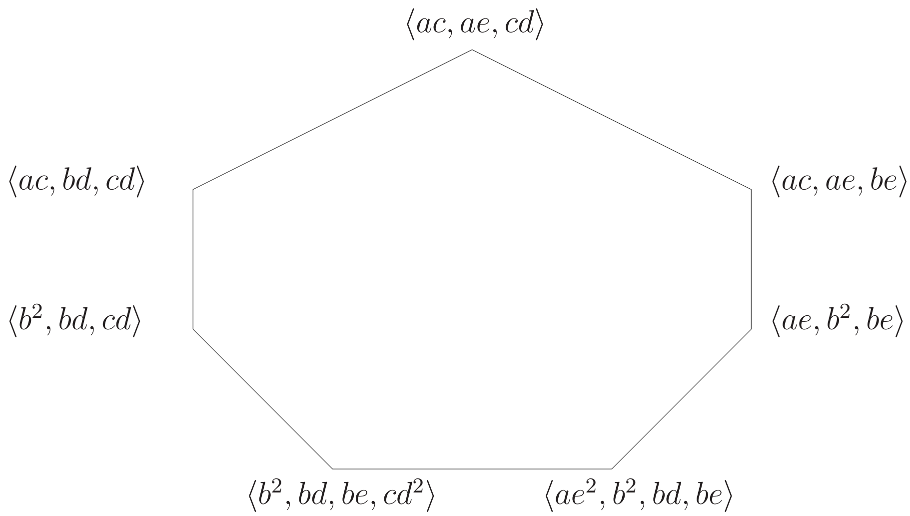


Figure 2.4: A state polytope for the ideal of Example 2.4.15

Remark 2.4.16. We note that while the Gröbner fan is defined for any ideal in the polynomial ring, the hypothesis in Proposition 2.4.11 that the ideal is homogeneous with respect to some positive grading cannot be removed. In [Jen05] Anders Jensen gives an example of an ideal with three generators in the polynomial ring $\mathbb{Q}[a, b, c, d]$ whose Gröbner fan is not the normal fan of any polyhedron. Positive grading is not strictly a necessary condition for the Gröbner fan to be the normal fan of a polyhedron, as there are examples of non-positively graded ideals with with this property.

2.5 Further reading

The material covered in this chapter is taken from Chapters 2 and 3 of [Stu96]. An excellent introduction to polytope theory is [Zie95]. Term orders in the polynomial ring were first classified by Robbiano in [Rob86]. The existence of the state polytope and the corresponding Gröbner fan was shown in two papers by Bayer and Morrison [BM88] and Mora and Robbiano [MR88] which both appeared in the same issue of the Journal of Symbolic Computation. Anders Jensen has new code Gfan [Jenb] to compute the list of all reduced Gröbner bases of a given ideal, and thus the Gröbner fan.

2.6 Tutorial 2

2.6.1 Polytopes and cones

The exercises in this section are designed to provide practice working with polytopes, polyhedral cones, and normal fans.

1. Consider the collection of points in $\mathbb{R}^2$:

$$S = \{(0,0), (-2,1), (0,2), (1,2), (2,4), (-1,1), (0,3), (1,0)\}.$$

   a) Draw the convex hull of S. What are the vertices?

   b) Find generators for all of the cones in the normal fan of the polygon you drew in part a).

2. Consider the convex hull of the points

$$\{(0,0,1), (1,0,0), (0,1,0), (0,-1,0), (-1,0,0), (0,0,-1)\} \subset \mathbb{R}^3.$$

   How many faces are there of dimension $0, 1, 2$, and 3 respectively? Describe the normal fan.

3. Prove that there is an inclusion reversing correspondence between the faces of a polytope $P \subset \mathbb{R}^n$ and the cones in its normal fan. Specifically, if $F_1 \subset F_2$ are two faces of P, then $\overline{C[F_1]} \supset \overline{C[F_2]}$. More generally for any two faces F_1 and F_2 show that $C[F_1 \cap F_2]$ is the smallest cone whose closure contains both $C[F_1]$ and $C[F_2]$.

4. The *lineality space* of a cone $C \subset \mathbb{R}^n$ is the largest linear subspace contained in C. Prove that if P is a polytope of dimension d in $\mathbb{R}^n$, then the closures of all the n-dimensional cones in the normal fan of P share a lineality space of dimension $n-d$.

2.6.2 Gröbner fan and state polytope

5. Given a non-zero polynomial $f = \sum_{i=1}^m c_i \mathbf{x}^{\mathbf{a}_i} \in S$, recall that $\mathrm{supp}(f) = \{\mathbf{a}_i \mid c_i \neq 0\}$. We call $\mathrm{New}(f) := \mathrm{conv}(\mathrm{supp}(f))$, the *Newton polytope* of f. Let $\mathrm{Vert}(f)$ be the subset of $\mathrm{supp}(f)$ consisting of the vertices of $\mathrm{New}(f)$.

(a) Prove that if $\mathbf{a} \in \mathrm{supp}(f) \setminus \mathrm{Vert}(f)$, then there is no weight vector $\mathbf{w}$ such that $\mathrm{in}_{\mathbf{w}}(f) = \mathbf{x}^{\mathbf{a}}$.

(b) Let $f = 3x^6y^2 + 2x^3y^3 - xy + 5x^3y^5 \in \mathbf{k}[x, y]$.

1) Find a weight vector $\mathbf{w}$ such that $\mathrm{in}_{\mathbf{w}}(f) = x^6y^2$.
2) Find a weight vector $\mathbf{w}$ such that $\mathrm{in}_{\mathbf{w}}(f) = x^3y^5$.
3) Show that $\mathrm{in}_{\succ}(f) \neq xy$ for every term ordering $\succ$.
4) Show that $\mathrm{in}_{\succ}(f) \neq x^3y^3$ for every term ordering $\succ$.

(c) If $f \in S$ is a homogeneous polynomial, then what is the state polytope of the principal ideal $\langle f \rangle$?

In the next two exercises you will need to compute the cones of Gröbner fans. Let $\mathcal{G} = \{g_1, \ldots, g_r\}$ be a reduced Gröbner basis of an ideal I with respect to some term order $\prec$. We can write $g_i = \mathbf{x}^{\mathbf{a}_i} + \sum c_{ij}\mathbf{x}^{\mathbf{b}_{ij}}$ where $\mathbf{x}^{\mathbf{a}_i}$ is the leading term of g_i and the various $\mathbf{x}^{\mathbf{b}_{ij}}$ are the other non-leading monomials in g_i. In the proof of Proposition 2.4.4 it is shown that these g_i suffice to determine the corresponding cone in the Gröbner fan. We can write the corresponding inequalities in the homogeneous setting as:

$$\begin{aligned} C[\prec] &= \{\mathbf{w} \in \mathbb{R}^n \;:\; \mathrm{in}_{\mathbf{w}}(g_i) = \mathbf{x}^{\mathbf{a}_i} \text{ for } i = 1, \ldots, r\} \\ &= \{\mathbf{w} \in \mathbb{R}^n \;: \mathbf{w} \cdot \mathbf{b}_{ij} < \mathbf{w} \cdot \mathbf{a}_i \text{ for } i = 1, \ldots, r \text{ and all } j\}. \end{aligned}$$

The set of inequalities above are called *defining inequalities* for the open polyhedral cone $C[\prec]$. The non-redundant inequalities in this set define the *facets* of the cone $C[\prec]$.

6. Let I be the ideal of Example 2.4.15.

a) For each of the seven given initial ideals find the corresponding Gröbner basis.

b) What is the three dimensional lineality space of the Gröbner fan? Write down a vector space basis.

c) Compute the defining inequalities for each of the cones in the Gröbner fan.

d) Find weight vectors $\mathbf{w} \in \mathbb{N}^5$ such that $\mathrm{in}_{\mathbf{w}}(I)$ is equal to each of the seven initial ideals.

7. Let $I = \langle x^2 + yz, xy + z^2 \rangle \subset \mathbb{Q}[x, y, z]$.

   a) Compute the grevlex and lex Gröbner bases of I with $x > y > z$.

   b) Find defining inequalities for the corresponding cones in the Gröbner fan.

   c) Some of these inequalities are redundant. For each cone find a non-redundant set. (Hint: There is a one-dimensional lineality space and each cone will have two non-redundant inequalities).

   d) Find all of the remaining initial ideals and cones in the Gröbner fan. One way to proceed is to compute lex and grevlex Gröbner bases with respect to different variable orderings then check if the corresponding cones cover. In section 2.6.3 of this tutorial we will see a more systematic way to compute the Gröbner fan.

   e) Draw the 2-dimensional state polytope of I with vertices labeled by initial ideals, as in Figure 2.4.

2.6.3 The Gröbner walk

One of the most important applications of the Gröbner fan is the *Gröbner walk* [Stu96, Chapter 3] , which is a general algorithm for converting a Gröbner basis from one term order to another. The lexicographic order is very useful when eliminating variables but its Gröbner bases are notoriously difficult to compute. The Gröbner walk allows us to start with an easier order such as graded reverse lex and transform the corresponding Gröbner basis.

Suppose we have a Gröbner basis $\mathcal{G}$ for I with respect to some starting order $\prec_s$. Let $\mathbf{w}_s$ be a weight vector realizing this order. The goal will be to compute a new Gröbner basis with respect to a target order $\prec_t$ represented by $\mathbf{w}_t$. To that end we pick a path from $\mathbf{w}_s$ to $\mathbf{w}_t$, which in sufficiently nice situations can be taken to be a straight line. We will assume that our path only passes through codimension-one cones of the Gröbner fan, and never enters lower-dimensional cones.

The Gröbner walk has two basic steps.

- Cross from one open cone C_i in the Gröbner fan to another C_{i+1} along the chosen path.

- Modify the current Gröbner basis to respect the new cone.

For the first step, suppose we are in some open cone C_i for which we know the Gröbner basis $\mathcal{G}_i$. Let $\mathbf{w}_{new}$ be the last point in $\overline{C_i}$ along the path, computed by intersecting the path with each facet of C_i.

For the second step consider the ideal $\langle \mathrm{in}_{\mathbf{w}_{new}}(\mathcal{G}_i)\rangle$ of initial forms of $\mathcal{G}_i$ with respect to the new weight vector. Since $\mathbf{w}_{new}$ is on the boundary of C_i, $\langle \mathrm{in}_{\mathbf{w}_{new}}(\mathcal{G}_i)\rangle$ will not be a monomial ideal, but for nice cases there will be mostly monomials with a couple of other polynomials (ideally just one binomial).

Define a new term order $\prec_{i+1}$ using the weight vector $\mathbf{w}_{new}$ but breaking ties with our target order $\prec_t$. The remarkable fact is that to compute the Gröbner basis $\mathcal{G}_{i+1}$ of I with respect to $\prec_{i+1}$ it is enough to compute the Gröbner basis of the far simpler ideal $\langle \mathrm{in}_{\mathbf{w}_{new}}(\mathcal{G}_i)\rangle = \mathrm{in}_{\mathbf{w}_{new}}(I)$.

Proposition 2.6.1. *[Stu96, page 23] Let $H = \{h_1, \ldots, h_s\}$ be a Gröbner basis of $\langle \mathrm{in}_{\mathbf{w}_{new}}(G_i)\rangle$ with respect to $\prec_{i+1}$. Write*

$$h_i = \sum_{g \in \mathcal{G}_i} p_g \, \mathrm{in}_{\mathbf{w}_{new}}(g),$$

then the set of all polynomials

$$\overline{h}_i = \sum_{g \in \mathcal{G}_i} p_g g$$

is a Gröbner basis for I with respect to $\prec_{i+1}$.

8. Use the Gröbner walk to convert the grevlex Gröbner basis to lex in Exercise 7a).

2.7 Solutions to Tutorial 2

2.7.1 Polytopes and cones

1. (a) The convex hull is the pentagon shown in Figure 2.5.

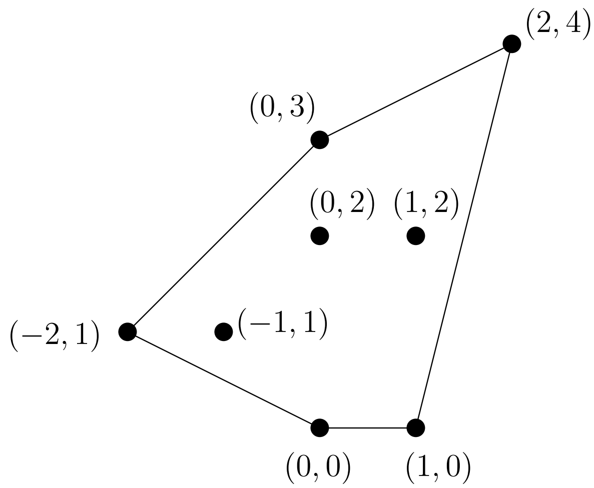


Figure 2.5:

The points $(0,0), (1,0), (2,4), (0,3)$, and $(-2,1)$ are the vertices and the remaining three points $(0,2), (1,2)$, and $(-1,1)$ are in the interior.

(b) There are five one-dimensional cones (rays) in the normal fan. They are spanned in counterclockwise order by $(-1,2)$, $(-1,1)$, $(-1,-2)$, $(0,-1)$, and $(4,-1)$. The five two-dimensional cones are generated by consecutive pairs of rays. A picture of the normal fan is shown in Figure 2.6

2. The convex hull of the given points is the octahedron shown in Figure 2.7. There are 6 vertices, 12 edges, 8 two-dimensional faces, and 1 three-dimensional face (the whole polytope). The normal fan has three-dimensional cones the cones over the facets of the cube with vertices $(\pm1, \pm1, \pm1)$.

3. The cone $C[F]$ was defined as the set of $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{c} \cdot \mathbf{x}$ is maximized for exactly $\mathbf{x} \in F$. To make this a closed set we must allow $\mathbf{c} \cdot \mathbf{x}$

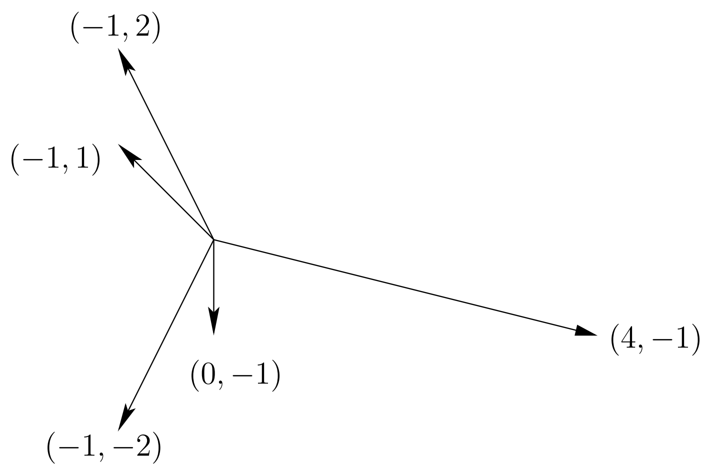


Figure 2.6:

to achieve its maximum on a larger set. Thus the closure of $C[F]$ is the set of $\mathbf{c} \in \mathbb{R}^n$ such that $\text{face}_{\mathbf{c}}(P) \supseteq F$.

It immediately follows that $\overline{C[F_1]} \supset \overline{C[F_2]}$ if and only if $F_1 \subset F_2$. More generally the closure of $C[F_1 \cap F_2]$ contains both $C[F_1]$ and $C[F_2]$, and conversely any other $\overline{C[F]}$ with this property has $F \subset F_1$ and $F \subset F_2$, thus $F \subset F_1 \cap F_2$ and $\overline{C[F]} \supset \overline{C[F_1 \cap F_2]}$.

4. If $P \subset \mathbb{R}^n$ is of dimension d, it spans an affine space of dimension d. Let V be the linear space of dimension $n-d$ orthogonal to this affine space. For any $\mathbf{v} \in V$, $\mathbf{v} \cdot \mathbf{p}$ is constant for $\mathbf{p} \in P$. Hence, $\text{face}_{\mathbf{v}}(P) = P$. Therefore, $V = C[P]$. By the previous exercise $V \subset C[F]$ for any smaller face F.

2.7.2 Gröbner fan and state polytope

5. (a) Let $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_r\} \subset \mathbb{Q}^n$ and $P = \text{conv}(\mathbf{v}_i : 1 \leq i \leq r)$. We use the fact that every element $\mathbf{v}$ of $\mathcal{P} \cap \mathbb{Q}^n$ has a representation $\mathbf{v} = \sum_{i=1}^r \lambda_i \mathbf{v}_i$ with $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{Q}_{\geq 0}$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_r = 1$. Let $U = \text{Vert (New } (f))$ and let $\mathbf{a} \in \text{supp }(f) \setminus U$. Then we can write $\mathbf{a} = \sum_{\mathbf{v} \in U} \lambda_{\mathbf{v}} \mathbf{v}$ with $\lambda_{\mathbf{v}} \in \mathbb{Q}_{\geq 0}$ and $\sum_{\mathbf{v} \in U} \lambda_{\mathbf{v}} = 1$. Let the least common multiple of the denominators be L. Then we have $L\mathbf{a} = \sum_{\mathbf{v} \in U} I_{\mathbf{v}} \mathbf{v}$, where $I_{\mathbf{v}} \in \mathbb{N}$ and $\sum_{\mathbf{v}} I_{\mathbf{v}} = L$. If $\mathbf{x}^{\mathbf{a}}$ is the leading term then $\mathbf{x}^{\mathbf{a}} \succ \mathbf{x}^{\mathbf{v}}$ for any $\mathbf{v} \in V$. This implies $\mathbf{x}^{L\mathbf{a}} \succ \mathbf{x}^{\sum_{\mathbf{v} \in V} I_{\mathbf{v}} \mathbf{v}}$

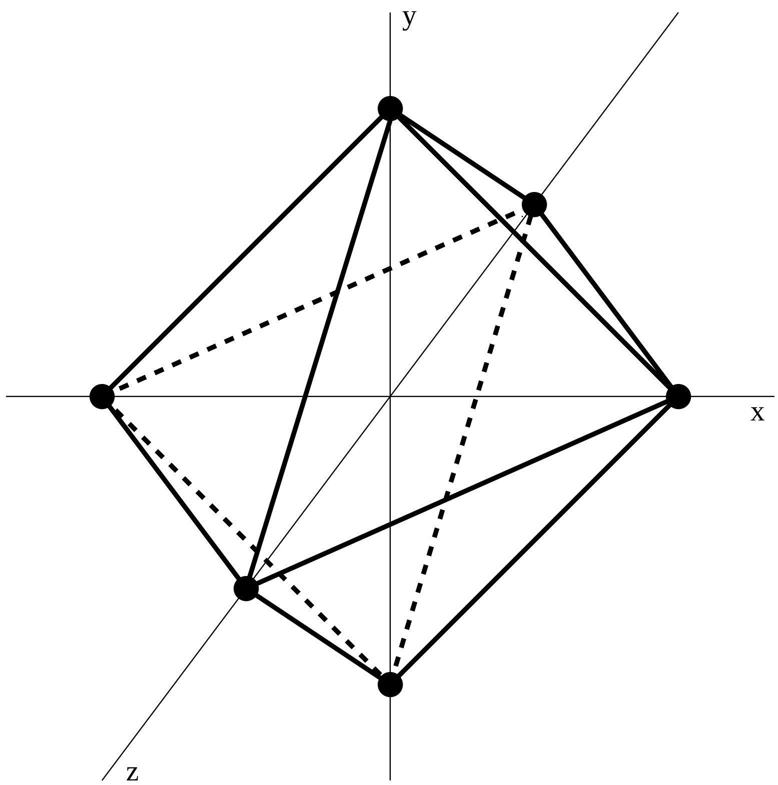


Figure 2.7:

(by the multiplicative axiom), which is a contradiction. Thus $\mathbf{x}^{\mathbf{a}}$ cannot be the leading term under any term order.

(b) 1) This is true for the lex ordering with $x > y$, so we take $\mathbf{w} = (10, 1)$.

2) This is true for the lex ordering with $y > x$, so we take $\mathbf{w} = (1, 10)$.

3) For any order $\prec$ we must have $1 \prec x^2y^2$, so $xy \prec x^3y^3$.

4) For any order $\prec$ we must have $1 \prec y^2$, so $x^3y^3 \prec x^3y^5$.

(c) Write $f = \sum c_i \mathbf{x}^{\mathbf{a}_i}$. We will show that if f is homogeneous of degree d, the initial ideals of f correspond exactly to the vertices of the Newton polytope. If that is the case then the Newton polytope is equal to $\text{State}_d(\langle f \rangle)$ which is equal to $\text{State}(\langle f \rangle)$ since the set consisting of the generator f of degree d is itself a universal Gröbner basis.

The vertices of the Newton polytope of f are those $\mathbf{a}_i$ such that there exists $\mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{w} \cdot \mathbf{a}_i > \mathbf{w} \cdot \mathbf{a}_j$ for all $j \neq i$. That is to say $\mathrm{in}_{\mathbf{w}}(f) = \mathbf{x}^{\mathbf{a}_i}$. Therefore, any leading monomial of f (corresponding to $\mathbf{w}$ nonnegative) is a vertex of the Newton polytope. Conversely, since f is homogeneous of degree d, $(1, \ldots, 1) \cdot a_j = d$ for all j. Therefore, we can take $\mathbf{w}' = \mathbf{w} + (k, \ldots, k)$ and still have $\mathrm{in}_{\mathbf{w}'}(f) = \mathbf{x}^{\mathbf{a}_i}$. Taking k sufficiently large, $\mathbf{w}'$ will be nonnegative. This results fails if f is not homogeneous as can be seen by considering the above nonhomogeneous polynomial.

6. a) The seven Gröbner bases, with leading terms underlined, are:

   (a) $\{\underline{ac} - b^2, \underline{ae} - bd, \underline{cd} - be\}$
   (b) $\{\underline{ac} - b^2, \underline{bd} - ae, \underline{cd} - be\}$
   (c) $\{\underline{b^2} - ac, \underline{bd} - ae, \underline{cd} - be\}$
   (d) $\{\underline{b^2} - ac, \underline{bd} - ae, \underline{be} - cd, \underline{cd^2} - ae^2\}$
   (e) $\{\underline{ae^2} - cd^2, \underline{b^2} - ac, \underline{bd} - ae, \underline{be} - cd\}$
   (f) $\{\underline{b^2} - ac, \underline{ae} - bd, \underline{be} - cd\}$
   (g) $\{\underline{ac} - b^2, \underline{ae} - bd, \underline{be} - cd\}$

   These can be computed using the software package Gfan [Jenb], or can be done by hand by finding polynomials in I with the given leading terms.

   b) The lineality space consists of those $\mathbf{w}$ such that $\mathrm{in}_{\mathbf{w}}(I) = I$ (do you see why?). Thus, for each binomial generator of I, the two monomials have equal weight. So, $w_1 + w_3 = 2w_2$, $w_1 + w_5 = w_2 + w_4$, and $w_3 + w_4 = w_2 + w_5$. The solution space is w_3, w_4, w_5 arbitrary, $w_1 = w_3 + 2w_4 - 2w_5$, and $w_2 = w_3 + w_4 - w_5$. A vector space basis is $\{(1, 1, 1, 0, 0), (2, 1, 0, 1, 0), (-2, -1, 0, 0, 1)\}$.

   c) To simplify computation we quotient out by the lineality space by making the change of coordinates $w_1' = w_1 - w_3 - 2w_4 + 2w_5$ and $w_2' = w_2 - w_3 - w_4 + w_5$. Therefore, any point $(w_1, w_2, w_3, w_4, w_5)$ can be transformed into $(w_1', w_2', 0, 0, 0)$ by adding something in the lineality space.

   The defining inequalities for the first initial ideal $\langle ac, ae, cd\rangle$ are

$$w_1' > 2w_2', w_1' > w_2', \text{ and } 0 > w_2'.$$

The first inequality is redundant (the sum of the second and third) so the cone is defined by $w'_1 > w'_2$ and $w'_2 < 0$. Similarly, for each of the other 6 initial ideals, removing redundant inequalities, we get

(b) $w'_1 > 2w'_2,\ w'_2 > w'_1$

(c) $2w'_2 > w'_1,\ w'_2 < 0$

(d) $w'_1 < 0,\ w'_2 > 0$

(e) $w'_1 > 0,\ w'_2 > w'_1$

(f) $w'_1 < 2w'_2,\ w'_1 > w'_2$

(g) $w'_1 > 2w'_2,\ w'_2 > 0.$

A picture of the Gröbner fan in the plane with coordinates w'_1 and w'_2 is shown in Figure 2.8

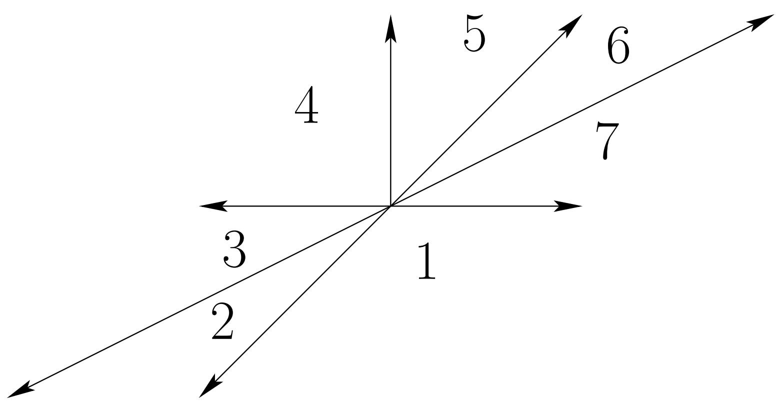


Figure 2.8:

d) Using the inequalities found above we can find weight vectors of the form $(w'_1, w'_2, 0, 0, 0)$ in each region. These may not be nonnegative so we can add an appropriate multiple of $(1, 1, 1, 1, 1)$ to get the following $\mathbf{w}$:

(a) $(2, 0, 1, 1, 1)$

(b) $(0, 1, 3, 3, 3)$

(c) $(0, 2, 3, 3, 3)$

(d) $(0, 2, 1, 1, 1)$

(e) $(1, 2, 0, 0, 0)$

(f) $(3, 2, 0, 0, 0)$

(g) $(3, 1, 0, 0, 0)$

7. a) The grevlex basis is $\langle x^2 + yz, xy + z^2, y^2z - xz^2\rangle$. The lex basis is $\langle x^2+yz, xy+z^2, xz^2-y^2z, y^3z+z^4\rangle$. These are computed by the following Macaulay 2 code:

```
i1 : R = QQ[x,y,z];
i2 : I = ideal(x^2 + y*z, x*y + z^2);
o2 : Ideal of R
i3 : gb(I)

o3 = | xy+z2 x2+yz y2z-xz2 |

i4 : R1 = QQ[x,y,z, MonomialOrder=>Lex];
i5 : I1 = substitute(I, R1);
i6 : gb(I1)

o6 = | xy+z2 x2+yz xz2-y2z y3z+z4 |
```

b) For the grevlex basis the defining inequalities are:

$$w_1 + w_2 > 2w_3, \quad 2w_1 > w_2 + w_3, \quad 2w_2 > w_1 + w_3$$

For the lex basis the inequalities are:

$$w_1 + w_2 > 2w_3, \quad 2w_1 > w_2 + w_3, \quad w_1 + w_3 > 2w_2, \quad w_2 > w_3$$

c) Make a change of variables $w_1' = w_1 - w_3$, and $w_2' = w_2 - w_3$ to account for the one-dimensional lineality space arising from the homogeneity. The inequalities for the grevlex cone become

$$w_1' + w_2' > 0, \quad 2w_1' > w_2', \quad 2w_2' > w_1',$$

and the first one is redundant as it is the sum of the second and third. The corresponding cone is generated by the vectors $(1, 2)$ and $(2, 1)$.

For the lex cone we get:

	Gröbner basis	Initial ideal
1)	$x^2+yz, xy+z^2, y^2z-xz^2$	x^2, xy, y^2z
2)	$x^2+yz, xy+z^2, xz^2-y^2x, y^3z+z^4$	x^2, xy, xz^2, y^3z
3)	$x^2+yz, xy+z^2, xz^2-y^2x, z^4+y^3z$	x^2, xy, xz^2, z^4
4)	x^2+yz, z^2+xy	x^2, z^2
5)	$yz+x^2, z^2+xy, x^2z-xy^2, x^4+xy^3$	yz, z^2, x^2z, x^4
6)	$yz+x^2, z^2+xy, x^2z-xy^2, xy^3+x^4$	yz, z^2, x^2z, xy^3
7)	$yz+x^2, z^2+xy, xy^2-x^2z$	yz, z^2, xy^2
8)	$yz+x^2, xy+z^2, z^3-x^3$	yz, xy, z^3
9)	$yz+x^2, xy+z^2, x^3-z^3$	yz, xy, x^3

$$w_1'+w_2'>0, \quad 2w_1'>w_2', \quad w_1'>2w_2', \quad w_2'>0$$

A little thought gives that the last two are nonredundant and the corresponding cone is generated by $(2,1)$ and $(1,0)$.

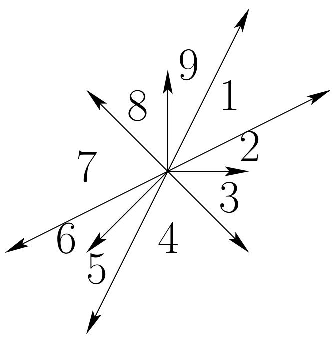
d) and e) There are 9 total reduced Gröbner bases all of which can be found as grevlex or lex orders with respect to some variable ordering. We list the 9 Gröbner bases and corresponding initial ideals. We leave it up to the reader to determine which ideals are determined by which orders.

Computing all the cones yields the Gröbner fan in Figure 2.9. The state polytope is also shown with vertices labeled by the corresponding initial ideals.

2.7.3 The Gröbner walk

8. Weight vectors realizing the grevlex and lex term orders are $(1,1,0)$ and $(3,1,0)$ respectively. Computing the grevlex cone as before, we find that we hit the boundary of the grevlex cone at $\mathbf{w}_{new}=(2,1,0)$. Our starting Gröbner basis is $\mathcal{G}=\{x^2-yz, xy-z^2, y^2z-xz^2\}$. The ideal of initial forms with respect to $\mathbf{w}_{new}$ is $\langle x^2, xy, y^2z-xz^2\rangle$. We next compute a Gröbner basis of this ideal of initial forms with respect to the new lex order, so the leading term of the binomial y^2z-xz^2 is now xz^2. There is one non-trivial S-pair:

$$z^2(xy)+y(-xz^2+y^2z)=y^3z.$$



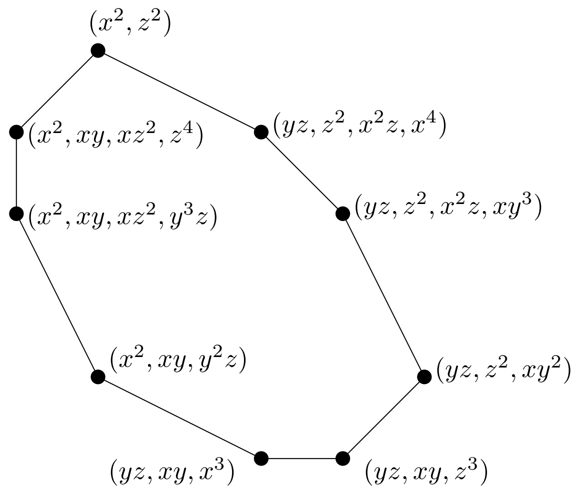


Figure 2.9: Gröbner fan and state polytope

Lifting this up to the original ideal yields

$$z^2(xy - z^2) + y(-xz^2 + y^2z) = y^3z - z^4.$$

It is not difficult to check that all other S-pairs reduce to zero.

So, our new Gröbner basis is

$$\{x^2 + yz, xy - z^2, xz^2 - y^2z, y^3z - z^4\}$$

which is already the desired reduced lex basis.

Chapter 3

Toric Ideals

3.1 Introduction

In this chapter we focus on a special class of polynomial ideals called toric ideals which allow interactions among algebra, geometry and combinatorics. They are the defining ideals of toric varieties which are a rich but fairly accessible class of varieties in algebraic geometry. See Chapter 6 for the connection to algebraic geometry. They also encode the combinatorics of polytopes and vector configurations, and have several applications. This chapter is based on Chapters 4, 5 and 12 of [Stu96].

To motivate the definition of a toric ideal, consider the following problem. Suppose you had an unlimited supply of coins in a certain currency, of denominations 5, 10, 25 and 50, and you would like to answer the following questions for any positive number b.

1. Is there a combination of coins that adds up to b?

2. How many coins of each type would you have to use if you wanted to make the combination in Problem 1 with as few coins as possible?

Problem 2 is the following integer program:

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 + x_3 + x_4 \\ \text{subject to} & 5x_1 + 10x_2 + 25x_3 + 50x_4 = b \\ & x_1, x_2, x_3, x_4 \in \mathbb{N} \end{array}$$

while Problem 1 is asking whether this program has a *feasible* solution, i.e., is $\{(x_1, x_2, x_3, x_4) \in \mathbb{N}^4 \,:\, 5x_1 + 10x_2 + 25x_3 + 50x_4 = b\}$ nonempty ?

Let us examine the possible ways of making change for $b = 100$ using the four types of coins, or equivalently all the feasible solutions to the above integer program with $b = 100$. There are 40 feasible solutions:

$(0,0,0,2)$	$(0,0,2,1)$	$(0,0,4,0)$	$(1,2,1,1)$	$(1,2,3,0)$	$(0,5,0,1)$
$(0,5,2,0)$	$(3,1,1,1)$	$(3,1,3,0)$	$(2,4,0,1)$	$(2,4,2,0)$	$(1,7,1,0)$
$(0,10,0,0)$	$(5,0,1,1)$	$(5,0,3,0)$	$(4,3,0,1)$	$(4,3,2,0)$	$(3,6,1,0)$
$(2,9,0,0)$	$(6,2,0,1)$	$(6,2,2,0)$	$(5,5,1,0)$	$(4,8,0,0)$	$(8,1,0,1)$
$(8,1,2,0)$	$(7,4,1,0)$	$(6,7,0,0)$	$(10,0,0,1)$	$(10,0,2,0)$	$(9,3,1,0)$
$(8,6,0,0)$	$(11,2,1,0)$	$(10,5,0,0)$	$(13,1,1,0)$	$(12,4,0,0)$	$(15,0,1,0)$
$(14,3,0,0)$	$(16,2,0,0)$	$(18,1,0,0)$	$(20,0,0,0)$		

In this example, it is easy to see that $(0,0,0,2)$ is the combination that minimizes the number of coins used. This chapter lays the foundation for a general strategy to solve integer programs which will be explained in detail in the tutorial. The method uses toric ideals which is the main topic of this chapter.

Continuing the example, we record each combination as a monomial in the polynomial ring $\mathbf{k}[x_1, x_2, x_3, x_4]$. For instance, $(6,2,0,1)$ is recorded as $x_1^6 x_2^2 x_4$. Letting $\mathcal{A} = \{5, 10, 25, 50\}$, we define the $\mathcal{A}$-degree of a monomial $x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}$ to be $5m_1 + 10m_2 + 25m_3 + 50m_4$. Thus the 40 monomials gotten from the above vectors are precisely all the monomials in $\mathbf{k}[x_1, x_2, x_3, x_4]$ of $\mathcal{A}$-degree $100 = b$. In general, Problem 1 is asking whether the bth graded part of $\mathbf{k}[x_1, x_2, x_3, x_4]$ is nontrivial. Theoretically, the answer is "yes" if and only if b lies in the semigroup $\mathbb{N}\mathcal{A} = \{5x_1 + 10x_2 + 25x_3 + 50x_4 : x_1, x_2, x_3, x_4 \in \mathbb{N}\}$. How do we check this in practice? More ambitiously, one could ask to enumerate all monomials in $\mathbf{k}[x_1, x_2, x_3, x_4]$ of $\mathcal{A}$-degree b. We have listed the exponents of all such monomials for $b = 100$. How can we do this in practice?

We will see in the exercises that toric ideals and their Gröbner bases give algorithms to answer both Problem 1 and 2.

3.2 Toric ideals

Fix a subset $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d \backslash \{\mathbf{0}\}$. We assume that the matrix $A = [\mathbf{a}_1\, \mathbf{a}_2\, \ldots\, \mathbf{a}_n] \in \mathbb{Z}^{d \times n}$ has rank d. Consider the following semigroup homomorphism:

$$\pi : \mathbb{N}^n \to \mathbb{Z}^d, \quad \mathbf{u} = (u_1, \ldots, u_n) \mapsto \sum_{i=1}^{n} \mathbf{a}_i u_i = A\mathbf{u}.$$

Then $\pi(\mathbb{N}^n) = \{A\mathbf{u} \,:\, \mathbf{u} \in \mathbb{N}^n\} =: \mathbb{N}\mathcal{A}$ is called the *monoid (semigroup)* generated by $\mathcal{A}$. The semigroup ring of $\mathbb{N}^n$ is $\mathbf{k}[\mathbf{x}] = \mathbf{k}[x_1, \ldots, x_n]$, and that of $\mathbb{Z}^d$ is the *Laurent polynomial ring* $\mathbf{k}[\mathbf{t}^{\pm\mathbf{1}}] := \mathbf{k}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. The map π lifts to the ring homomorphism:

$$\hat{\pi} \,:\, \mathbf{k}[\mathbf{x}] \to \mathbf{k}[\mathbf{t}^{\pm 1}], \;\; x_j \mapsto \mathbf{t}^{\mathbf{a}_j} := t_1^{a_{1j}} t_2^{a_{2j}} \cdots t_d^{a_{dj}}.$$

Definition 3.2.1. The *toric ideal* of $\mathcal{A}$, denoted as $I_{\mathcal{A}}$, is the kernel of the map $\hat{\pi}$.

In this situation it is natural to grade the polynomial ring $\mathbf{k}[\mathbf{x}]$ by setting $\deg(x_i) = \mathbf{a}_i$ for $i = 1, \ldots, n$. Then the set of all degrees of polynomials in $\mathbf{k}[\mathbf{x}]$ is $\mathbb{N}\mathcal{A}$. A polynomial in $\mathbf{k}[\mathbf{x}]$ is $\mathcal{A}$-homogeneous if it is homogeneous under this multigrading.

Proposition 3.2.2. *1. The toric ideal $I_{\mathcal{A}}$ is a prime ideal in $\mathbf{k}[\mathbf{x}]$.*

2. *[Stu96, Lemma 4.1, Corollary 4.3] The ideal $I_{\mathcal{A}}$ is generated as a $\mathbf{k}$-vector space by the infinitely-many binomials $\{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \,:\, \pi(\mathbf{u}) = \pi(\mathbf{v}),\; \mathbf{u}, \mathbf{v} \in \mathbb{N}^n\}$, and hence $I_{\mathcal{A}} = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \,:\, \pi(\mathbf{u}) = \pi(\mathbf{v})\rangle$.*

3. *[Stu96, Lemma 4.2] The ring $\mathbf{k}[\mathbf{x}]/I_{\mathcal{A}}$ has Krull dimension d.*

4. *[Stu96, Corollary 4.4] For every term order $\succ$ the reduced Gröbner basis of $I_{\mathcal{A}}$ with respect to $\succ$ consists of a finite set of binomials of the form $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_{\mathcal{A}}$.*

Proof. 1. Since $\mathbf{k}[\mathbf{x}]/I_{\mathcal{A}} \cong \hat{\pi}(\mathbf{k}[\mathbf{x}]) = \mathbf{k}[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}]$, which is an integral domain, $I_{\mathcal{A}}$ is a prime ideal.

2. First note that a binomial $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_{\mathcal{A}}$ if and only if it is $\mathcal{A}$-homogeneous, so $\pi(\mathbf{u}) = \pi(\mathbf{v})$. It therefore suffices to show that each polynomial $f \in I_{\mathcal{A}}$ is a $\mathbf{k}$-linear combination of these binomials. Fix a term order $\succ$ on $\mathbf{k}[\mathbf{x}]$, and suppose f cannot be written as a $\mathbf{k}$-linear combination of $\mathcal{A}$-homogeneous binomials. Choose such an f for which $\mathrm{in}_{\succ}(f) = \mathbf{x}^{\mathbf{u}}$ is minimal with respect to $\succ$, which we call a "minimal criminal". Since $f \in I_{\mathcal{A}} = \ker(\hat{\pi})$, $f(\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}) = 0$. In particular, $\hat{\pi}(\mathbf{x}^{\mathbf{u}}) = \mathbf{t}^{\pi(\mathbf{u})}$ must cancel in this expansion. Hence there is a monomial $\mathbf{x}^{\mathbf{v}} \prec \mathbf{x}^{\mathbf{u}}$ in f such that $\pi(\mathbf{u}) = \pi(\mathbf{v})$. Then the polynomial $f' = f - \mathbf{x}^{\mathbf{u}} + \mathbf{x}^{\mathbf{v}}$ cannot be written as a $\mathbf{k}$-linear combination of binomials in $I_{\mathcal{A}}$. However since $\mathrm{in}_{\succ}(f') \prec \mathrm{in}_{\succ}(f)$, this contradicts the assumption. Minimal criminal arguments are a staple of Gröbner bases proofs.

3. Theorem A in Chapter 8 of [Eis94] states that if R is an affine domain over a field $\mathbf{k}$ then the Krull dimension of R is the transcendence degree of R over $\mathbf{k}$. Thus the Krull dimension of $\mathbf{k}[\mathbf{x}]/I_{\mathcal{A}} \cong \mathbf{k}[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}]$ is the maximum number of algebraically independent monomials $\mathbf{t}^{\mathbf{a}_i}$ which in turn is equal to the maximum number of linearly independent vectors in $\mathcal{A}$ which is d by assumption.

4. By the Hilbert basis theorem and part 2 there exists a finite list of binomials from the list $\{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \pi(\mathbf{u}) = \pi(\mathbf{v})\}$ that generate $I_{\mathcal{A}}$. We use this generating set as the input for Buchberger's algorithm. It can be checked that the S-pair of two binomials is again a binomial, and that the normal form of a homogeneous binomial with respect to a set of homogeneous binomials with marked leading terms is also a homogeneous binomial. Thus each reduced Gröbner basis of $I_{\mathcal{A}}$ computed from the binomial generating set mentioned above consists of a finite set of $\mathcal{A}$-homogeneous binomials.

$\square$

Example 3.2.3. For $\mathcal{A} = \{5, 10, 25, 50\}$, $I_{\mathcal{A}} = \langle x_3^2 - x_4, x_1x_2^2 - x_3, x_1^2 - x_2 \rangle$. This toric ideal has 12 distinct reduced Gröbner bases that one can compute using the software package CaTS [Jena]. Each reduced Gröbner basis is numbered as a vertex of the state polytope of $I_{\mathcal{A}}$. In each case, the first list of binomials give the facets of the Gröbner cone and the second list is the reduced Gröbner basis. CaTS uses $a, b, c, \ldots$ for variables in a ring. The following computation requires that the matrix A be stored in a file. See the CaTS homepage for various acceptable formats. The command needed to obtain the following output is `cats -p1 -i filename`.

```
Vtx: 0 (3 facets/4 binomials/degree 3)
{# c^2-d, # b^3-a*c, # a^2-b}
{c^2-d, b^3-a*c, a*b^2-c, a^2-b}
Vtx: 1 (3 facets/4 binomials/degree 3)
{# a^2-b, # b^3-a*c, # d-c^2}
{a^2-b, a*b^2-c, b^3-a*c, d-c^2}
Vtx: 2 (3 facets/3 binomials/degree 5)
{# a^5-c, # b-a^2, # d-c^2}
{a^5-c, b-a^2, d-c^2}
Vtx: 3 (3 facets/5 binomials/degree 5)
{# b^5-c^2, # a*c-b^3, # d-c^2}
{a^2-b, a*b^2-c, b^5-c^2, a*c-b^3, d-c^2}
```

```
Vtx: 4 (3 facets/5 binomials/degree 3)
{# a*b^2-c, # c^2-b^5, # d-b^5}
{a^2-b, a*b^2-c, a*c-b^3, c^2-b^5, d-b^5}
Vtx: 5 (3 facets/3 binomials/degree 2)
{# a^2-b, # c-a*b^2, # d-b^5}
{a^2-b, c-a*b^2, d-b^5}
Vtx: 6 (3 facets/3 binomials/degree 1)
{# b-a^2, # c-a^5, # d-a^10}
{b-a^2, c-a^5, d-a^10}
Vtx: 7 (3 facets/6 binomials/degree 5)
{# a*b^2-c, # a*c-b^3, # b^3*c-a*d}
{a^2-b, a*b^2-c, b^5-d, a*c-b^3, b^3*c-a*d, c^2-d}
Vtx: 8 (3 facets/3 binomials/degree 5)
{# a^2-b, # b^5-d, # c-a*b^2}
{a^2-b, b^5-d, c-a*b^2}
Vtx: 9 (3 facets/3 binomials/degree 10)
{# a^10-d, # b-a^2, # c-a^5}
{a^10-d, b-a^2, c-a^5}
Vtx: 10 (3 facets/6 binomials/degree 5)
{# b^5-d, # c^2-d, # a*d-b^3*c}
{a^2-b, a*b^2-c, b^5-d, a*c-b^3, c^2-d, a*d-b^3*c}
Vtx: 11 (3 facets/3 binomials/degree 5)
{# a^5-c, # b-a^2, # c^2-d}
{a^5-c, b-a^2, c^2-d}
```

3.3 Algorithms for toric ideals

Since $I_{\mathcal{A}}$ is a prime ideal that does not contain any monomials, if $\mathbf{x}^{\mathbf{w}}(\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}) \in I_{\mathcal{A}}$ then $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_{\mathcal{A}}$. Thus every minimal generating set and reduced Gröbner basis of $I_{\mathcal{A}}$ consists of $\mathcal{A}$-homogeneous binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ with $\mathrm{supp}(\mathbf{u}) \cap \mathrm{supp}(\mathbf{v}) = \emptyset$. We may record $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ as the vector $\mathbf{u} - \mathbf{v}$. Conversely, given any vector $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n$, write $\mathbf{p} = \mathbf{p}^+ - \mathbf{p}^-$ where $\mathbf{p}^+, \mathbf{p}^- \in \mathbb{N}^n$ are the unique vectors obtained as follows: $(\mathbf{p}^+)_i = p_i$ when $p_i > 0$ and $(\mathbf{p}^+)_i = 0$ otherwise while $(\mathbf{p}^-)_i = -p_i$ when $p_i < 0$ and $(\mathbf{p}^-)_i = 0$ otherwise. For example, if $\mathbf{p} = (3, -4, 0, 5, 1)$, then $\mathbf{p}^+ = (3, 0, 0, 5, 1)$ and $\mathbf{p}^- = (0, 4, 0, 0, 0)$. If $\mathbf{p} \in \mathbb{Z}^n$ such that $A\mathbf{p} = \mathbf{0}$, then the binomial $\mathbf{x}^{\mathbf{p}^+} - \mathbf{x}^{\mathbf{p}^-}$ is $\mathcal{A}$-homogeneous and lies in $I_{\mathcal{A}}$.

We now outline algorithms for computing reduced Gröbner bases of the toric ideal $I_\mathcal{A}$ starting with the configuration $\mathcal{A}$. We compute reduced Gröbner bases with respect to a weight vector $\mathbf{w}$ which, as we saw in Chapter 2, is more general than computing reduced Gröbner bases with respect to term orders.

The Conti-Traverso algorithm [CT91]

Input: A vector configuration $\mathcal{A} \subset \mathbb{Z}^d$ and a term order given by the weight vector $\mathbf{w}$.

Output: The reduced Gröbner basis of $I_\mathcal{A}$ with respect to $\mathbf{w}$.

1. Introduce $n + d + 1$ indeterminates $t_0, t_1, \ldots, t_d, x_1, \ldots, x_n$. Let $\succ$ be any elimination order such that $\{t_i\} \succ \{x_j\}$ and the x variables are ordered by $\mathbf{w}$.

2. Compute the reduced Gröbner basis $\mathcal{G}_\succ(J)$ of the ideal

$$J = \langle t_0 t_1 \cdots t_d - 1,\ x_j \mathbf{t}^{\mathbf{a}_j^-} - \mathbf{t}^{\mathbf{a}_j^+},\ j = 1, \ldots, n\rangle.$$

3. Output the set $\mathcal{G}_\succ(J) \cap \mathbf{k}[\mathbf{x}]$ which is the reduced Gröbner basis of $I_\mathcal{A}$ with respect to $\mathbf{w}$.

Proof. The correctness of this algorithm follows from Theorem 2 in Section 3.3 of [CLO97] on computing equations of the smallest variety containing a rationally parameterized set of points. Note that $I_\mathcal{A}$ is such a variety given by the rational parameterization $\hat{\pi}$. $\square$

Buchberger's algorithm is very sensitive to the number of variables used, and in that light the above algorithm is not ideal as it requires $d + 1$ extra variables beyond the variables $x_1, \ldots, x_n$ of $I_\mathcal{A}$. However, it has the advantage that a generating set for $I_\mathcal{A}$ is not needed to begin the algorithm. There are examples of toric ideals for which a minimal generating set is also a minimal universal Gröbner basis, and hence the problem of finding a minimal generating set is not an easy task. Several fast algorithms for computing a generating set for $I_\mathcal{A}$ that do not require additional variables are known. Many of these are implemented in CoCoA [COC] and 4ti2 [Hem] which currently have the fastest codes for computing toric ideals and their Gröbner bases. We describe one of these algorithms here. See [Stu96, Chapter 12] for more details.

The Hoşten-Sturmfels algorithm [Stu96, Algorithm 12.3]

Input: $\mathcal{A}$ and a term order $\mathbf{w}$.

Output: The reduced Gröbner basis of $I_\mathcal{A}$ with respect to $\mathbf{w}$.

1. Find a lattice basis $\mathcal{B}$ for the lattice $\ker_\mathbb{Z}(\mathcal{A}) := \{\mathbf{u} \in \mathbb{Z}^n\ :\ A\mathbf{u} = 0\}$.

2. (optional) Replace $\mathcal{B}$ by a *reduced* basis in the sense of Lenstra, Lenstra and Lovász [Sch86, Chapter 6.2]. Call it $\mathcal{B}$ as well.

3. Let $J_0 = \langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \,:\, \mathbf{u} \in \mathcal{B} \rangle$.
4. For $i = 1, 2, \ldots, n$: Compute $J_i := (J_{i-1} : x_i^\infty)$.
5. Compute the reduced Gröbner basis of $J_n = I_{\mathcal{A}}$ with respect to $\mathbf{w}$.

Proof. The proof of the correctness of this algorithm follows from Lemma 3.3.1 which was proved originally in [HS95]. $\square$

Lemma 3.3.1. *[Stu96, Lemma 12.2] A set $\mathcal{B}$ is a lattice basis for* $\ker_{\mathbb{Z}}(\mathcal{A})$ *if and only if $(J_{\mathcal{B}} : (x_1 \cdots x_n)^\infty) = I_{\mathcal{A}}$, where $J_{\mathcal{B}} := \langle \mathbf{x}^{\mathbf{b}^+} - \mathbf{x}^{\mathbf{b}^-} \,:\, \mathbf{b} \in \mathcal{B} \rangle$.*

Proof. Clearly $J_{\mathcal{B}} \subseteq I_{\mathcal{A}}$. Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_r\}$ is a basis for $\ker_{\mathbb{Z}}(\mathcal{A})$. Let $\mathbf{u} \in \ker_{\mathbb{Z}}(\mathcal{A})$. Then $\mathbf{u} = \sum_{i=1}^r \lambda_i \mathbf{b}_i$ for some $\lambda_i \in \mathbb{Z}$. This implies that

$$\frac{\mathbf{x}^{\mathbf{u}^+}}{\mathbf{x}^{\mathbf{u}^-}} - 1 = \prod_{i=1}^r \Big(\frac{\mathbf{x}^{\mathbf{b}_i^+}}{\mathbf{x}^{\mathbf{b}_i^-}} \Big)^{\lambda_i} - 1.$$

Clearing denominators we get that

$$\prod_{i=1}^r (\mathbf{x}^{\mathbf{b}_i^-})^{\lambda_i} (\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}) = \mathbf{x}^{\mathbf{u}^-} \Big(\prod_{i=1}^r (\mathbf{x}^{\mathbf{b}_i^+})^{\lambda_i} - \prod_{i=1}^r (\mathbf{x}^{\mathbf{b}_i^-})^{\lambda_i} \Big).$$

If we show that the right hand side lies in $J_{\mathcal{B}}$ then we will get that a monomial multiple of $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ lies in $J_{\mathcal{B}}$ which will finish the "only-if" direction of the argument. Note that if $\mathbf{x}^{\mathbf{p}^+} - \mathbf{x}^{\mathbf{p}^-}$ and $\mathbf{x}^{\mathbf{q}^+} - \mathbf{x}^{\mathbf{q}^-}$ lie in an ideal then $\mathbf{x}^{\mathbf{q}^+}(\mathbf{x}^{\mathbf{p}^+} - \mathbf{x}^{\mathbf{p}^-}) + \mathbf{x}^{\mathbf{p}^-}(\mathbf{x}^{\mathbf{q}^+} - \mathbf{x}^{\mathbf{q}^-}) = \mathbf{x}^{\mathbf{q}^+}\mathbf{x}^{\mathbf{p}^+} - \mathbf{x}^{\mathbf{p}^-}\mathbf{x}^{\mathbf{q}^-}$ also lies in the ideal. Applying this argument to $\{\mathbf{x}^{\mathbf{b}_i^+} - \mathbf{x}^{\mathbf{b}_i^-} \,:\, i = 1, \ldots, r\} \subset J_{\mathcal{B}}$ we get that $\prod_{i=1}^r (\mathbf{x}^{\mathbf{b}_i^+})^{\lambda_i} - \prod_{i=1}^r (\mathbf{x}^{\mathbf{b}_i^-})^{\lambda_i}$ lies in $J_{\mathcal{B}}$.

To argue the "if" direction, we have to show that if $\mathbf{u} \in \ker_{\mathbb{Z}}(\mathcal{A})$, then $\mathbf{u} = \sum_{i=1}^r \lambda_i \mathbf{b}_i$ for some integers λ_i. Now $\mathbf{u} \in \ker_{\mathbb{Z}}(\mathcal{A})$ implies that $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_{\mathcal{A}} = (J_{\mathcal{B}} : (x_1 \cdots x_n)^\infty)$. Hence $\mathbf{x}^{\mathbf{a}}(\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}) \in J_{\mathcal{B}}$. Hence, we can connect the lattice points $\mathbf{a} + \mathbf{u}^+$ and $\mathbf{a} + \mathbf{u}^-$ by a sequence of vectors in $\mathcal{B}$ which shows that $\mathbf{u} = \mathbf{a} + \mathbf{u}^+ - \mathbf{a} + \mathbf{u}^-$ is an integer combination of vectors in $\mathcal{B}$.

$\square$

In Tutorial 1 we saw how to compute $(I : x_i^\infty)$ for a homogeneous ideal I and hence we can compute $(J_{\mathcal{B}} : (x_1 \cdots x_n)^\infty)$. Lattice bases can be computed using a computer algebra package such as Maple or Macaulay 2.

3.4 Universal Gröbner bases for toric ideals

In the rest of this chapter we construct a universal Gröbner basis for $I_{\mathcal{A}}$ called the Graver basis of $I_{\mathcal{A}}$, and derive a bound on the maximum degree of elements in this universal Gröbner basis. This basis plays a fundamental role in the study and applications of toric ideals.

Recall from Chapter 2 that a particular choice of universal Gröbner basis of $I_{\mathcal{A}}$ is the union (up to sign) of all the reduced Gröbner bases of $I_{\mathcal{A}}$. We denote this set by $\mathcal{U}_{\mathcal{A}}$.

Example 3.4.1. Taking the union over all reduced Gröbner bases of $I_{\{5,10,25,50\}}$ from Example 3.2.3, we get $\mathcal{U}_{\mathcal{A}} = \{x_3^2 - x_4, x_2^3 - x_1x_3, x_1x_2^2 - x_3, x_1^2 - x_2, x_1^5 - x_3, x_2^5 - x_3^2, x_4 - x_2^5, x_4 - x_1^{10}, x_2^3x_3 - x_1x_4\}$. This is computed easily from the CaTS output from earlier.

From Chapter 2 and Proposition 3.2.2(4) we know that $\mathcal{U}_{\mathcal{A}}$ is finite and consists of $\mathcal{A}$-homogeneous binomials. A binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_{\mathcal{A}}$ is *primitive* if there is no binomial $\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-} \in I_{\mathcal{A}}$ such that $\mathbf{x}^{\mathbf{v}^+}$ properly divides $\mathbf{x}^{\mathbf{u}^+}$ and $\mathbf{x}^{\mathbf{v}^-}$ properly divides $\mathbf{x}^{\mathbf{u}^-}$. The following is an easy fact.

Lemma 3.4.2. *(Lemma 4.6 [Stu96]) Every binomial in $\mathcal{U}_{\mathcal{A}}$ is primitive.*

Definition 3.4.3. The *Graver basis* of $I_{\mathcal{A}}$, denoted by $Gr_{\mathcal{A}}$, is the set of all primitive binomials in $I_{\mathcal{A}}$.

There are many examples of $\mathcal{A}$ for which $Gr_{\mathcal{A}}$ is a proper superset of $\mathcal{U}_{\mathcal{A}}$. This is the case for our running example where $Gr_{\mathcal{A}} = \{x_1^2 - x_2, x_1x_2^2 - x_3, x_1^3x_2 - x_3, x_1^5 - x_3, x_2^3 - x_1x_3, x_3^2 - x_4, x_1x_2^2x_3 - x_4, x_1^3x_2x_3 - x_4, x_1^5x_3 - x_4, x_2^5 - x_3^2, x_2^5 - x_4, x_1^2x_2^4 - x_4, x_1^4x_2^3 - x_4, x_1^6x_2^2 - x_4, x_1^8x_2 - x_4, x_1^{10} - x_4, x_2^3x_3 - x_1x_4\}$. Graver bases can be computed using the software package 4ti2.

For a collection of vectors $\mathcal{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subset \mathbb{Z}^d$, recall that $\mathrm{pos}(\mathcal{P})$ is the cone generated by $\mathcal{P}$.

Definition 3.4.4. Let D denote the semigroup $\mathrm{pos}(\mathcal{P}) \cap \mathbb{Z}^d$ for $\mathcal{P} \subset \mathbb{Z}^d$. A *Hilbert basis* of $\mathrm{pos}(\mathcal{P})$ is a finite generating set for the semigroup D.

Example 3.4.5. The Hilbert basis of $\mathrm{pos}(\{(1,0),(1,3),(1,4),(1,6)\})$ is

$$\{(1,0),(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\}.$$

The following lemma shows that Definition 3.4.4 makes sense.

Lemma 3.4.6. *Every rational polyhedral cone has a Hilbert basis.*

Proof. Assume that $\mathbf{p}_1, \dots, \mathbf{p}_k$ are primitive vectors generating the cone $\mathrm{pos}(\mathcal{P})$. Consider the finite set $D' = \{\mathbf{x} \in \mathbb{Z}^d : \mathbf{x} = \sum_i \lambda_i \mathbf{p}_i \text{ with } 0 \leq \lambda_i < 1\}$. Note that if $\mathbf{y} \in D$, then $\mathbf{y} = \sum_j \mu_j \mathbf{p}_j$ for some $\mu_j \in \mathbb{R}_{\geq 0}$. Write this as $\mathbf{y} = \sum_i \lfloor \mu_i \rfloor \mathbf{p}_i + \sum_i (\mu_i - \lfloor \mu_i \rfloor)\mathbf{p}_i$. The second sum belongs to D', so, since $\mathbf{y}$ was arbitrary, this shows that $\{\mathbf{p}_1, \dots, \mathbf{p}_k\} \cup D'$ generate D. We may delete redundant elements in this set to minimalize the generating set and obtain a Hilbert basis for $\mathrm{pos}(\mathcal{P})$. □

When $\mathrm{pos}(\mathcal{P})$ is *pointed*, which means that there is no line in the cone, then $\mathrm{pos}(\mathcal{P})$ has a unique minimal Hilbert basis.

We use Hilbert bases to give an explicit (theoretical) construction of $Gr_{\mathcal{A}}$. For each sign pattern $\sigma \in \{+, -\}^n$, consider the pointed polyhedral cone $C_\sigma := \ker(\mathcal{A}) \cap \mathbb{R}^n_\sigma$, where $\ker(\mathcal{A})$ is the vector space $\{\mathbf{u} \in \mathbb{R}^n : A\mathbf{u} = 0\}$. Here $\mathbb{R}^n_\sigma$ is the orthant of $\mathbb{R}^n$ with sign pattern σ. Let $\mathcal{H}_\sigma$ be the Hilbert basis of C_σ which is a pointed cone. The Graver basis $Gr_{\mathcal{A}} = \{\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} : \mathbf{u} \in \bigcup_\sigma \mathcal{H}_\sigma \backslash \{\mathbf{0}\}\}$.

Lemma 3.4.7. *The above construction yields the Graver basis $Gr_{\mathcal{A}}$. By construction it is finite.*

Proof. If $\mathbf{u} \in C_\sigma$, then $\mathbf{u}$ is a $\mathbb{N}$-linear combination of elements of $\mathcal{H}_\sigma$ all of which are sign-compatible to $\mathbf{u}$. If $\mathbf{u} \notin \mathcal{H}_\sigma$, then every element $\mathbf{v} \in \mathcal{H}_\sigma$ involved in the combination has the property that $\mathbf{v}^+ \leq \mathbf{u}^+$ and $\mathbf{v}^- \leq \mathbf{u}^-$ which implies that $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ is not primitive. Thus if $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_{\mathcal{A}}$ is primitive then $\mathbf{u} \in \cup_\sigma \mathcal{H}_\sigma \backslash \{\mathbf{0}\}$. On the other hand, if $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ is such that $\mathbf{u} \in \mathcal{H}_\sigma$, then it is primitive by construction. □

Algorithm 7.2 in [Stu96] is a more practical method for constructing $Gr_{\mathcal{A}}$. See Tutorial 4 for an example.

Our next goal is to provide bounds for the maximum degree of an element in the Graver basis. In order to do this, we isolate a special subset of the Graver basis known as the set of circuits of $I_{\mathcal{A}}$. A bound on the maximum degree of a circuit can be derived from linear algebra. This in turn yields a bound on the maximum degree of an element in $\mathcal{G}r_{\mathcal{A}}$.

Definition 3.4.8. A primitive binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_{\mathcal{A}}$ is called a *circuit* if its support is minimal among all binomials in $I_{\mathcal{A}}$. The set of circuits of $I_{\mathcal{A}}$ is denoted as $\mathcal{C}_{\mathcal{A}}$.

We refer to the set of vectors $\{\mathbf{u} : \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in \mathcal{C}_{\mathcal{A}}\}$ as the circuits of $\mathcal{A}$. This set will also be denoted as $\mathcal{C}_{\mathcal{A}}$. Recall that circuits of a matrix were also defined in Chapter 1 in the context of linear ideals. The same notion is being used here. We now state a few facts about circuits. See [Stu96, Lemmas 4.8, 4.9, 4.10 and 4.11] for details.

Lemma 3.4.9. *1. The circuits $\mathcal{C}_{\mathcal{A}}$ are precisely the generators of the pointed rational polyhedral cones C_σ in the construction of the Graver basis.*

2. *Every vector $\mathbf{v} \in \ker(\mathcal{A})$ can be written as a non-negative rational combination of $n-d$ circuits, each of which is sign-compatible to $\mathbf{v}$.*

3. *The support of a circuit has cardinality at most $d+1$.*

4. *Every circuit $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ of $I_{\mathcal{A}}$ appears in some reduced Gröbner basis of $I_{\mathcal{A}}$. Thus $\mathcal{C}_{\mathcal{A}} \subseteq \mathcal{U}_{\mathcal{A}} \subseteq Gr_{\mathcal{A}}$.*

5. *Let $D(\mathcal{A}) := \max\{|\det[\mathbf{a}_{i_1} \ldots \mathbf{a}_{i_d}]| \,:\, 1 \leq i_1 < i_2 < \cdots < i_d \leq n\}$. If $\mathbf{u} \in \mathcal{C}_{\mathcal{A}}$ then $\|\mathbf{u}\|_1 \leq (d+1)D(\mathcal{A})$.*

Proof. (Sketch of the main ideas.)

1. Recall that circuits of $\mathcal{A}$ are vectors in $\ker_{\mathbb{Z}}(\mathcal{A})$ of minimal support. Thus they lie at the intersection of as many coordinate hyperplanes as possible. The extreme rays of the cones C_σ are cut out by a maximal number of coordinate hyperplanes.

2. Caratheodory's Theorem for polyhedral cones says that every vector in a d-dimensional rational polyhedral cone is a non-negative rational combination of d generators of the cone [Zie95].

3. The circuits of $\mathcal{A}$ are the minimal dependencies among the columns of A. Recall that $\mathrm{rank}(A) = d$.

4. This fact follows from an argument very similar to the minimality argument in Proposition 4.3 (2) of Chapter 1.

5. By part 3 it suffices to show that $|u_i| \leq D(\mathcal{A})$ for each $i = 1, \ldots, n$. Let $\mathrm{supp}(\mathbf{u}) = \{i_1, \ldots, i_r\}$. Then the $d \times r$ matrix $(\mathbf{a}_{i_1} \ldots \mathbf{a}_{i_r})$ has rank $r-1$ since $\mathbf{u}$ is a minimal dependency on the columns of A. Choose columns $\mathbf{a}_{i_{r+1}}, \ldots, \mathbf{a}_{i_{d+1}}$ of A such that $B = (\mathbf{a}_{i_1} \ldots \mathbf{a}_{i_{d+1}})$ has rank d. Hence the kernel of B is one-dimensional, and by Cramer's rule is spanned by the vector
$$\sum_{j=1}^{d+1} (-1)^j \det(\mathbf{a}_{i_1} \ldots \mathbf{a}_{i_{j-1}} \mathbf{a}_{i_{j+1}} \ldots \mathbf{a}_{i_{d+1}}) \mathbf{e}_{i_j} \qquad (*).$$

Thus $\mathbf{u}$ is a rational multiple of $(*)$ (extended by zeros to be a vector in $\mathbb{R}^n$). However, since $\mathbf{u}$ is a circuit and $(*)$ is integral, $(*)$ is in fact an integer multiple of $\mathbf{u}$ which proves the claim.

□

Using Lemma 3.4.9 we obtain a bound on the maximum degree of a Graver basis element.

Theorem 3.4.10. *Let* $D(\mathcal{A}) := \max\{|\det[\mathbf{a}_{i_1} \ldots \mathbf{a}_{i_d}]| : 1 \leq i_1 < i_2 < \cdots < i_d \leq n\}$. *The total degree of any element of* $Gr_{\mathcal{A}}$ *is less than* $(d+1)(n-d)D(\mathcal{A})$.

Proof. Note that the ideal $I_{\mathcal{A}}$ need not be homogeneous in the total degree grading. The "total degree" of an element $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ is taken to be $\max\{\|\mathbf{u}^+\|_1, \|\mathbf{u}^-\|_1\}$.

If $\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-} \in Gr_{\mathcal{A}}$, then apply Lemma 3.4.9 (2) to find $n-d$ circuits sign-compatible to $\mathbf{v}$ and non-negative rational numbers $\lambda_1, \ldots, \lambda_{n-d}$ such that

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \cdots + \lambda_{n-d}\mathbf{u}_{n-d}.$$

The sign-compatibility implies that $\mathbf{v}^+ = \lambda_1 \mathbf{u}_1^+ + \cdots + \lambda_{n-d}\mathbf{u}_{n-d}^+$ and $\mathbf{v}^- = \lambda_1 \mathbf{u}_1^- + \cdots + \lambda_{n-d}\mathbf{u}_{n-d}^-$. Further, since $\mathbf{v}$ is primitive, $\lambda_i < 1$ for each i. Assuming that the total degree of $\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-}$ equals $\|\mathbf{v}^+\|_1$, we get from Lemma 3.4.9 (5) that

$$\|\mathbf{v}^+\|_1 \leq \sum_{j=1}^{n-d} \lambda_j \|\mathbf{u}_j{}^+\|_1 < (n-d)(d+1)D(\mathcal{A}).$$

□

It has been conjectured by Sturmfels that the factor $(n-d)$ in the bound proved above is unnecessary.

Conjecture 3.4.11. *The total degree of any element of* $Gr_{\mathcal{A}}$ *is less than or equal to* $(d+1)D(\mathcal{A})$.

Apart from intrinsic mathematical interest, tighter bounds on the degree of Graver basis elements have important consequences in integer programming. Several bounds in this field appear as functions of this maximum degree. The maximum degree of elements in a universal Gröbner basis of an ideal is a measure of the complexity of the ideal and the complexity of Buchberger's algorithm working on this ideal. The single-exponential upper bound on the degree of elements in $Gr_{\mathcal{A}}$ should be contrasted with the double-exponential lower bounds on the elements in a universal Gröbner basis of a class of binomial ideals found by Mayr and Meyer [MM82]. These ideals have been used in the literature to show that Gröbner bases calculations can be difficult from the view of computational complexity.

3.5 Tutorial 3

In Tutorial 1 the program Macaulay 2 was introduced. In this tutorial we will create our own functions in Macaulay 2 to implement the algorithms in Lecture 3. A function in Macaulay 2 has the syntax

> *function-name* = (*names for input(s) separated by commas*) ->
> (*computations to execute separated by semi-colons*)

Below are two examples. The first function `toBinomial` creates a binomial from an integer vector. The second function `HSAlg` encodes the Hoşten-Sturmfels algorithm described in Lecture 3. The package "LLL.m2" allows us to do the optional reduction of the basis in the sense of Lenstra, Lenstra and Lovász from step two of the Hoşten-Sturmfels algorithm.

```
load "LLL.m2" -- load a package for doing LLL reduction

toBinomial = (b,R) -> (
  -- take a vector b of the form {*,*,...,*}, and
  -- a ring R having length of b number of variables
  -- return binomial x^(b-) - x^(b+)
  pos := 1_R;
  neg := 1_R;
  scan(#b, i-> if b_i > 0 then pos = pos * R_i^(b_i)
       else if b_i < 0 then neg = neg * R_i^(-b_i));
  pos - neg);

HSAlg = (A,w) -> (
  -- take list of rows of A written {{*},{*},...,{*}}
  -- and take a weight vector w
  -- return a reduced GB for I_A with respect to w
  n := #(A_0);   -- NOTE: A is not a matrix
  R = QQ[x_1..x_n,Degrees=>transpose A,MonomialSize=>16,
         Weights=>w];
  B := transpose LLL syz matrix A;
  J := ideal apply(entries B, b -> toBinomial(b,R));
  scan(gens ring J, f -> J = saturate(J,f));
  gens gb J)
```

Notice that the prefix '`--`' is used to denote the start of a comment; the entire rest of the line is ignored by the program. A list in Macaulay 2 is denoted by curly braces and its elements are separated by commas. In the function `HSAlg` instead of using one of the predefined orders and the command `MonomialOrder=>`*name-of-order*, we use `Weights=>w` which tells Macaulay 2 to use a weight order determined by the vector `w`. The command `saturate(J,f)` computes $(J : f^\infty)$. The command `#L` returns the size of the list `L`. More help with commands is available by typing `help` *name-of-command* in Macaulay 2, or better, by using the html-based help either locally or at `http://www.math.uiuc.edu/Macaulay2/Manual`.

In one of the functions above, we used a monomial order defined by a weight vector. Another useful order is an elimination order. For an ideal $I \subset \mathbf{k}[\mathbf{t}, \mathbf{x}]$, generators for the ideal $I \cap \mathbf{k}[\mathbf{x}]$ are found by calculating a Gröbner basis of I with respect to an elimination order for the t_i variables and then throwing out any generators which contain t_i's. In Macaulay 2 including the setting `MonomialOrder => Eliminate n` in the definition of a ring will force the use of an elimination order eliminating the first n variables. Note that an elimination order need not be a total order, just one where the variables to be eliminated are bigger than those that are not to be eliminated. Macaulay 2 refines this partial order by using the graded reverse lexicographic order whenever necessary to attain a total order on the monomials.

1. (a) Working by hand, use the Conti-Traverso algorithm to find a reduced Gröbner basis $\mathcal{G}_{(3,2,1)}$ of $I_\mathcal{A}$ for $A = [1\ 2\ 3]$. Use grevlex as a tiebreaking term order.

   (b) Now, using Macaulay 2, implement the Conti-Traverso algorithm by replacing the question marks in the following function by your own code to complete the algorithm.

```
CTAlg = (A,w) -> (
  -- take a list of rows of matrix A of the form
  --   {{*},{*},...,{*}}
  --   and a weight vector w
  -- return a reduced GB of I_A
  n := #(A_0);
  d := #A;
  R := QQ[t_0..t_d,x_1..x_n, MonomialSize=>16,
        MonomialOrder=>    ???    ];
  J := ideal (    ???    );
  I := selectInSubring(1, gens gb J); -- select entries
    -- without vars from 1st ‘part’ of mon. order
  S :=   ???    ; -- the ring I should end up in
  gens gb substitute(I, S))
```

2. Recall that while defining the toric ideal, we defined the map $\pi : \mathbb{N}^n \to \mathbb{Z}^d$ by $\mathbf{u} \mapsto A\mathbf{u}$, and its image was $\mathbb{N}\mathcal{A}$. For any $\mathbf{b} \in \mathbb{N}\mathcal{A}$, we define the *fiber* of π over $\mathbf{b}$ to be the set $\pi^{-1}(\mathbf{b}) = \{\mathbf{u} \in \mathbb{N}^n \ : \ \pi(\mathbf{u}) = \mathbf{b}\}$. We also sometimes use the term *fiber* to refer to the set of monomials having the same $\mathcal{A}$-degree, that is, $\{\mathbf{x}^{\mathbf{u}} \ : \ \pi(\mathbf{u}) = \mathbf{b}\}$ which is also a basis for the vector space $\mathbf{k}[\mathbf{x}]_{\mathbf{b}}$.

   For any subset $H \subset \ker(\pi)$, we may define a graph on the fiber $\pi^{-1}(\mathbf{b})$ by taking the vertices to be the vectors in the fiber, and connecting two vertices $\mathbf{u}$ and $\mathbf{v}$ when $\mathbf{u} - \mathbf{v} \in H$. Furthermore, given a term order $\succ$ on $\mathbb{N}^n$, we can create a directed graph from the undirected one by orienting an edge from $\mathbf{u}$ to $\mathbf{v}$ when $\mathbf{u} \succ \mathbf{v}$.

   (a) Let $H \subset \ker(\pi)$. Prove that $\{\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \ : \ \mathbf{u} \in H\}$ is a generating set of $I_{\mathcal{A}}$ if and only if for every $\mathbf{b} \in \mathbb{N}\mathcal{A}$ the graph on the fiber $\pi^{-1}(\mathbf{b})$ defined by H is connected.

   (b) Let $H \subset \ker(\pi)$, and let $\succ$ be a term order on $\mathbb{N}^n$. Prove that $\mathcal{G} = \{\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \ : \ \mathbf{u} \in H\}$ is a Gröbner basis of $I_{\mathcal{A}}$ with respect to $\succ$ if and only if for every $\mathbf{b} \in \mathbb{N}\mathcal{A}$ the directed graph on the fiber $\pi^{-1}(\mathbf{b})$ defined by H and $\succ$ is connected as an undirected graph and has a unique sink (a vertex with only incoming edges) at the unique minimal vector in the fiber under the ordering by $\succ$. Note that a term order $\succ$ on $\mathbb{N}^n$ also defines a term order on monomials where $\mathbf{x}^{\mathbf{u}} \succ \mathbf{x}^{\mathbf{v}}$ if $\mathbf{u} \succ v$.

3. (a) Describe an algorithm using toric ideals to solve the integer program $\min\{\mathbf{c} \cdot \mathbf{x} \ : \ A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n, \mathbf{x} \geq 0\}$ where $\mathbf{c}$ is the cost (weight) vector and $\mathbf{b} \in \mathbb{N}\mathcal{A} = \{A\mathbf{u} \ : \ \mathbf{u} \in \mathbb{Z}^n, \mathbf{u} \geq 0\}$. That is, find a vector or set of vectors $\mathbf{x}$ that minimize $\mathbf{c} \cdot \mathbf{x}$ subject to the constraints above.

   (b) In order to do the calculations from your algorithm in part a using Macaulay 2 it may be helpful to know that you can reduce a monomial m modulo an ideal I using the command `m % I`.

   Use your algorithm to solve the integer program about currency from the introduction to Lecture 3.

4. The programs 4ti2 [Hem] and CoCoA [COC] both have specialized packages for toric computations. To use the program 4ti2, create a file *matrix-file* for each A matrix with the following lines in the file:

line 1: *number-of-rows-in-A number-of-columns-in-A*
remaining lines: *entries of row of A, each separated by spaces*

A Gröbner basis may be calculated by running `groebner` *matrix-file*. This process will return a file *matrix-file.gro* in the same format as the input. The output file can be formatted with the command `output`. To list the output as binomials, for example, type `output bin` *matrix-file.gro*. Caution: you must include the extension *.gro*, or else you will get binomials from the input matrix A. For further information, see the web site `http://www.4ti2.de`.

In CoCoA, `Toric(`*matrix*`)` returns the toric ideal corresponding to the matrix. CoCoA also has an integer programming package. Details on using CoCoA may be found at `http://cocoa.dima.unige.it`.

(a) Use one or both of the above programs to find a Gröbner basis for the toric ideal generated by the matrix

$$\begin{pmatrix} 3 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 2 & 3 \end{pmatrix}.$$

(b) For more practice, check your previous calculations using these programs.

5. (a) Show that the result in Exercise 2b can be used to enumerate all the elements of a fiber.

   (b) The program CaTS will enumerate all lattice points in a fiber using the command `cats_fiber`. The input file is a list of the columns of A in the form $\{(\text{col } 1)(\text{col2}) \cdots (\text{col n})\}$ followed by a vector of the fiber in the form $(d_1, d_2, \ldots, d_n)$. For more information see `http://www.soopadoopa.dk/anders/cats/cats.html`.

   Use CaTS to find all the lattice points in the fiber necessary to solve the currency problem in the introduction to Lecture 3.

6. [*From [CLO98]*, p. 359] Suppose a small local trucking firm has two customers, A and B, that generate shipments to the same location. Each shipment from A is a pallet weighing exactly 400 kilos and taking up 2 cubic meters of volume. Each pallet from B weighs 500 kilos and takes up 3 cubic meters of volume. The shipping firm uses small trucks that can carry any load up to 3700 kilos, and up to 20 cubic meters. B's

product is more perishable, though, and they are willing to pay a higher price for on-time delivery: \$15 per pallet verses \$11 per pallet from A. How many pallets from each of the two companies should be included in each truckload to maximize the revenues generated?

7. For each of the following matrices, calculate, by hand, a Hilbert basis of the cone spanned by the columns of A.

   (a) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 3 & 4 \end{pmatrix}$

8. As we saw in Exercise 4, the program 4ti2 can compute a generating set and Gröbner basis for $I_{\mathcal{A}}$. It can also calculate Graver bases by running `graver` *matrix-file*.

   (a) For each of the following matrices, use 4ti2 to calculate the Graver basis for the lattice ideal defined by A.

      i. $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$

      ii. $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 3 & 4 \end{pmatrix}$

   (b) For the above matrices, check, by hand or using Macaulay 2, that the conjectured bound on the total degree of any element of the Graver basis (Conjecture 3.4.11) holds, that is, the total degree is always less than $(d+1)D(A)$ where

$$D(A) = \max\{|\det[\mathbf{a}_{i_1} \cdots \mathbf{a}_{i_d}]| \ : \ 1 \leq i_1 \leq i_2 \leq \ldots \leq i_d \leq n\}.$$

3.6 Solutions to Tutorial 3

1. (a) A reduced Gröbner basis of $I_{\mathcal{A}}$ is $\{x_1^2 - x_2,\ x_1x_2 - x_3,\ x_2^2 - x_1x_3\}$.

   (b) Here is one way to fill in the gaps to create a program which implements the Conti-Traverso algorithm.

```
CTAlg = (A,w) -> (
  -- take a list of rows of matrix A of the form
  --   {{*},{*},...,{*}}
  --   and a weight vector w
  -- return a reduced GB of I_A
  n := # (A_0);
  d := #A;
  R := QQ[t_0..t_d,x_1..x_n, MonomialSize=>16,
        MonomialOrder=>Eliminate(d+1)];
  J := ideal (append(
        apply(n, j -> (
        firstmon = x_(j+1);
        secondmon = 1_R;
        scan(d, i-> (
            if A_i_j < 0
            then firstmon = firstmon*(t_(i+1))^(-A_i_j)
            else secondmon = secondmon*(t_(i+1))^(A_i_j)));
        firstmon - secondmon)),
        product(toList (t_0..t_d))-1));
  I := selectInSubring(1, gens gb J);
  S := QQ[x_1..x_n, Degrees => transpose A, Weights=>w];
  gens gb substitute(I, S))
```

2. (a) Assume $\{\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \ :\ \mathbf{u} \in H\}$ generates $I_{\mathcal{A}}$. Fix a vector $\mathbf{b}$. Then we want to show the graph on $\pi^{-1}(\mathbf{b})$ defined by H is connected. Consider any two vertices p, and q in the graph $\pi^{-1}(\mathbf{b})$. We want to show that $\mathbf{p} - \mathbf{q} \in H$. Proposition 3.2.2 gives us that $\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{q}} \in I_{\mathcal{A}}$, so we can write this binomial in terms of our generating set for $I_{\mathcal{A}}$ as $\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{q}} = \sum_{i=1}^{m} \mathbf{x}^{\mathbf{w}_i}(\mathbf{x}^{{\mathbf{u}_i}^+} - \mathbf{x}^{{\mathbf{u}_i}^-})$, where the vectors $\mathbf{u}_i$ are not necessarily unique. We proceed by induction. If $m = 1$ then up to relabeling $\mathbf{p} = \mathbf{w}_1 + {\mathbf{u}_1}^+$ and $\mathbf{q} = \mathbf{w}_1 + {\mathbf{u}_1}^-$. So $\mathbf{p} - \mathbf{q} = \mathbf{u} \in H$, and hence there is an edge between $\mathbf{p}$ and $\mathbf{q}$. If $m > 1$, then by relabeling, we may assume $\mathbf{x}^{\mathbf{p}} = \mathbf{x}^{\mathbf{w}_1}\mathbf{x}^{{\mathbf{u}_1}^+}$. Since $\mathbf{u} \in H$, this means that $\mathbf{p}$ and $(\mathbf{w}_1 + {\mathbf{u}_1}^-)$ are connected by an edge in the graph. So we have reduced the problem to showing that $(\mathbf{w}_1 + {\mathbf{u}_1}^-)$ and $\mathbf{q}$ are

connected. However, $\mathbf{x}^{\mathbf{w}_1}\mathbf{x}^{\mathbf{u}_1{}^-} - \mathbf{x}^{\mathbf{q}} = \sum_{i=2}^{m} \mathbf{x}^{\mathbf{w}_i}(\mathbf{x}^{\mathbf{u}_i{}^+} - \mathbf{x}^{\mathbf{u}_i{}^-})$ is a smaller sum, so by induction $\mathbf{w}_1 + \mathbf{u}_1{}^-$ and $\mathbf{q}$ are connected.

For the other direction, assume that H defines a connected graph on $\pi^{-1}(\mathbf{b})$ for every $\mathbf{b} \in \mathbb{N}\mathcal{A}$. Take any $\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{q}} \in I_{\mathcal{A}}$. Set $\mathbf{b} = \pi(\mathbf{p}) = \pi(\mathbf{q})$. By the hypothesis, the graph on $\pi^{-1}(\mathbf{b})$, the vertices $\mathbf{p}$ and $\mathbf{q}$ must have a path connecting them, say $\mathbf{p}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{q}$. So the vectors $\mathbf{p} - \mathbf{v}_1, \mathbf{v}_1 - \mathbf{v}_2, \ldots, \mathbf{v}_m - \mathbf{q}$ are elements of H. But then we can write $\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{q}} = (\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{v}_1}) + (\mathbf{x}^{\mathbf{v}_1} - \mathbf{x}^{\mathbf{v}_2}) + \cdots + (\mathbf{x}^{\mathbf{v}_m} - \mathbf{x}^{\mathbf{q}})$, hence H generates $I_{\mathcal{A}}$.

(b) Assume that $\mathcal{G}$ is a Gröbner basis of $I_{\mathcal{A}}$ with respect to some ordering $\succ$. We can think of this as an ordering on the monomials or the exponent vectors. By part a, each fiber graph is connected. So we want to show there is a unique sink when $\succ$ is used to orient the graph. Suppose there are two sinks $\mathbf{p}$ and $\mathbf{q}$. Since they are in the same fiber, the binomial $\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{q}} \in I_{\mathcal{A}}$ by Proposition 3.2.2. We may assume $\mathrm{in}(\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{q}}) = \mathbf{x}^{\mathbf{p}}$. Since $\mathcal{G}$ is a reduced Gröbner basis of $I_{\mathcal{A}}$, $\mathbf{x}^{\mathbf{p}} = \mathbf{x}^{\mathbf{w}}\,\mathrm{in}(g)$ for some $g = \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in \mathcal{G}$. Without loss of generality $\mathrm{in}(g) = \mathbf{x}^{\mathbf{u}^+}$ so $\mathbf{p} - (\mathbf{w} + \mathbf{u}^-) \in H$ and $\mathbf{p} \succ (\mathbf{w} + \mathbf{u}^-)$. Therefore there an edge from p to $(\mathbf{w} + \mathbf{u}^-)$, and hence $\mathbf{p}$ cannot be a sink.

For the other direction, assume that the directed graph on each fiber defined by H and $\succ$ has a unique sink at the minimal vector in that fiber. We want to show that $\mathcal{G}$ is a Gröbner basis of $I_{\mathcal{A}}$ with respect to $\succ$. By part a, the connectedness of each fiber graph implies that $\mathcal{G}$ generates $I_{\mathcal{A}}$, so $\langle \mathrm{in}(g) \ : \ g \in \mathcal{G}\rangle \subset \mathrm{in}(I_{\mathcal{A}})$. To verify equality, take a monomial $\mathbf{x}^{\mathbf{p}} \in \mathrm{in}(I_{\mathcal{A}})$. Either $\mathbf{p}$ is the unique sink of the graph on $\pi^{-1}(\pi(\mathbf{p}))$ or there is an edge leaving $\mathbf{p}$ and going to some other vertex $\mathbf{q}$. If it is the sink, then it is minimal in the term order and hence cannot have been the initial term of some polynomial in $I_{\mathcal{A}}$. If it is not the sink, then we have $\mathbf{p} - \mathbf{q} \in H$ and $\mathbf{x}^{\mathbf{p}} \succ \mathbf{x}^{\mathbf{q}}$. So there is some element of $g = \{\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in G$ such that $\mathbf{u}^+ - \mathbf{u}^- = \mathbf{p} - \mathbf{q}$, hence $\mathrm{in}(g)$ divides $\mathbf{x}^{\mathbf{p}}$. This proves that $\mathbf{x}^{\mathbf{p}} \in \langle\{\mathrm{in}(g) \ : \ g \in \mathcal{G}\}\rangle$.

3. (a) First calculate $I_{\mathcal{A}}$ in the polynomial ring using a term order defined by the weight vector $\mathbf{c}$. Using the grading defined by $\deg(x_i) = \mathbf{a}_i$, find some monomial of degree $\mathbf{b}$. Now reduce the monomial modulo $I_{\mathcal{A}}$ using the weight vector term order, and take the exponent vector to get the $\mathbf{x}$ which minimizes the integer program.

(b) For this problem $\mathbf{c} = (1, 1, 1, 1)$, $A = [5\ 10\ 25\ 50]$, and $b = 100$. To find $I_{\mathcal{A}}$ in Macaulay 2 using the Hoşten-Sturmfels algorithm type `I = ideal HSAlg({{5,10,25,50}},{1,1,1,1})`. A monomial of degree 100 in $I_{\mathcal{A}}$ is x_1^{20}. If you didn't know this, you could find such a monomial in Macaulay 2 by typing `basis(100, ring I)` and taking any entry of the resulting matrix. To reduce the monomial modulo $I_{\mathcal{A}}$ in Macaulay 2, type `x_1^(20)= % I`. The result is x_4^2, so the solution to the integer program is the vector $\mathbf{x} = (0, 0, 0, 2)$. In terms of coins, this solution vector means that the smallest number of coins adding up to a value of 100 is two coins each of value 50.

4. Using `4ti2`, the output of the command `output bin exercise.gro`, where *exercise* is the file containing the matrix and *exercise.gro* is the result of running `groebner exercise`, is

```
[
x[5]*x[6]-x[4]*x[8],
x[7]^2-x[6]*x[8],
x[3]*x[6]-x[2]*x[7],
x[4]^2-x[2]*x[6],
x[8]^2-x[7]*x[9],
x[2]*x[4]-x[1]*x[6],
x[2]*x[5]-x[1]*x[8],
x[3]*x[7]-x[2]*x[8],
x[3]*x[4]-x[1]*x[7],
x[2]^2-x[1]*x[4],
x[5]^2-x[3]*x[9],
x[5]*x[7]-x[4]*x[9],
x[3]*x[5]-x[1]*x[9],
x[4]*x[5]-x[2]*x[8],
x[7]*x[8]-x[6]*x[9],
x[3]*x[8]-x[2]*x[9],
x[3]^2-x[1]*x[5],
x[2]*x[3]*x[9]-x[1]*x[5]*x[8]
]
```

5. (a) Given a reduced Gröbner basis $\mathcal{G}_{\succ}$ of $I_{\mathcal{A}}$, Exercise 2b shows that the exponent vectors form connected directed graphs on the elements of each fiber. The idea is to start with some element of the fiber and then wander around the graph to find all the other elements of

the fiber. Computationally, start with a fiber element $\mathbf{u} \in \pi^{-1}(\mathbf{b})$. Reduce $\mathbf{x}^{\mathbf{u}}$ modulo $\mathcal{G}_{\succ}$ to get a monomial whose exponent is the unique minimal element of the fiber. Now the elements of $\mathcal{G}_{\succ}$ can be used to move backward from the unique sink to find all vertices of the graph.

One method for effectively finding all the vertices in a graph without using large amounts of memory to keep track of which vertices have already been visited is the reverse search algorithm by Avis and Fukuda [AF92]. This reverse search has been implemented in CaTS.

(b) Create a file containing the following line:

```
{(5)(10)(25)(50)} (20,0,0,0)
```

Then run the program `cats_fiber` on it to get the elements of the fiber. The output should be the same as the fiber elements listed in the beginning of Lecture 3.

6. Let a be the number of pallets of A's product shipped and b be the number of pallets of B's product shipped. We must have $a, b \geq 0$ and they both must be integers. We want to maximize $15b + 11a$ subject to the constraints $400a + 500b \leq 3700$ and $2a + 3b \leq 20$. Solving this integer program using the method of Exercise 4 or Exercise 5, you should find the revenues will be maximized with 4 pallets of each. Note that it is helpful to add "slack variables" in order to change the inequalities to equalities before solving.

7. To find the Hilbert basis for a set of vectors $\mathcal{P}$, we draw the cone pos$(\mathcal{P})$ and the lattice $\mathbb{Z}\mathcal{P}$. Then we find a minimal set of vectors whose $\mathbb{N}$-linear combinations give the vectors of pos$(\mathcal{P}) \cap \mathbb{Z}^2$.

(a) $\mathcal{P} = \{(1,0),(1,1),(1,3),(1,4)\}$ can be drawn as the black dots in Figure 3.1. The cone pos$(\mathcal{P})$ is the gray shaded region.

The intersection of the cone with $\mathbb{Z}^2$ is the semigroup of all integer lattice points within the cone. Therefore the Hilbert basis is $\{(1,0),(1,1),(1,2),(1,3),(1,4)\}$.

(b) $\mathcal{P} = \{(1,0),(2,1),(2,3),(3,4)\}$ can be drawn as the black dots in Figure 3.2. The cone pos$(\mathcal{P})$ is the gray shaded region.

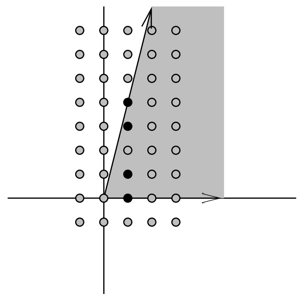


Figure 3.1: The cone $\mathrm{pos}(\mathcal{P})$ for $\mathcal{P}$ in 7a.

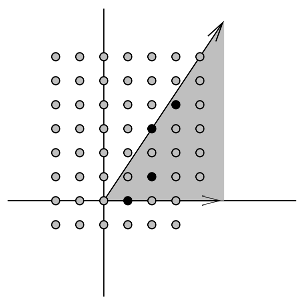


Figure 3.2: The cone $\mathrm{pos}(\mathcal{P})$ for $\mathcal{P}$ in 7b.

Again the intersection of the cone with $\mathbb{Z}^2$ is the semigroup of all integer lattice points within the cone, so the Hilbert basis is $\{(1,0),(1,1),(2,3)\}$.

8. (a) i. Run the command `graver` *filename* from 4ti2 on the matrix A and use `output bin` *filename.gra* to format the output file. The result is that the Graver basis is $\{x_2^3 - x_1^2x_3,\ x_2^4 - x_1^3x_4,\ x_2x_3 - x_1x_4,\ x_2^2x_4 - x_1x_3^2,\ x_2x_4^2 - x_3^3,\ x_3^4 - x_1x_4^3\}$.

   ii. Again, using 4ti2, we find that the Graver basis is $\{x_2^3 - x_1^4x_3,\ x_2^4 - x_1^5x_4,\ x_2x_3 - x_1x_4,\ x_2^2x_4 - x_1^3x_3^2,\ x_2x_4^2 - x_1^2x_3^3,\ x_1x_3^4 - x_4^3,\ x_2x_3^5 - x_4^4\}$.

   (b) i. First we calculate the total degrees of the elements of the Graver basis.

element	total degree
$x_2^3 - x_1^2x_3$	3
$x_2^4 - x_1^3x_4$	4
$x_2x_3 - x_1x_4$	2
$x_2^2x_4 - x_1x_3^2$	3
$x_2x_4^2 - x_3^3$	3
$x_3^4 - x_1x_4^3$	4

Now we calculate the determinants of the two by two minors of A. This can be done by hand, or using Macaulay 2 with the command `minors(2,matrix {{1,1,1,1},{0,1,3,4}})`. Either way, we find that $D(A) = 4$. So now we check that indeed $4 \leq (3)(4)$.

ii. First we calculate the total degrees of the elements of the Graver basis.

element	total degree
$x_2^3 - x_1^4x_3$	5
$x_2^4 - x_1^5x_4$	6
$x_2x_3 - x_1x_4$	2
$x_2^2x_4 - x_1^3x_3^2$	5
$x_2x_4^2 - x_1^2x_3^3$	5
$x_1x_3^4 - x_4^3$	5
$x_2x_3^5 - x_4^4$	6

Now we calculate the determinants of the two by two minors of A and find that $D(A) = 5$. So now we check that indeed $6 \leq (3)(5)$.

Chapter 4

Triangulations

The main point of this chapter is to discuss the relationship between initial ideals of toric ideals and triangulations of the vector configurations.

4.1 Background on triangulations

Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a collection of nonzero integral vectors in $\mathbb{Z}^d$. We use the polyhedral language defined in Chapter 2. We will always assume that $\mathrm{pos}(\mathbf{a}_i : 1 \leq i \leq n)$ is d-dimensional. We denote by $[n]$ the set $\{1, \ldots, n\}$. A *simplicial complex* is a collection $\{\sigma \subseteq [n]\}$ that is closed under taking subsets.

A *triangulation* of $\mathcal{A}$ is a simplicial complex Δ of subsets of $[n]$ such that the cones $\{\mathrm{pos}(\mathbf{a}_i : i \in \sigma) : \sigma \in \Delta\}$ form a simplicial fan whose support is $\mathrm{pos}(\mathbf{a}_i : 1 \leq i \leq n)$. A geometric realization of a triangulation has vertices at some of the points of $\mathcal{A}$.

Example 4.1.1. Let $\mathcal{A} \subset \mathbb{Z}^3$ be the collection $\{(1,0,0),(1,1,0),(1,2,0),(1,0,1),(1,1,1)\}$. These five vectors all have first coordinate one, so a cross-section of $\mathrm{pos}(\mathbf{a}_i : 1 \leq i \leq 5)$ is obtained by taking the convex hull of the points $\{(0,0),(1,0),(2,0),(0,1),(1,1)\}$. Two triangulations of $\mathcal{A}$ are shown in cross-section in Figure 4.1, where the points are labeled $1, \ldots, 5$ is the order listed here. The first consists of all subsets of sets in $\{\{1,2,4\},\{2,4,5\},\{2,3,5\}\}$, while the second is all subsets of sets in $\{\{1,2,4\},\{2,3,4\},\{3,4,5\}\}$.

We focus on a subclass of triangulations known as regular or coherent triangulations. These subclasses were first introduced by Gelfand, Kapranov, and Zelevinsky [GKZ94].

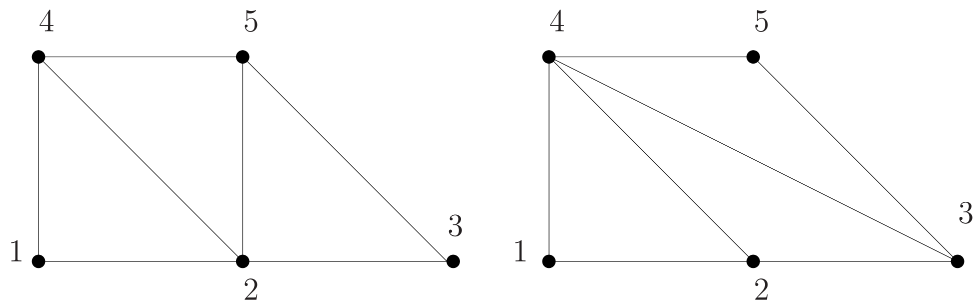


Figure 4.1: Triangulations of the configuration in Example 4.1.1

Definition 4.1.2. A triangulation Δ of $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is *regular* or *coherent* if there is a weight vector $\mathbf{w} \in \mathbb{R}^n$ for which the following condition holds: a subset $\{i_1, \dots, i_k\}$ is a face of Δ if and only if there exists a vector $\mathbf{c} \in \mathbb{R}^d$ with $\mathbf{a}_j \cdot \mathbf{c} = w_j$ for $j \in \{i_1, \dots, i_k\}$, and $\mathbf{a}_j \cdot \mathbf{c} < w_j$ otherwise. In this case we denote by $\Delta_{\mathbf{w}}$ the triangulation Δ.

An alternative description of $\Delta_{\mathbf{w}}$ is that it is the triangulation of $\mathcal{A}$ induced by the lower faces of the cone in $\mathbb{R}^{d+1}$ formed by taking the polyhedral cone generated by the vectors $(\mathbf{a}_j, w_j)$. A *lower face* of a cone in $\mathbb{R}^{d+1}$ is one visible from the point $-N\mathbf{e}_{d+1}$ for $N \gg 0$. To see the equivalence, note that $\operatorname{pos}((\mathbf{a}_j, w_j) : j \in \sigma)$ is a lower face of this cone if and only if there is a vector $\mathbf{c} \in \mathbb{R}^d$ with $(\mathbf{c}, -1) \cdot (\mathbf{a}_j, w_j) = 0$ for $j \in \sigma$ and $(\mathbf{c}, -1) \cdot (\mathbf{a}_j, w_j) < 0$ for $j \notin \sigma$. But this means $\mathbf{c} \cdot \mathbf{a}_j = w_j$ for $j \in \sigma$ and $\mathbf{c} \cdot \mathbf{a}_j < w_j$ for $j \notin \sigma$, so $\operatorname{pos}(\mathbf{a}_j, w_j)$ is a face of $\Delta_{\mathbf{w}}$.

Note that for some $\mathbf{w}$ the lower faces will not be simplicial cones, so we will not get a triangulation of $\mathcal{A}$, but rather a subdivision. The requirement that $\Delta_{\mathbf{w}}$ be a triangulation of $\mathcal{A}$ places some conditions on $\mathbf{w}$, which are satisfied for most $\mathbf{w} \in \mathbb{R}^n$.

Example 4.1.3. The two triangulations shown in Figure 4.1 are both regular. The triangulation on the left is $\Delta_{\mathbf{w}}$ for $\mathbf{w} = (1, 0, 1, 0, 0)$. To check this, note that a vector $\mathbf{c}$ corresponding to the face $\{1, 2, 4\}$ is $(1, -1, -1)$, a $\mathbf{c}$ for $\{2, 4, 5\}$ is $(0, 0, 0)$, and a $\mathbf{c}$ for $\{2, 3, 5\}$ is $(-1, 1, 0)$. The triangulation on the left is $\Delta_{\mathbf{w}}$ for $\mathbf{w} = (1, 0, 0, 0, 1)$.

A non-regular triangulation is shown in Figure 4.2. This is the smallest such example in terms of dimension d, number of vectors n, and codimension $n - d$. Checking that this triangulation is not regular involves writing down a list of inequalities on the components of a vector $\mathbf{w}$ that would satisfy the conditions of Definition 4.1.2, and observing that no solution to these inequalities exists.

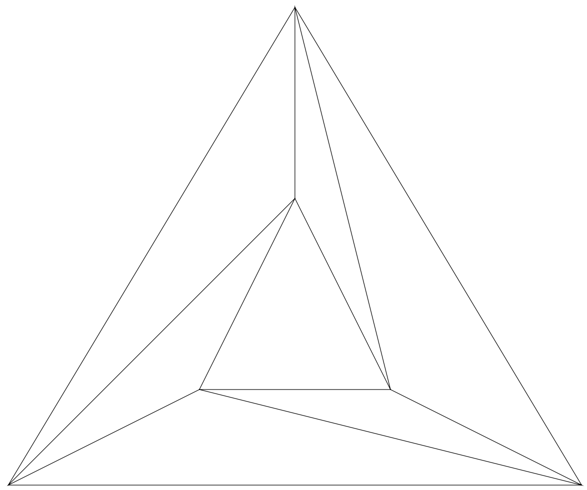


Figure 4.2: The smallest non-regular triangulation.

Simplicial complexes have an intimate connection to commutative algebra via the Stanley-Reisner ideal.

Definition 4.1.4. Let Δ be a simplicial complex of subsets of $[n]$. The *Stanley-Reisner ideal* $I(\Delta)$ is the squarefree monomial ideal

$$I(\Delta) = \Big\langle \prod_{i \in \sigma} x_i : \sigma \notin \Delta \Big\rangle = \bigcap_{\sigma \in \Delta} \langle x_i : i \notin \sigma \rangle.$$

Example 4.1.5. Let Δ be the first triangulation of Example 4.1.1. Then Δ consists of all subsets of the sets $\{\{1,2,4\},\{2,4,5\},\{2,3,5\}\}$, and the Stanley-Reisner ideal $I(\Delta) = \langle x_1x_3, x_1x_5, x_3x_4 \rangle$.

The map from simplicial complexes to squarefree monomial ideals given by taking the Stanley-Reisner ideal is a bijection. It associates a simplicial complex, the Stanley-Reisner complex $\Delta(I)$, to any squarefree monomial ideal I. Specifically, $\Delta(I) = \{\sigma \subseteq [n] : \prod_{i \in \sigma} x_i \notin I\}$.

4.2 The connection with initial ideals of $I_{\mathcal{A}}$

The main result of this lecture is a natural map from initial ideals of the toric ideal $I_{\mathcal{A}}$ to regular triangulations of the configuration $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. We denote by $\mathrm{supp}(\mathbf{u})$ the set $\{i : u_i \neq 0\}$ of a vector $\mathbf{u} \in \mathbb{R}^n$, which we call the *support* of $\mathbf{u}$. Recall the *radical* of an ideal J is the ideal $\mathrm{rad}(J) = \langle f : f^n \in J \text{ for some } n \in \mathbb{N}\rangle$. We say $\mathbf{w}$ is a *generic weight vector* if $\mathrm{in}_{\mathbf{w}}(I_{\mathcal{A}})$ is a monomial ideal.

Theorem 4.2.1. *Let A be a $d \times n$ integer-valued matrix with $\ker(A) \cap \mathbb{N}^n = \{0\}$, and let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be the columns of A. Let $J = \mathrm{in}_{\mathbf{w}}(I_{\mathcal{A}})$ for some generic weight vector $\mathbf{w}$. Then $\mathrm{rad}(J)$ is the Stanley-Reisner ideal of the simplicial complex $\Delta_{\mathbf{w}}$.*

Proof. Let $F \subseteq [n]$. Then F is a face of the simplicial complex associated to $\Delta_{\mathbf{w}}$ if and only if there is a $\mathbf{c} \in \mathbb{R}^d$ with $\mathbf{a}_i \cdot \mathbf{c} \leq \mathbf{w}_i$ for $1 \leq i \leq n$, with equality exactly when $i \in F$. Let P be the polyhedron $\{\mathbf{c} : \mathbf{a}_i \cdot \mathbf{c} \leq \mathbf{w}_i\}$. Candidates for F thus correspond to faces of P. Choose a cost vector $\mathbf{b} \in \mathbb{R}^d$ that is maximized at the face of P corresponding to F. Then we can find an optimal solution $\mathbf{c}$ to the linear program

$$\text{maximize } \mathbf{c} \cdot \mathbf{b} \text{ subject to } \mathbf{c} \in \mathbb{R}^d,\ \mathbf{a}_i \cdot \mathbf{c} \leq \mathbf{w}_i \text{ for } 1 \leq i \leq n,$$

with $\mathbf{a}_i \cdot \mathbf{c} = \mathbf{w}_i$ exactly when $i \in F$.

Now linear programming duality and complementary slackness imply that this $\mathbf{c}$ exists if and only there is an optimal solution $\mathbf{u} \in \mathbb{R}^n$ to the dual program

$$\text{minimize } \mathbf{u} \cdot \mathbf{w} \text{ subject to } \mathbf{u} \in \mathbb{R}^n,\ \mathbf{u} \geq \mathbf{0},\ \sum_{i=1}^{n} u_i \mathbf{a}_i = \mathbf{b}$$

with $\mathrm{supp}(\mathbf{u}) = F$. Details and definitions of linear programming duality and complementary slackness may be found in any text on linear programming, such as in Corollary 7.1g and Section 7.9 of [Sch86]. In general the solution $\mathbf{u} \in \mathbb{R}^n$ to the dual program will have rational entries (since the vectors $\mathbf{a}_i$ are integral), but we can replace $\mathbf{b}$ by a suitably large multiple to ensure that $\mathbf{u}$ lies in $\mathbb{N}^n$. But this means that the monomial $\mathbf{x}^{\mathbf{u}}$ has $\mathrm{supp}(\mathbf{u}) = F$, and $\mathbf{x}^{\mathbf{u}} \notin J$, since the solution to this integer program is the unique standard monomial of J of degree $\mathbf{b}$ (see Exercise 3 of Tutorial 3). In fact, no power of $\mathbf{x}^{\mathbf{u}}$ lies in J, since if $\mathbf{u}$ is an integral solution to this dual program then $l\mathbf{u}$ is a solution to the dual program with $\mathbf{b}$ replaced by $l\mathbf{b}$. This condition on $\mathbf{x}^{\mathbf{u}}$ is exactly the condition that $\mathbf{x}^{\mathbf{u}} \notin \mathrm{rad}(J)$. Thus any monomial $\mathbf{x}^{\mathbf{v}}$ in S with $\mathrm{supp}(\mathbf{v}) = F$ does not lie in $\mathrm{rad}(J)$. Each of these implications is reversible, so we conclude that $\mathrm{rad}(J)$ is the Stanley-Reisner ideal of the simplicial complex $\Delta_{\mathbf{w}}$. $\square$

Example 4.2.2. Let $\mathcal{A} = \{(1,0,0), (1,1,0), (1,2,0), (1,0,1), (1,1,1), (1,0,2)\}$. Then $I_{\mathcal{A}} = \langle cf - e^2, de - bf, af - d^2, ae - bd, cd - be, ac - b^2\rangle \subseteq \mathbf{k}[a,b,c,d,e,f]$.

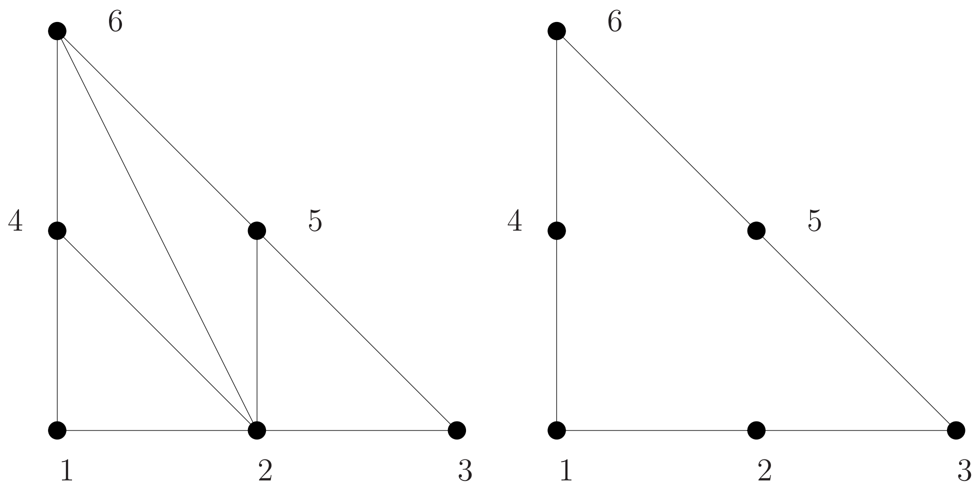


Figure 4.3: The triangulations of Example 4.2.2

For $\mathbf{w} = (10, 1, 10, 5, 1, 1)$, we have $\mathrm{in}_{\mathbf{w}}(I_{\mathcal{A}}) = \langle cf, de, af, ae, cd, ac\rangle$. This is already a radical ideal, and it is the Stanley-Reisner ideal of the triangulation shown in cross-section on the left in Figure 4.3.

When $\mathbf{w} = (1, 10, 1, 10, 10, 1)$, $\mathrm{in}_{\mathbf{w}}(I_{\mathcal{A}}) = \langle e^2, de, be, d^2, bd, b^2\rangle$. The radical of this ideal is $\langle b, d, e\rangle$. This is the Stanley-Reisner ideal of the second triangulation in Figure 4.3.

We leave it to the reader to check that each of the triangulations in Figure 4.3 is $\Delta_{\mathbf{w}}$ for the corresponding $\mathbf{w}$.

4.3 The secondary fan and the Gröbner fan

In the same way that initial ideals of an ideal correspond to vertices of the state polytope, regular triangulations of a vector configuration correspond to vertices of a polytope known as the secondary polytope. Its normal fan is called the secondary fan of $\mathcal{A}$.

Definition 4.3.1. Let $\mathcal{A}$ be a collection of integral vectors in $\mathbb{Z}^d$. The *secondary fan* of $\mathcal{A}$ is the polyhedral fan $\mathcal{S}_{\mathcal{A}}$ whose top-dimensional open cones consist of all $\mathbf{w}'$ for which $\Delta_{\mathbf{w}'} = \Delta_{\mathbf{w}}$, as $\Delta_{\mathbf{w}}$ varies over the finitely-many regular triangulations of $\mathcal{A}$.

Remark 4.3.2. In giving Definition 4.3.1 we are omitting a proof that the secondary fan is actually a polyhedral fan. In principle a proof can be given

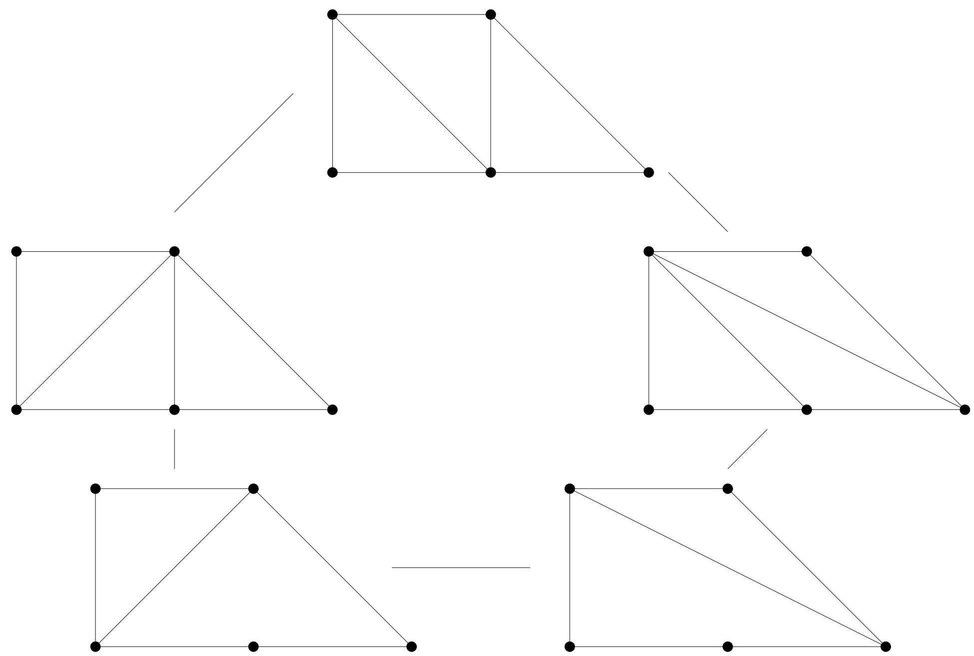


Figure 4.4: The secondary polytope of the configuration of Example 4.1.1.

like the one given for the Gröbner fan in Chapter 2, where we define sets of weight vectors for each triangulation, check that their closures are polyhedral cones, and finally check that these cones fit together to form a polyhedral fan.

For example, to check that the closure of these sets are at least convex, note that if $\Delta_{\mathbf{w}} = \Delta_{\mathbf{w}'}$, and $\mathbf{c} \cdot \mathbf{a}_i \leq \mathbf{w}_i$, and $\mathbf{c}' \cdot \mathbf{a}_i \leq \mathbf{w}'_i$ with equality for the same values of i, then $(\lambda\mathbf{c} + (1-\lambda)\mathbf{c}') \cdot \mathbf{a}_i \leq (\lambda\mathbf{w} + (1-\lambda)\mathbf{w}')_i$, for any $0 \leq \lambda \leq 1$, with equality for the same values of i.

Example 4.3.3. For the configuration in Example 4.1.1 the secondary fan has five cones. The corresponding secondary polytope (a pentagon) appears in Figure 4.4, with each vertex replaced by the corresponding triangulation.

Note that although the secondary fan lives in $\mathbb{R}^5$ we have drawn a two-dimensional polytope. This is because the secondary fan has a three dimensional lineality space. In general, if $\mathcal{A}$ is a configuration of n vectors in $\mathbb{Z}^d$ not lying in any hyperplane, then lineality space of the secondary fan is the row space of the corresponding matrix A with columns the vectors of $\mathcal{A}$, so the secondary polytope of $\mathcal{A}$ will be $(n-d)$-dimensional. Our example consists of five vectors in $\mathbb{R}^3$ (which we draw as five points in $\mathbb{R}^2$), so we have a two-dimensional secondary polytope.

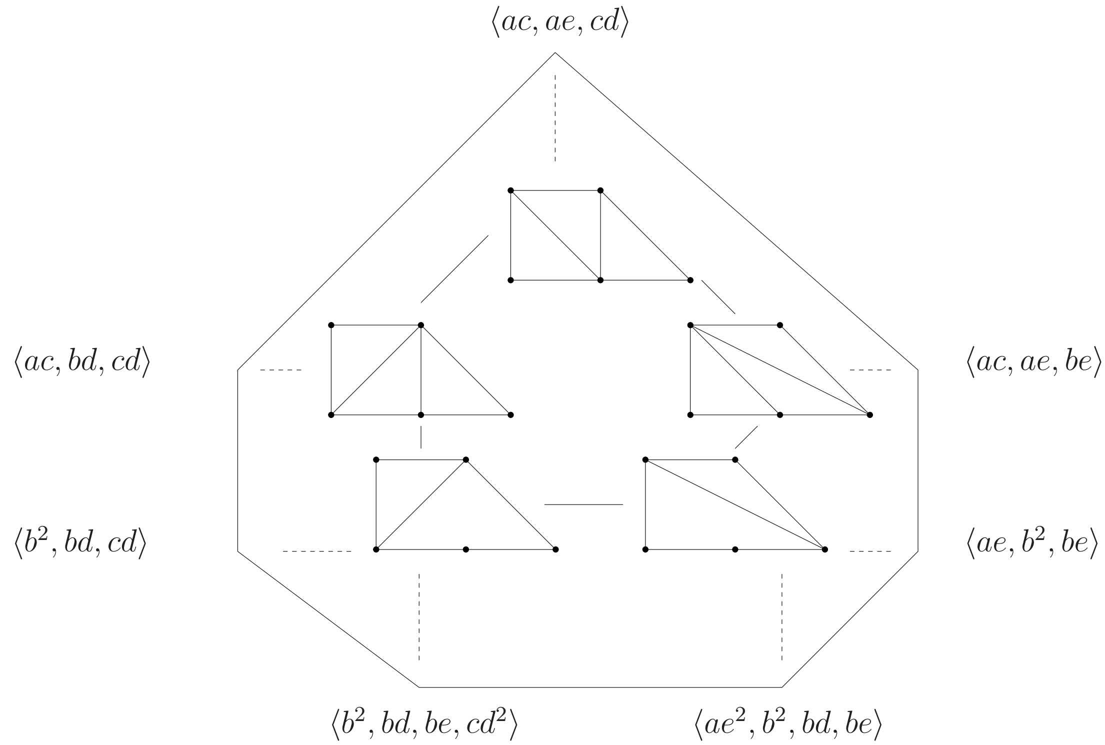


Figure 4.5: The relation between the state and secondary polytope of the configuration of Example 4.1.1.

A polyhedral fan $\mathcal{F}'$ *refines* a polyhedral fan $\mathcal{F}$ if every cone of $\mathcal{F}'$ is contained inside some cone of $\mathcal{F}$.

Proposition 4.3.4. *The Gröbner fan of $I_{\mathcal{A}}$ refines the secondary fan of $\mathcal{A}$.*

Proof. If $\mathrm{in}_{\mathbf{w}}(I_{\mathcal{A}}) = \mathrm{in}_{\mathbf{w}'}(I_{\mathcal{A}})$, then Theorem 4.2.1 implies that $\Delta_{\mathbf{w}} = \Delta_{\mathbf{w}'}$. $\square$

Example 4.3.5. Let $\mathcal{A}$ be the configuration of Example 4.1.1. The relationship between the state and secondary polytopes of this configuration is shown in Figure 4.5. Here the outer heptagon is the state polytope, with the corresponding initial ideal of $I_{\mathcal{A}}$ written next to the vertex, and the inner pentagon is the secondary polytope. A dotted line connecting an inner vertex to an outer vertex means that the corresponding cone in the secondary fan contains the corresponding cone in the Gröbner fan.

In general this refinement is proper. However there are some special cases where a cone in the secondary fan is not subdivided in the Gröbner cone.

Definition 4.3.6. A triangulation Δ of a collection $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of integral vectors is *unimodular* if every simplex $\sigma \in \Delta$ satisfies $\mathbb{N}\{\mathbf{a}_i : i \in \sigma\} = \mathrm{pos}(\mathbf{a}_i : i \in \sigma) \cap \mathbb{Z}\{\mathbf{a}_i : 1 \leq i \leq n\}$.

The configuration $\mathcal{A}$ is called *unimodular* if *every* triangulation of $\mathcal{A}$ is unimodular.

Proposition 4.3.7. *If Δ is a regular unimodular triangulation of $\mathcal{A}$, then the cone of the secondary fan corresponding to Δ is not subdivided in the Gröbner fan.*

Proof. Let J be an initial ideal of $I_{\mathcal{A}}$ corresponding to Δ. It suffices to show that J is squarefree, so x_i^2 does not divide any generator for $1 \leq i \leq n$. This suffices because then J is the Stanley-Reisner ideal of a triangulation Δ of $\mathcal{A}$. Since every initial ideal of $I_{\mathcal{A}}$ corresponding to Δ must be contained in the Stanley-Reisner ideal of Δ, and we cannot have proper containment of initial ideals by Corollary 2.2.3, this proves the proposition.

We now show that J is squarefree. Let $\mathbf{x}^{\mathbf{u}}$ be a monomial not in J. Let $\mathbf{b} = \sum_i u_i \mathbf{a}_i$. Let $\sigma \in \Delta$ be a top-dimensional simplex with $\mathbf{b} \in \mathrm{pos}(\mathbf{a}_i : i \in \sigma)$, so $\mathbf{b} = \sum_{i \in \sigma} \lambda_i \mathbf{a}_i$ for some $\lambda_i \geq 0$. The fact that Δ is a unimodular triangulation of $\mathcal{A}$ means that since $\mathbf{b} \in \mathbb{Z}\mathcal{A}$ we have $\lambda_i \in \mathbb{N}$.

By construction we have $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\lambda} \in I_{\mathcal{A}}$. We cannot have $\mathbf{x}^{\lambda} \in J$, since $\mathrm{supp}(\lambda) \subseteq \sigma \in \Delta$, and by assumption $\mathbf{x}^{\mathbf{u}} \notin J$. Since J is an initial ideal of $I_{\mathcal{A}}$ the only way this is possible is if $\mathbf{x}^{\mathbf{u}} = \mathbf{x}^{\lambda}$. This means that $\mathrm{supp}(\mathbf{u}) \in \Delta$, so we conclude that $\mathbf{x}^{l\mathbf{u}} \notin J$ for any $l \in \mathbb{N}$. Thus $\mathbf{x}^{\mathbf{u}} \notin \mathrm{rad}(J)$. Since $\mathbf{x}^{\mathbf{u}}$ was an arbitrary monomial not in J, this shows that any monomial not in J is not in $\mathrm{rad}(J)$, so we must have $J = \mathrm{rad}(J)$. Since J is monomial, we conclude that J is squarefree. $\square$

The converse, however, is not true because there is a configuration with a nonunimodular triangulation whose secondary cone is not subdivided in the Gröbner fan. In fact, the following is an open problem.

Problem 4.3.8. *How does the Gröbner fan refine a given cone in the secondary fan?*

When $n-d = 2$, so the state and secondary polytopes are two-dimensional, a two-dimensional cone in the secondary fan is subdivided by adding the rays corresponding to the Hilbert basis of the cone. No such answer is known when $n - d \geq 3$.

Problem 4.3.9. *Characterize the top-dimensional open cones in the secondary fan that are not subdivided in the Gröbner fan.*

Proposition 4.3.7 says that the cones corresponding to unimodular triangulations are not subdivided, and when $n - d = 2$ these are the only non-subdivided cones. When $n - d \geq 3$ more complicated cones can fail to subdivide.

4.4 Further reading

This lecture covers Chapter 8 of [Stu96].

The Stanley-Reisner ideal was introduced in the work of Stanley and Reisner. A notable early application was Stanley's proof of the Upper Bound Theorem for simplicial spheres, which bounds the number of faces of each dimension a d-dimensional simplicial sphere can have in terms of the number of vertices. Much more information about Stanley-Reisner ideals can be found in [Sta96]. Stanley calls the quotient of the polynomial ring by the Stanley-Reisner ideal the *face ring* of the simplicial complex. Other aspects of squarefree monomial ideals in combinatorics and commutative algebra can be found in [MS05].

The original motivation for regular triangulations can be found in [GKZ94]. A good reference for general information about triangulations will be the forthcoming book [LRS]. The program `TOPCOM` by Jörg Rambau is the current best method to compute all regular triangulations, and thus the secondary fan. When $\mathcal{A}$ is unimodular, the program CaTS [Jena] computes the Stanley-Reisner ideals of all regular triangulations. CaTS can also be used for arbitrary $\mathcal{A}$ by applying Theorem 4.2.1, though there may be much redundancy in this case. The secondary polytope is a special case of a fiber polytope [BS92].

4.5 Tutorial 4

1. Let $I = \langle x^2y, y^2z, u^3\rangle$ be a monomial ideal in a polynomial ring $k[x, y, z, u]$.

   (a) Find the radical $\sqrt{I}$ of I.

   (b) Find the primary decomposition of $\sqrt{I}$.

   (c) Draw the Stanley-Reisner complex of $\sqrt{I}$.

2. Let $\mathcal{A} = \{(2, 0, 0), (1, 1, 0), (0, 2, 0), (1, 0, 1), (0, 1, 1), (0, 0, 2)\}$. The ideals I_1, I_2 and I_3 described below are three initial ideals of the toric ideal $I_{\mathcal{A}}$ in the polynomial ring $k[a, b, c, d, e, f]$, where the variable a corresponds to the first point $(2, 0, 0)$ and so on. Find the triangulation corresponding to each of these initial ideals.
   $I_1 = \langle e^2, de, d^2, cd, bd, b^2\rangle$
   $I_2 = \langle b^2, bd, be, ae^2, af, bf, cf\rangle$
   $I_3 = \langle ac, cd, ae, de, af, cf\rangle$

Algorithm 4.5.1 ([Stu96, Algorithm 7.2]). Computing the Graver basis of a toric ideal using Lawrence liftings.

Consider a matrix A in $\mathbb{Z}^{d\times n}$ and construct the enlarged matrix called the *Lawrence lifting* of A

$$\Lambda(A) = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$$

where $\mathbf{0}$ and $\mathbf{1}$ are the $d \times n$ zero and $n \times n$ identity matrices, respectively. The matrices A and $\Lambda(A)$ have isomorphic kernels, and so the toric ideal is a homogeneous prime ideal

$$I_{\Lambda(A)} = \langle \mathbf{x}^{\mathbf{u}^+}\mathbf{y}^{\mathbf{u}^-} - \mathbf{x}^{\mathbf{u}^-}\mathbf{y}^{\mathbf{u}^+} \ : \ \mathbf{u} \in \ker(A)\rangle$$

in the polynomial ring $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

For a Lawrence type matrix $\Lambda(A)$, the Graver basis and universal Gröbner basis are the same, and they are equal to any reduced Gröbner basis for the ideal $I_{\Lambda(A)}$ [Stu96, Theorem 7.1]. This fact gives us the following algorithm for computing the Graver basis of a given matrix.

Step 1. Choose any term order on $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$, and compute the reduced Gröbner basis $\mathcal{G}$ of $I_{\Lambda(A)}$.

Step 2. Substitute 1 for $y_1, \ldots, y_n$ in $\mathcal{G}$. The resulting subset of $k[x_1, \ldots, x_n]$ is the Graver basis $Gr_{\mathcal{A}}$.

In the following exercises, let $\mathcal{A}$ be a set of vectors $\{(3,0),(2,1),(1,2),(0,3)\}$ in $\mathbb{Z}^2$. The toric ideal $I_{\mathcal{A}} = \langle xz - y^2, xw - yz, yw - z^2\rangle$ in the polynomial ring $k[x,y,z,w]$ corresponds to the projective twisted cubic.

3. Use Algorithm 4.5.1 and the Hoşten-Sturmfels Algorithm described in Tutorial 3 to compute the Graver basis of $I_{\mathcal{A}}$.

4. Show that the universal Gröbner basis $\mathcal{U}_{\mathcal{A}}$ for $I_{\mathcal{A}}$ is the same as the Graver basis found in Exercise 3. To argue that each element of $Gr_{\mathcal{A}}$ is in $\mathcal{U}_{\mathcal{A}}$ you may use the fact that $\mathcal{C}_{\mathcal{A}} \subseteq \mathcal{U}_{\mathcal{A}} \subseteq Gr_{\mathcal{A}}$ (where $\mathcal{C}_{\mathcal{A}}$ is the set of circuits of $I_{\mathcal{A}}$; see Definition 3.4.8), and in the case of a non-circuit compute some initial ideal to show that the presence of that element in $\mathcal{U}_{\mathcal{A}}$ is necessary. *Hint:* Try initial ideals with respect to some term orders or simple weight vectors.

5. Use the program CaTS [Jena], which was introduced in Chapter 3, to compute all initial ideals of $I_{\mathcal{A}}$.

6. Suppose $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$ and $\pi : \mathbb{N}^n \longrightarrow \mathbb{Z}^d$ as defined in Section 3.2 is a semigroup homomorphism such that $\pi(\mathbf{u}) = u_1\mathbf{a}_1 + \ldots + u_n\mathbf{a}_n$. A vector $\mathbf{b} \in \mathbb{N}\mathcal{A}$ is called a *Gröbner degree* if for some binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in \mathcal{U}_{\mathcal{A}}$, $\pi(\mathbf{u}^+) = \pi(\mathbf{u}^-) = \mathbf{b}$. If $\mathbf{b}$ is a Gröbner degree, the polytope $\mathrm{conv}(\pi^{-1}(\mathbf{b}))$ is called a *Gröbner fiber*.

   The *inner normal fan* of a polytope P is defined similarly to the outer normal fan, but this time the cones consist of minimizing vectors. In other words, for each face F of P, the *inner normal cone* is the set $\{\mathbf{c} \in \mathbb{R}^n \ : \ \mathrm{face}_{-\mathbf{c}}\, P = F\}$.

   (a) ([Stu96, Theorem 7.15]) For $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$, let P be the Minkowski sum of all Gröbner fibers $\mathrm{conv}(\pi^{-1}(\mathbf{b}))$. Show that the inner normal fan of P is the Gröbner fan of $I_{\mathcal{A}}$.

   Now let $\mathcal{A}$ correspond to the projective twisted cubic as above.

   (b) Find the Gröbner fibers of $\mathcal{A}$.

   (c) Compute the Minkowski sum of the Gröbner fibers that you found and compare your answer with the combinatorial description of the state polytope given by CaTS in Exercise 5.

7. (a) Find the initial complexes of $I_{\mathcal{A}}$ (an *initial complex* of $I_{\mathcal{A}}$ is the Stanley-Reisner complex corresponding to the radical of an initial ideal of $I_{\mathcal{A}}$).

   (b) Find all triangulations of $\mathcal{A}$.

   (c) Show that all triangulations of $\mathcal{A}$ are regular, and find a vector corresponding to each triangulation.

   (d) Examine your findings above to see how they fit with Theorem 4.2.1.

   (e) Find the secondary polytope of $\mathcal{A}$.

4.6 Solutions to Tutorial 4

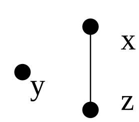
1. (a) $\sqrt{I} = \langle xy, yz, u\rangle$. In general, if an ideal in a polynomial ring is generated by monomials $M_1, \dots, M_n$, then its radical is generated by $M_1', \dots, M_n'$, where each M_i' is obtained by replacing every power greater than 1 on the variables in M_i by 1 (prove this!).

   (b) $\sqrt{I} = \langle x, z, u\rangle \cap \langle y, u\rangle$. There are several algorithms for finding primary decomposition of monomial ideals; see [Eis94, Chapter 3] and [Vil01, Corollary 5.1.13]. In Macaulay 2 the relevant code would be

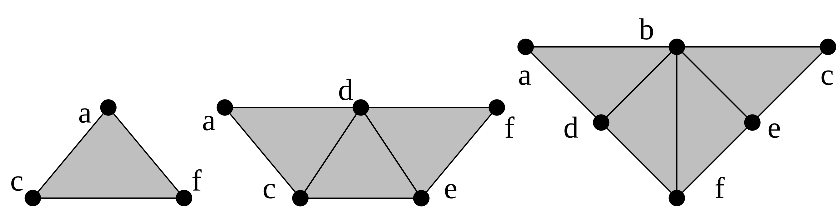
```
i1 :  R = QQ[x,y,z,u];
i2 : I = monomialIdeal(x*y,y*z,u);
o2 : MonomialIdeal of R
i3 : ass I
o3 = {monomialIdeal (y, u), monomialIdeal (x, z, u)}
o3 : List
i4 : primaryDecomposition I
o4 = {monomialIdeal (y, u), monomialIdeal (x, z, u)}
o4 : List
```

   The *associated primes* of I are the radicals of the primary ideals appearing in a primary decomposition of I; these are given by `ass I` in the code above.

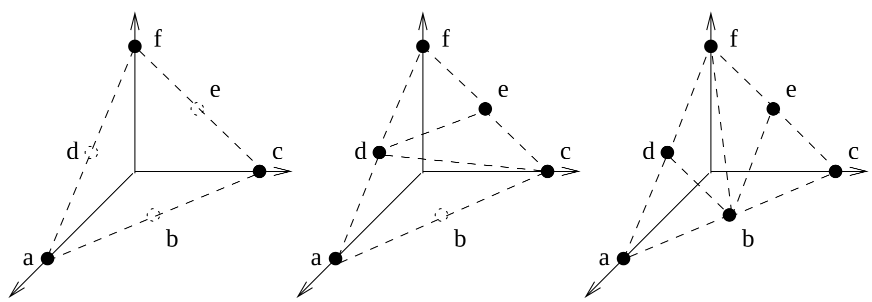
   (c)



2. The radicals of these ideals are: $\sqrt{I_1} = \langle e, b, d\rangle$, $\sqrt{I_2} = \langle b, ae, af, cf\rangle$ and $\sqrt{I_3} = \langle ac, cd, ae, de, af, cf\rangle$, which are respectively from left to right the Stanley-Reisner ideals of the following three simplicial complexes:



These simplicial complexes correspond, respectively, to the three following triangulations of $\mathcal{A}$:



3.

$$\Lambda(A) = \begin{pmatrix} 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Pick a weight vector $\mathbf{w}$ and apply HSAlg from Tutorial 3 to the list LA that contains the rows of the Lawrence matrix as follows. In the following code we add a row of 1's to the matrix, as Macaulay 2 will not accept multidegrees that start with 0. This does not affect the toric ideal, since the vector $(1, 1, 1, 1, 1, 1, 1, 1, 1)$ is in the row span of $\Lambda(A)$, and hence the kernel of the matrix will remain the same.

```
i1 : LA = {{1,1,1,1,1,1,1,1}, {3,2,1,0,0,0,0,0},
           {0,1,2,3,0,0,0,0}, {1,0,0,0,1,0,0,0},
           {0,1,0,0,0,1,0,0}, {0,0,1,0,0,0,1,0},
           {0,0,0,1,0,0,0,1}};

i2 : w={1,1,1,1,1,1,1,1};

i3 : HSAlg(LA,w)

o3 = | x_2x_4x_7^2-x_3^2x_6x_8 x_1x_4x_6x_7-x_2x_3x_5x_8
       x_1x_3x_6^2-x_2^2x_5x_7 x_1x_4^2x_7^3-x_3^3x_5x_8^2
       x_1^2x_4x_6^3-x_2^3x_5^2x_8 |
```

```
               1       5
o3 : Matrix R  <--- R
```

The output is the set

$$\{x_1x_3x_6^2-x_2^2x_5x_7,\ x_1x_4x_6x_7-x_2x_3x_5x_8,\ x_3^2x_6x_8-x_2x_4x_7^2,\ x_1^2x_4x_6^3-x_2^3x_5^2x_8,$$
$$x_1x_4^2x_7^3-x_3^3x_5x_8^2\}.$$

After setting $x_5 = x_6 = x_7 = x_8 = 1$, and $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = w$, we have the Graver basis

$$Gr_{\mathcal{A}} = \{xz - y^2,\ xw - yz,\ z^2 - yw,\ x^2w - y^3,\ xw^2 - z^3\}.$$

4. The four elements $z^2 - yw$, $xz - y^2$, $xw^2 - z^3$ and $x^2w - y^3$ are circuits, and so they belong to the universal Gröbner basis $\mathcal{U}_{\mathcal{A}}$.

   To show the remaining element $xw - yz$ is in $\mathcal{U}_{\mathcal{A}}$, we need to find an initial ideal of $I_{\mathcal{A}}$ for which one of the terms of this binomial is a minimal generator. If we use the reverse lexicographic order $x \succ y \succ w \succ z$, we get $\mathrm{in}_{\succ}(I_{\mathcal{A}}) = \langle yw, xw, y^2\rangle$, and so we need the basis element $xw - yz$ to obtain the generator xw for the initial ideal.

   Alternatively, one could use a weight vector, say $\mathbf{w} = (1, 1, 0, 1)$. Then by Lemma 2.4.2, since $Gr_{\mathcal{A}}$ contains a universal Gröbner basis, $\mathrm{in}_{\mathbf{w}}(I_{\mathcal{A}})$ is generated by $\mathrm{in}_{\mathbf{w}}(g)$ where $g \in Gr_{\mathcal{A}}$. So

$$\mathrm{in}_{\mathbf{w}}(I_{\mathcal{A}}) = \langle yw, xw, y^2, xw^2, x^2w - y^3\rangle = \langle yw, xw, y^2\rangle,$$

   so the binomial $xw - yz$ must be in $\mathcal{U}_{\mathcal{A}}$.

5. To use the program CaTS, put $\mathcal{A}$ into a file called “twisted.dat” (so the content of twisted.dat is simply: {(3,0) (2,1) (1,2) (0,3)}). Then run the command:

```
cats -p2 -e -i  twisted.dat
```

   CaTS will read the input file twisted.dat, and write the output in a file, which in this case is called “twisted.list”. This file will contain the following data (the term `Vtx` refers to a vertex of the state polytope):

```
Vtx: 0 (2 facets/3 binomials/degree 2)
    Initial ideal:{z^2, y*z, y^2}
  Facet Binomials:{# z^2-y*w,# y^2-x*z}
Vtx: 1 (2 facets/4 binomials/degree 3)
    Initial ideal:{y^2, y*z, z^3, y*w}
  Facet Binomials:{# z^3-x*w^2,# y*w-z^2}
Vtx: 2 (2 facets/4 binomials/degree 3)
    Initial ideal:{y^2, y*z, y*w, x*w^2}
  Facet Binomials:{# y*z-x*w,# x*w^2-z^3}
Vtx: 3 (2 facets/3 binomials/degree 2)
    Initial ideal:{y^2, x*w, y*w}
  Facet Binomials:{# y^2-x*z,# x*w-y*z}
Vtx: 4 (2 facets/3 binomials/degree 2)
    Initial ideal:{x*z, x*w, y*w}
  Facet Binomials:{# x*z-y^2,# y*w-z^2}
Vtx: 5 (2 facets/4 binomials/degree 3)
    Initial ideal:{y^3, x*z, y*z, z^2}
  Facet Binomials:{# y^3-x^2*w,# x*z-y^2}
Vtx: 6 (2 facets/4 binomials/degree 3)
    Initial ideal:{x*z, y*z, z^2, x^2*w}
  Facet Binomials:{# y*z-x*w,# x^2*w-y^3}
Vtx: 7 (2 facets/3 binomials/degree 2)
    Initial ideal:{x*z, z^2, x*w}
  Facet Binomials:{# z^2-y*w,# x*w-y*z}
```

6. (a) It is enough to show that the two fans have the same maximal cones. Suppose $\mathbf{w}$ and $\mathbf{w}'$ are two vectors in the interior of a maximal cone of the inner normal fan of P. We have

$$\text{face}_{-\mathbf{w}}(P) = \text{face}_{-\mathbf{w}'}(P).$$

This equality transfers to each summand of the Minkowski sum, so that we have, equivalently, for each Gröbner degree $\mathbf{b}$

$$\text{face}_{-\mathbf{w}}(\text{conv}(\pi^{-1}(\mathbf{b}))) = \text{face}_{-\mathbf{w}'}(\text{conv}(\pi^{-1}(\mathbf{b}))) = \mathbf{u} \in \text{conv}(\pi^{-1}(\mathbf{b})).$$

In other words

$$\mathbf{w}\cdot\mathbf{u} < \mathbf{w}\cdot\mathbf{v},\ \ \mathbf{w}'\cdot\mathbf{u} < \mathbf{w}'\cdot\mathbf{v} \text{ for all } \mathbf{v} \in \text{conv}(\pi^{-1}(\mathbf{b}))$$

which is equivalent to saying that $\mathbf{w}$ and $\mathbf{w}'$ pick exactly the same (unique) standard monomial $\mathbf{x}^{\mathbf{u}}$ in Gröbner degree $\mathbf{b}$. This means that that $\mathrm{in}_{\mathbf{w}}(I_{\mathcal{A}})$ and $\mathrm{in}_{\mathbf{w}'}(I_{\mathcal{A}})$ have the same generating set in Gröbner degree $\mathbf{b}$. By varying the choice of $\mathbf{b}$ among all Gröbner degrees, we get, equivalently, $\mathrm{in}_{\mathbf{w}}(I_{\mathcal{A}}) = \mathrm{in}_{\mathbf{w}'}(I_{\mathcal{A}})$. This means that $\mathbf{w}$ and $\mathbf{w}'$ belong to the same maximal cone of the Gröbner fan, which proves our claim.

(b) We find the Gröbner degrees:

$$\begin{array}{rll} yw - z^2 & \mathbf{u}^+ = (0,1,0,1) & \pi(\mathbf{u}^+) = (2,4) \\ xw - yz & \mathbf{u}^+ = (1,0,0,1) & \pi(\mathbf{u}^+) = (3,3) \\ xz - y^2 & \mathbf{u}^- = (0,2,0,0) & \pi(\mathbf{u}^-) = (4,2) \\ xw^2 - z^3 & \mathbf{u}^- = (0,0,3,0) & \pi(\mathbf{u}^-) = (3,6) \\ x^2w - y^3 & \mathbf{u}^- = (0,3,0,0) & \pi(\mathbf{u}^-) = (6,3) \end{array}$$

We can easily compute each of the fibers:

$$\begin{array}{ll} (1) & \pi^{-1}((2,4)) = \{(0,0,2,0),(0,1,0,1)\} \\ (2) & \pi^{-1}((3,3)) = \{(0,1,1,0),(1,0,0,1)\} \\ (3) & \pi^{-1}((4,2)) = \{(0,2,0,0),(1,0,1,0)\} \\ (4) & \pi^{-1}((3,6)) = \{(0,1,1,1),(0,0,3,0),(1,0,0,2)\} \\ (5) & \pi^{-1}((4,2)) = \{(1,1,1,0),(0,3,0,0),(2,0,0,1)\} \end{array}$$

(c) We are looking at the Minkowski sum:

$$\mathrm{conv}(\pi^{-1}((2,4))) + \mathrm{conv}(\pi^{-1}((3,3))) + \mathrm{conv}(\pi^{-1}((4,2))) + \\ \mathrm{conv}(\pi^{-1}((3,6))) + \mathrm{conv}(\pi^{-1}((6,3))).$$

To draw this polytope, first note that it is enough to consider a projection onto $\mathbb{R}^2$. This is because the last two coordinates of the vectors in each fiber determine that particular vector. To see this, suppose $\mathbf{u}$ and $\mathbf{v}$ are two vectors in the fiber $\pi^{-1}(\mathbf{b})$ with $u_3 = v_3$ and $u_4 = v_4$. This means that $A\mathbf{u} = A\mathbf{v}$, where

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

is the matrix of the twisted cubic. It follows that, as the last two coordinates of $\mathbf{u}$ and $\mathbf{v}$ are equal,

$$\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

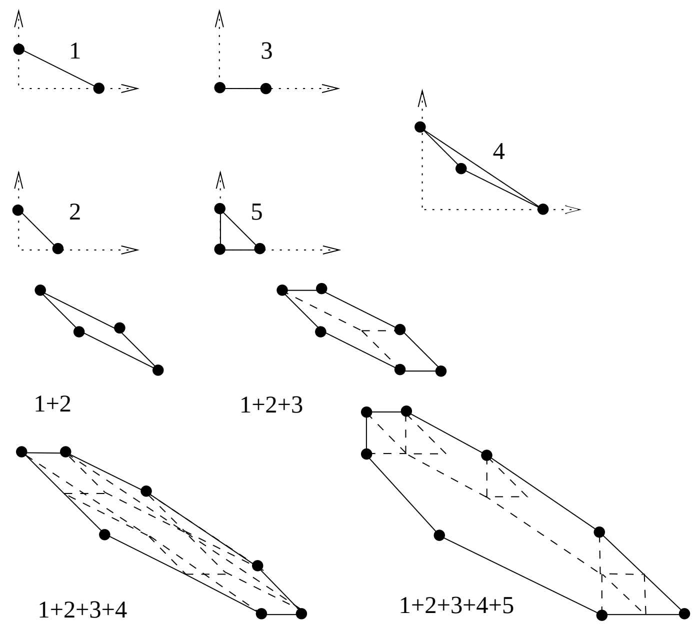


Figure 4.6: The Minkowski sum of the Gröbner fibers

and since the matrix $\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$ is nonsingular, we conclude that $u_1 = v_1$ and $u_2 = v_2$, and therefore $\mathbf{u} = \mathbf{v}$.

In Figure 4.6 all the fibers as well as the Minkowski sum are drawn in $\mathbb{R}^2$.

The following explanation clarifies the correspondence between the polytope P and the state polytope. Consider the first initial ideal $\langle z^2, yz, y^2 \rangle$ that CaTS gave us in Exercise 5. The Gröbner cone corresponding to this initial ideal, is the closure of the cone

$$C = \{\mathbf{w} \in \mathbb{R}^n_{\geq 0} \;:\; \mathrm{in}_{\mathbf{w}}(I) = \langle z^2, yz, y^2 \rangle\}.$$

By Lemma 2.4.2, we can use the Graver basis we found in Exercise 3 to describe C as

$$C = \{\mathbf{w} \in \mathbb{R}^n_{\geq 0} \;:\; 2w_3 > w_2 + w_4,\ 2w_2 > w_1 + w_3,\ 3w_3 > w_1 + 2w_4,$$
$$3w_2 > 2w_1 + w_4,\ w_2 + w_3 > w_1 + w_4\}.$$

Every vertex of P is a sum of vertices of each Gröbner fiber. The inequalities appearing in the description of C tell us which vertex of each Gröbner fiber we should take:

$$\mathbf{v}_1 = (0,0,2,0),\ \mathbf{v}_2 = (0,2,0,0),\ \mathbf{v}_3 = (0,0,3,0),$$
$$\mathbf{v}_4 = (0,3,0,0),\ \mathbf{v}_5 = (0,1,1,0).$$

The vertex $\mathbf{v}_1 + \ldots + \mathbf{v}_5$ of P corresponds to the vertex of the state polytope corresponding to the initial ideal $\langle z^2, y^2, yz\rangle$. Similarly, one can match the remaining 7 initial ideals with the remaining 7 vertices of P.

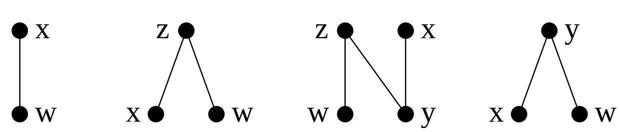
7. (a) We have already found the initial ideals in Exercise 5. It is also possible to get a list of the initial ideals without any information about the state polytope by using the file "twisted.dat" created earlier and running the command `cats -p2+ < twisted.dat`. The output will be the following.

```
{{z^2, y*z, y^2},
{y^2, y*z, z^3, y*w},
{y^2, y*z, y*w, x*w^2},
{y^2, x*w, y*w},
{x*z, x*w, y*w},
{y^3, x*z, y*z, z^2},
{x*z, y*z, z^2, x^2*w},
{x*z, z^2, x*w}}
```

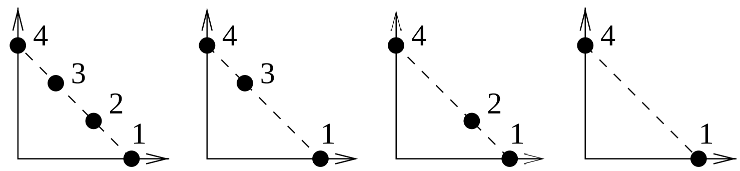
So the initial ideals, up to radicals, are:

$$\langle y,z\rangle,\ \langle xw,y\rangle,\ \langle xz,xw,yw\rangle,\ \langle z,xw\rangle,$$

which are the Stanley-Reisner ideals of the (initial) complexes (from left to right)



(b) Let the numbers $1,\ldots,4$ represent the points $\mathbf{a}_1 = (3,0)$, $\mathbf{a}_2 = (2,1)$, $\mathbf{a}_3 = (1,2)$ and $\mathbf{a}_4 = (0,3)$ of $\mathcal{A}$, respectively. Then the four triangulations are $T_1 = \{\{1,2\},\{2,3\},\{3,4\}\}$, $T_2 = \{\{1,3\},\{3,4\}\}$, $T_3 = \{\{1,2\},\{2,4\}\}$ and $T_4 = \{\{1,4\}\}$. These are shown below respectively, from left to right:



(c) The four triangulations in part (7a) correspond respectively to the following weight vectors: $\mathbf{w}_1 = (1,0,0,1)$, $\mathbf{w}_2 = (1,1,0,1)$, $\mathbf{w}_3 = (1,0,1,1)$ and $\mathbf{w}_4 = (0,1,1,0)$. These choices for the $\mathbf{w}_i$ are not unique; in fact, there are infinitely many options.

We show that, for example, $\mathbf{w}_2 = (1,1,0,1)$ is the appropriate weight vector for $T_2 = \{\{1,3\},\{3,4\}\}$; i.e. $T_2 = \Delta_{\mathbf{w}_2}$. A similar method works for T_1, T_3 and T_4.

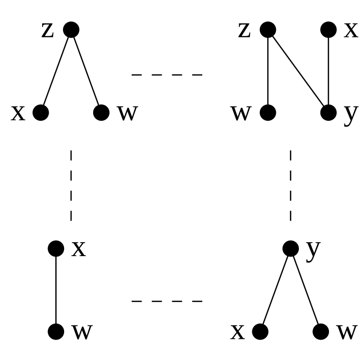
Consider the face $\{1,3\}$. We would like to find $\mathbf{c} = (c_1, c_2) \in \mathbb{R}$ such that

$$\begin{array}{lll} \mathbf{a}_1 \cdot \mathbf{c} = 3c_1 & = (\mathbf{w}_2)_1 & = 1 \\ \mathbf{a}_2 \cdot \mathbf{c} = 2c_1 + c_2 & < (\mathbf{w}_2)_2 & = 1 \\ \mathbf{a}_3 \cdot \mathbf{c} = c_1 + 2c_2 & = (\mathbf{w}_2)_3 & = 0 \\ \mathbf{a}_4 \cdot \mathbf{c} = 3c_2 & < (\mathbf{w}_2)_4 & = 1 \end{array}$$

Solving this system, we arrive at $\mathbf{c} = (1/3, -1/6)$.

Now we need to show that $\{1\}$, $\{3\}$, $\{3,4\}$ and $\{4\}$ are also faces by showing that for each one of them there exists such a $\mathbf{c}$ satisfying a similar set of equations above (the only thing that changes in each case is that the (in)equality signs move around depending which face you are looking at). For these four faces, we see that, respectively, the pairs $\mathbf{c} = (1/3, -1/3)$, $\mathbf{c} = (-1/3, 1/6)$, $\mathbf{c} = (-2/3, 1/3)$ and $\mathbf{c} = (-1, 1/3)$ work.

(d)



Chapter 5

Resolutions

5.1 Basics of minimal free resolutions

An important object in commutative algebra is the (minimal) free resolution of an R-module where R is a commutative ring with identity. In this chapter we will study minimal free resolutions of toric ideals in $S = \mathbf{k}[x_1, \ldots, x_n]$. For a detailed treatment of the material we touch upon here, see the book by Miller and Sturmfels [MS05]. See Chapters 5 and 6 of [CLO98] for a thorough introduction to free resolutions of polynomial ideals and quotients of polynomial rings, algorithms for their computation, and applications.

Given an ideal I of S, a free resolution $\mathcal{F}$ of the S-module S/I is an exact sequence of the form

$$\mathcal{F} :\ 0 \longleftarrow S/I \longleftarrow F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} F_2 \xleftarrow{\phi_3} \cdots \xleftarrow{\phi_r} F_r \cdots$$

where each F_i is a free S-module (hence of the form S^{β_i} for some $\beta_i \in \mathbb{N}$). If $F_l \neq 0$ and $0 = F_{l+1} = F_{l+2} = \cdots$, then $\mathcal{F}$ is finite of *length* l:

$$\mathcal{F} :\ 0 \longleftarrow S/I \longleftarrow F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} F_2 \xleftarrow{\phi_3} \cdots \xleftarrow{\phi_l} F_l \longleftarrow 0.$$

In this chapter we will consider only minimal free resolutions of ideals. Before we define this precisely, let us see an example using Macaulay 2.

```
i1 : S = QQ[x,y];
i2 : I = ideal(x^2-x, y^2-y, x*y)
i3 : F = res coker gens I
      1      3      2
o3 = S  <-- S  <-- S  <-- 0
```

```
     0      1      2      3

o3 : ChainComplex
i4 : F.dd
          1                        3
o4 = 0 : S  <------------------- S  : 1
               | x2-x xy y2-y |
          3                        2
     1 : S  <------------------- S  : 2
               {2} | -y  0    |
               {2} | x-1 -y+1 |
               {2} | 0   x    |
          2
     2 : S  <----- 0 : 3
               0
o4 : ChainComplexMap
```

This computation produces a minimal free resolution of length two of S/I where $I = \langle x^2 - x, y^2 - y, xy \rangle$ with $F_0 = S$, $F_1 = S^3$, $F_2 = S^2$. The rank of F_i in this minimal resolution is called the ith *Betti number* of S/I. We can choose bases for the S-modules F_i allowing the maps ϕ_1 and ϕ_2 to be given by matrices. The entries in ϕ_1 are the three minimal generators of I and thus, $\text{image}(\phi_1) = I = \text{kernel}(S \longrightarrow S/I)$. Each column in the matrix ϕ_2 is called a *syzygy* (relation) of I. For instance, the first column gives a relation on the generators of I:

$$(-y)(x^2 - x) + (x - 1)(xy) + 0(y^2 - y) = 0.$$

The computation shows that the module of (first) syzygies of I, which is $\text{kernel}(\phi_1) = \text{image}(\phi_2)$, is generated by the two columns of ϕ_2. However, $\text{kernel}(\phi_2) = 0$ since if

$$\begin{pmatrix} -y \\ x-1 \\ 0 \end{pmatrix} h_1 + \begin{pmatrix} 0 \\ -y+1 \\ x \end{pmatrix} h_2 = 0$$

then $h_1 = h_2 = 0$. Thus the resolution stops after two steps. The degrees of the entries (columns) in the maps ϕ_1 and ϕ_2 are recorded in the *Betti diagram* of $\mathcal{F}$:

```
i5 : betti F

o5 = total: 1 3 2
         0: 1 . .
         1: . 3 2
```

The Betti diagram is to be thought of as a matrix with rows and columns indexed by $0, 1, 2, \ldots$. The column index i denotes the step of the resolution. The top row "`o5 = total:  1 3 2`" indicates the Betti numbers of the steps of the resolution. The row index j is used as follows: the entry in position $\{j, i\}$ of the Betti diagram is the number of entries (columns) of degree $j+i$ in the map ϕ_i. Thus the bottom right entry "`2`" indicates that ϕ_2 has two degree $(3 = 1+2)$ entries (columns). The degree of the syzygy $(-y, x-1, 0)$ is three since in the syzygy relation

$$(-y)(x^2 - x) + (x-1)(xy) + 0(y^2 - y) = 0,$$

the highest degree of a term is three. In Macaulay 2 display format, the degrees of the syzygies in ϕ_1 are recorded on the left side of the matrix giving ϕ_2.

In the 1890s Hilbert proved that every ideal in the polynomial ring has a finite free resolution which is known as Hilbert's Syzygy Theorem.

Theorem 5.1.1. (Hilbert's Syzygy Theorem) *Let $S = \mathbf{k}[x_1, \ldots, x_n]$. Then every finitely generated S-module has a free resolution of length at most n.*

This theorem implies that every ideal I in S has a resolution with at most $n+1$ free S-modules in it. The length of a shortest finite free resolution of S/I is called the *projective dimension* of S/I. This is an important invariant of S/I and can be computed in Macaulay 2 using the command `pdim`.

```
i6 : pdim coker gens I
o6 = 2
```

In this chapter, we are interested in graded minimal free resolutions of ideals. If the polynomial ring S is graded by an abelian group A, then we can write $S = \oplus_{a \in A} S_a$ where S_a is the $\mathbf{k}$-vector space of all homogeneous polynomials of degree a. In the usual total degree grading where $\deg(x_i) = 1$ for $i = 1, \ldots, n$, the group $A = \mathbb{Z}$ and $S = \oplus_{t \in \mathbb{Z}} S_t$ where $S_t = 0$ for all $t < 0$ and $S_0 = \mathbf{k}$. An S-module M is graded by A if there exists subgroups $M_a, a \in A$ such that (i) $M = \oplus_{a \in A} M_a$ and (ii) $S_a M_b \subset M_{a+b}$. When $A = \mathbb{Z}$, graded ideals are just homogeneous ideals under the usual total degree grading.

In this chapter we only consider gradings of S by abelian groups of the form $\mathbb{Z}^n/\mathcal{L}$ where $\mathcal{L}$ is a sublattice of $\mathbb{Z}^n$ such that $\mathcal{L} \cap \mathbb{N}^n = \{\mathbf{0}\}$. See Chapter 8 in [MS05] for a discussion of valid multigradings in the context of resolutions.

The module M *twisted* by $d \in A$ is defined to be

$$M(d) = \oplus_{a \in A} M(d)_a \text{ where } M(d)_a := M_{d+a}.$$

Example 5.1.2. For a simple example, consider $S(-2) = \oplus_{t \in \mathbb{Z}} S_{-2+t}$. This twisted ring is generated (as an S-module) by 1 which lies in $S_0 = S_{-2+2}$ and hence has degree two in $S(-2)$.

The twisted free modules we will see are of the form $S(-d_1) \oplus S(-d_2) \oplus \cdots \oplus S(-d_m)$ generated by the standard basis elements $e_1, \ldots, e_m$ of degrees $d_1, \ldots, d_m$ respectively.

Definition 5.1.3. Let M and N be S-modules graded by A. A homomorphism $\phi : M \to N$ is a *graded homomorphism* of degree d if $\phi(M_a) \subset N_{a+d}$.

Example 5.1.4. If M is a graded S-module generated by homogeneous elements $f_1, \ldots, f_m$ of degrees $d_1, \ldots, d_m$ respectively, then we get the graded homomorphism $\phi : S(-d_1) \oplus S(-d_2) \oplus \cdots \oplus S(-d_m) \longrightarrow M$ of degree zero which sends $e_i \mapsto f_i$.

We saw earlier that homomorphisms in free resolutions are given by matrices. A graded homomorphism of degree zero

$$S(-d_1) \oplus \cdots \oplus S(-d_p) \longrightarrow S(-c_1) \oplus \cdots \oplus S(-c_m)$$

is defined by an $m \times p$ matrix U whose ijth entry u_{ij} is a homogeneous element of S of degree $d_j - c_i$ for all i, j. We call such a matrix a *graded* matrix over S.

Definition 5.1.5. If M is a graded S-module, then a *graded resolution* of M is a resolution of the form

$$0 \longleftarrow M \longleftarrow F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} F_2 \cdots \longleftarrow$$

where each F_i is a twisted free graded module $S(-d_1) \oplus \cdots \oplus S(-d_{p_i})$ and each map ϕ_i is a graded homomorphism of degree zero (given by graded matrices).

Remark 5.1.6. In the Macaulay 2 display format, maps in resolutions always go from right to left as in Definition 5.1.5. However, it is more usual in the literature to have maps between modules go from left to right. We use the Macaulay 2 format in this chapter when talking about a full resolution to match the Macaulay 2 outputs on examples.

The Hilbert Syzygy Theorem holds also for graded resolutions which guarantees finite graded free resolutions of length at most n for every S-module. Among these the most useful are the minimal graded resolutions defined as follows.

Definition 5.1.7. Let $\mathcal{F}$ be a graded resolution

$$0 \longleftarrow M \longleftarrow F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} F_2 \longleftarrow \cdots$$

of a graded S-module M.

1. The resolution $\mathcal{F}$ is *minimal* if the non-zero entries in the graded matrices ϕ_l have positive degree for every $l \geq 1$.

2. If $\mathcal{F}$ is a minimal graded resolution of M, then the number of copies of $S(-d)$ in F_i is the graded Betti number $\beta_{i,d}$.

Theorem 3.13 in [CLO98, Chapter 6] proves that any two minimal resolutions of M are isomorphic. The above examples and explanations are taken from Chapter 6 of [CLO98]. We refer the reader to that chapter for more details.

The ideal $I_{\{5,10,25,50\}}$ from Chapter 3 is homogeneous under the grading $\deg(a) = 5, \deg(b) = 10, \deg(c) = 25$, and $\deg(d) = 50$. Computing the minimal graded free resolution of this homogeneous ideal using Macaulay 2, we get the following.

```
i1 : S = QQ[a..d, Degrees => {5,10,25,50}, MonomialSize => 16]
i2 : I = ideal(c^2-d, a*b^2-c, a^2-b)
i3 : G = res coker gens I
i4 : G.dd

          1                            3
o4 = 0 : S  <----------------------- S  : 1
               | a2-b ab2-c c2-d |

          3                                    3
     1 : S  <------------------------------- S  : 2
               {10} | -ab2+c -c2+d 0      |
               {25} | a2-b   0     -c2+d |
               {50} | 0      a2-b  ab2-c |
```

```
       3                        1
2 : S  <------------------- S  : 3
          {35} | c2-d    |
          {60} | -ab2+c  |
          {75} | a2-b    |

       1
3 : S  <----- 0 : 4
          0

i5 : betti G
o5 = total: 1 3 3 1
         0: 1 . . .
         9: . 1 . .
        24: . 1 . .
        33: . . 1 .
        49: . 1 . .
        58: . . 1 .
        73: . . 1 .
        82: . . . 1
```

Note that in line `i1` we prescribe the grading on S. Here $F_0 = S$, $F_1 = S(-10) \oplus S(-25) \oplus S(-50)$, $F_2 = S(-35) \oplus S(-60) \oplus S(-75)$ and $F_3 = S(-85)$. Check that each of the matrices displayed above is a graded matrix with respect to the above degrees. The Betti diagram has been truncated to remove all rows with no non-zero entries. The projective dimension of S/I is three and *regularity* (which is the index of the last row in the Betti diagram) is 82. The values of all the non-zero Betti numbers are one. The above is a graded resolution where S has been graded by the semigroup $\mathbb{N}\{5, 10, 25, 50\}$ or alternately by the group $\mathbb{Z}^4/\mathcal{L}$ where $\mathcal{L} = \ker_{\mathbb{Z}}([5\ 10\ 25\ 50])$.

5.2 Free resolutions of toric ideals

Our goal in the rest of this chapter is to see what can be said about minimal free graded resolutions of toric ideals using the rich polyhedral and combinatorial information they carry. If $I_{\mathcal{A}}$ is a toric ideal, we assume that the lattice $\mathcal{L} =$

$\ker_{\mathbb{Z}}(A)$ has the property that $\mathcal{L} \cap \mathbb{N}^n = \{\mathbf{0}\}$. The abelian group $\mathbb{Z}^n/\mathcal{L}$ grades both the polynomial ring S and the S-module $S/I_{\mathcal{A}}$ via $\deg(x_i) := \mathbf{e}_i + \mathcal{L}$ where $\mathbf{e}_i$ is the ith unit vector in $\mathbb{Z}^n$. Note that this implies that the set of degrees of polynomials in S is the monoid $\mathbb{N}^n/\mathcal{L}$. This monoid is isomorphic to $\mathbb{N}\mathcal{A}$ via the map

$$\pi : \mathbb{Z}^n/\mathcal{L} \to \mathbb{Z}\mathcal{A}, \;\; \mathbf{u} + \mathcal{L} \mapsto A\mathbf{u}.$$

Thus $\deg(x_i) = \mathbf{a}_i$, and $I_{\mathcal{A}}$ and $S/I_{\mathcal{A}}$ are graded by $\mathbb{N}\mathcal{A}$. A monomial $\mathbf{x}^{\mathbf{u}}$ is said to be of $\mathcal{A}$-degree $A\mathbf{u}$.

We first show that the graded Betti numbers of $I_{\mathcal{A}}$ can be calculated from homology groups of certain simplicial complexes that come from polyhedra associated to $\mathcal{L}$. For each $\mathbf{b} \in \mathbb{N}\mathcal{A} \,(\cong \mathbb{N}^n/\mathcal{L})$ define a simplicial complex $\Delta_{\mathbf{b}}$ as follows:

$$\begin{aligned} \Delta_{\mathbf{b}} &:= \{\sigma \subseteq [n] : \sigma \subseteq \operatorname{supp}(\mathbf{u}) \text{ for some } \mathbf{u} \in \pi^{-1}(\mathbf{b})\} \\ &= \{\sigma \subseteq [n] : \mathbf{b} - \textstyle\sum_{i \in \sigma} \mathbf{a}_i \in \mathbb{N}\mathcal{A}\}. \end{aligned}$$

Consider the minimal multigraded free resolution of $S/I_{\mathcal{A}}$ where $\deg(x_i) = \mathbf{a}_i$ for $i = 1, \ldots, n$. Then the Betti numbers are of the form $\beta_{i,\mathbf{b}}$ which denotes the size of a basis for the ith syzygy module of degree $\mathbf{b}$ or equivalently, the rank of $S(-\mathbf{b})$ in F_i.

Theorem 5.2.1. *[Stu96, Theorem 12.12] The multigraded Betti number $\beta_{i,\mathbf{b}}$ equals the rank of the $(i-1)$th reduced homology group $\widetilde{H}_{i-1}(\Delta_{\mathbf{b}}, \mathbf{k})$ of the simplicial complex $\Delta_{\mathbf{b}}$.*

This theorem was proved originally in [CM91, CP93].

Corollary 5.2.2. *The toric ideal $I_{\mathcal{A}}$ has a minimal generator of degree $\mathbf{b}$ if and only if $\beta_{1,\mathbf{b}} = rank(\widetilde{H}_0(\Delta_{\mathbf{b}}, \mathbf{k})) > 0$. The latter condition is equivalent to the simplicial complex $\Delta_{\mathbf{b}}$ being disconnected.*

Example 5.2.3. Consider the ideal $I_{\mathcal{A}}$ where $\mathcal{A} = \{5, 10, 25, 50\}$ graded by $\mathbb{N}\mathcal{A}$. We saw that $a^2 - b$, $ab^2 - c$, $c^2 - d$ are minimal generators of $I_{\mathcal{A}}$. The monomials of the same $\mathcal{A}$-degree as the generators can be computed either via CaTS or via Macaulay 2 when the $\mathcal{A}$-degree is small.

```
i1 : S = QQ[a..d, Degrees => {5, 10,25,50}, MonomialSize => 16]
i2 : basis({10},S)
o2 = | a2 b |

i3 : basis({25},S)
```

```
o3 = | a5 a3b ab2 c |

i4 : basis({50},S)
o4 = | a10 a8b a6b2 a5c a4b3 a3bc a2b4 ab2c b5 c2 d |
```

Listing just the maximal cells in the simplicial complexes, we get: $\Delta_{10} = \{\{1\},\{2\}\}$, $\Delta_{25} = \{\{1,2\},\{3\}\}$ and $\Delta_{50} = \{\{1,2,3\},\{4\}\}$, all of which are disconnected.

From the Betti diagram we see that there is a syzygy of degree 85 in the third step of the resolution. Thus we expect that $\mathrm{rank}(\widetilde{H}_2(\Delta_{85}, \mathbf{k})) > 0$.

```
i5 : basis({85},S)

o5 = | a17 a15b a13b2 a12c a11b3 a10bc a9b4 a8b2c a7b5 a7c2 a7d
a6b3c a5b6 a5bc2 a5bd a4b4c a3b7 a3b2c2 a3b2d a2b5c a2c3 a2cd
ab8 ab3c2 ab3d b6c bc3 bcd |
```

Then $\Delta_{85} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$ which is an empty tetrahedron and hence $\widetilde{H}_2(\Delta_{85}, \mathbf{k}) \neq 0$.

Peeva and Sturmfels extend Theorem 5.2.1 to lattice ideals graded by $\mathbb{N}^n/\mathcal{L}$. Toric ideals are special examples of lattice ideals. As a corollary they obtain the following theorem.

Theorem 5.2.4. *[PS98b, Theorem 2.3] The projective dimension of $S/I_\mathcal{A}$ as an S-module is at most $2^{n-d} - 1$.*

Note that $n - d = \mathrm{codim}(I_\mathcal{A}) = \mathrm{rank}(\mathcal{L})$. In order to prove the above theorem and for the rest of our discussion, it is worthwhile to look at the fiber $P_\mathbf{b} := \pi^{-1}(\mathbf{b}) = \{\mathbf{x} \in \mathbb{N}^n : A\mathbf{x} = \mathbf{b}\}$. Choose a matrix $B \in \mathbb{Z}^{n \times (n-d)}$ such that $\mathcal{L} = \{B\mathbf{z} : \mathbf{z} \in \mathbb{Z}^{n-d}\}$. Let $\mathbf{u}_0 \in P_\mathbf{b}$. Then

$$\begin{aligned} P_\mathbf{b} &= \{\mathbf{u} \in \mathbb{N}^n : \mathbf{u} + \mathcal{L} = \mathbf{u}_0 + \mathcal{L}\} = \{\mathbf{u} \in \mathbb{N}^n : \mathbf{u}_0 - \mathbf{u} \in \mathcal{L}\} \\ &= \{\mathbf{u} \in \mathbb{N}^n : \mathbf{u}_0 - \mathbf{u} = B\mathbf{z} \text{ for some } \mathbf{z} \in \mathbb{Z}^{n-d}\} \\ &\cong \{\mathbf{z} \in \mathbb{Z}^{n-d} : \mathbf{u} = \mathbf{u}_0 - B\mathbf{z} \geq 0\} \\ &= \{\mathbf{z} \in \mathbb{Z}^{n-d} : B\mathbf{z} \leq \mathbf{u}_0\} =: P'_\mathbf{b} \end{aligned}$$

Thus we have created $P'_\mathbf{b} \subset \mathbb{Z}^{n-d}$, a bijective copy of $P_\mathbf{b}$ in $\mathbb{N}^n$, via the affine linear transformation that takes $\mathbf{u} \in P_\mathbf{b} \mapsto \mathbf{z} \in \mathbb{Z}^{n-d}$ such that $\mathbf{u} = \mathbf{u}_0 - B\mathbf{z}$. It can be checked that a different choice of $\mathbf{u}_0 \in P_\mathbf{b}$ will lead to the same isomorphic copy $P'_\mathbf{b}$ up to translation. So we can pick any $\mathbf{u}_0$ in this construction.

The set $P'_{\mathbf{b}} = Q_{\mathbf{b}} \cap \mathbb{Z}^{n-d}$ where $Q_{\mathbf{b}}$ is the polyhedron

$$Q_{\mathbf{b}} := \{\mathbf{z} \in \mathbb{R}^{n-d} \,:\, B\mathbf{z} \leq \mathbf{u}_0\}.$$

This implies that

$$\Delta_{\mathbf{b}} = \{\sigma \subseteq [n] \,:\, \sigma \subseteq \operatorname{supp}(\mathbf{u}_0 - B\mathbf{z}),\, \mathbf{z} \in P'_{\mathbf{b}}\}.$$

Thus $\mathbf{z} \in P'_{\mathbf{b}}$ contributes $\sigma = \{j \in [n] \,:\, B_j \cdot \mathbf{z} < (\mathbf{u}_0)_j\}$ which is the index set of all inequalities in $B\mathbf{z} \leq \mathbf{u}_0$ that do not hold at equality at $\mathbf{z}$.

Sketch of proof of Theorem 5.2.4. The projective dimension of $I_{\mathcal{A}}$ is the maximum integer l such that $\beta_{l,\mathbf{b}} \neq 0$ for some $\mathbf{b} \in \mathbb{N}\mathcal{A}$. By Theorem 5.2.1, $\beta_{l,\mathbf{b}} = \operatorname{rank}(\widetilde{H}_{l-1}(\Delta_{\mathbf{b}}, \mathbf{k}))$. If $Q_{\mathbf{b}}$ has a lattice point $\mathbf{z}$ in its interior, then $[n] \in \Delta_{\mathbf{b}}$ which implies that $\Delta_{\mathbf{b}}$ has the same homology groups as the n-ball and $\widetilde{H}_{l-1}(\Delta_{\mathbf{b}}, \mathbf{k}) = 0$ for all l. Thus if $\beta_{l,\mathbf{b}} \neq 0$ then all lattice points in $P'_{\mathbf{b}}$ lie on the boundary of $Q_{\mathbf{b}}$. Such polyhedra are said to be *empty* or *lattice point free*.

Let $F_1, \ldots, F_s$ be the facets of $\Delta_{\mathbf{b}}$. Then by definition there exists $\mathbf{z}_1, \ldots, \mathbf{z}_s$ in $P'_{\mathbf{b}}$ such that $F_i = \operatorname{supp}(\mathbf{u}_0 - B\mathbf{z}_i)$. Suppose $s > 2^{n-d}$. Then there exists two vectors, say $\mathbf{z}_1$ and $\mathbf{z}_2$, that have the same odd-even parity for their components. This implies that $\mathbf{z} = \frac{1}{2}(\mathbf{z}_1 + \mathbf{z}_2) \in \mathbb{Z}^{n-d}$. Further, $B\mathbf{z}_1 \leq \mathbf{u}_0$, $B\mathbf{z}_2 \leq \mathbf{u}_0$ implies that $B\mathbf{z} = B(\frac{1}{2}(\mathbf{z}_1 + \mathbf{z}_2)) \leq \mathbf{u}_0$. Thus $\mathbf{z} \in P'_{\mathbf{b}}$ which corresponds to $\frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2) \in P_{\mathbf{b}}$ where $\mathbf{u}_1 = \mathbf{u}_0 - B\mathbf{z}_1$ and $\mathbf{u}_2 = \mathbf{u}_0 - B\mathbf{z}_2$. Therefore, $\operatorname{supp}(\frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2)) = \operatorname{supp}(\mathbf{u}_1) \cup \operatorname{supp}(\mathbf{u}_2) \in \Delta_{\mathbf{b}}$ which contradicts that $F_1 = \operatorname{supp}(\mathbf{u}_1), F_2 = \operatorname{supp}(\mathbf{u}_2)$ are facets in $\Delta_{\mathbf{b}}$. Thus $s \leq 2^{n-d}$. Now computing homology of $\Delta_{\mathbf{b}}$ we see that homology of $\Delta_{\mathbf{b}}$ vanishes above dimension 2^{n-d}. $\square$

The above discussions show that lattice point free polytopes of the form $Q_{\mathbf{b}}$ play a fundamental role in the resolution of the toric ideal $I_{\mathcal{A}}$. Such polytopes have been studied extensively in discrete geometry and optimization. They play a central role in the work on integer programming due to the mathematical economist Scarf at Yale University [BSS95, Sca81, Sca86]. Using Scarf's work, Bayer, Peeva and Sturmfels gave explicit resolutions of generic monomial and lattice ideals in [BPS98, BS98, PS98a, PS98b]. The complete material in these papers is beyond the scope of this chapter. However, we will conclude with the main results in [PS98a] which deals with minimal free resolutions of generic lattice ideals. A lattice ideal is *generic* if it is generated by binomials of full support.

Example 5.2.5. [PS98a, Example 4.5] The toric ideal $I_{\{20,24,25,31\}}$ is generic. It is minimally generated by $\{a^4 - bcd,\ a^3c^2 - b^2d^2,\ a^2b^3 - c^2d^2,\ ab^2c - d^3,\ b^4 - a^2cd,\ b^3c^2 - a^3d^2,\ c^3 - abd\}$.

Definition 5.2.6. For a $\mathbf{b} \in \mathbb{N}\mathcal{A}$, let $\gcd(\mathbf{b})$ denote the greatest common divisor of $\{\mathbf{x}^{\mathbf{u}},\ \mathbf{u} \in P_{\mathbf{b}}\}$. Then $P_{\mathbf{b}}$ is called *basic* if $\gcd(\mathbf{b}) = 1$ and $\gcd(\{\mathbf{x}^{\mathbf{u}} : \mathbf{u} \in P_{\mathbf{b}}\} \backslash \{\mathbf{x}^{\mathbf{a}}\}) \neq 1$ for any $\mathbf{x}^{\mathbf{a}}$ such that $\mathbf{a} \in P_{\mathbf{b}}$.

Example 5.2.7. Resolving $S/I_{\{20,24,25,31\}}$ when S is graded by $\mathbb{N}\{20, 24, 25, 31\}$, we get the following.

```
i7 : F = res coker gens I
      1      7      12      6
o7 = S  <-- S  <-- S   <-- S  <-- 0

     0      1      2       3      4
   1                                                           7
0:S <---------------------------------------------------------- S
    |c3-abd a4-bcd ab2c-d3 b4-a2cd a3c2-b2d2 a2b3-c2d2 b3c2-a3d2|

   7                                                          12
1:S <--------------------------------------------------------- S
    {75} | -bd -a3 0   d2  0   0   -ab2 -a2d -b3 0   0    0    |
    {80} | -c2 -bd -cd 0   0   d2  0    0    0   -b3 -b2c 0    |
    {93} | 0   0   0   -ab -b2 -bc c2   0    0   0   a3   -a2c |
    {96} | 0   0   -a2 0   ac  0   0    -c2  -ad -cd 0    d2   |
    {110}| a   c   0   0   d   0   0    0    0   0   0    b2   |
    {112}| 0   0   b   c   0   0   -d   0    0   a2  0    0    |
    {122}| 0   0   0   0   0   a   0    b    c   0   -d   0    |

     12                                 6
2 : S   <---------------------------- S  : 3
         {130} | d  0  0  0   -b2 0   |
         {135} | 0  -d 0  0   0   -b2 |
         {136} | -c 0  -d 0   0   0   |
         {137} | b  0  0  -a2 0   0   |
         {141} | -a c  0  0   0   0   |
         {142} | 0  -b -c 0   0   0   |
         {143} | 0  0  -b 0   0   a2  |
         {146} | 0  a  0  -d  0   0   |
         {147} | 0  0  a  0   d   0   |
         {152} | 0  0  0  c   0   d   |
```

```
          {153} | 0  0  0  -b  c   0   |
          {158} | 0  0  0  0   a   c   |
     6
3 : S  <----- 0 : 4
         0
```

The truncated Betti diagram of this graded minimal free resolution is:

```
i9 : betti F
total: 1 7 12 6
    0: 1 .  . .
   74: . 1  . .
   79: . 1  . .
   92: . 1  . .
   95: . 1  . .
  109: . 1  . .
  111: . 1  . .
  121: . 1  . .
  128: . .  1 .
  133: . .  1 .
  134: . .  1 .
  135: . .  1 .
  139: . .  1 .
  140: . .  1 .
  141: . .  1 .
  144: . .  1 .
  145: . .  1 .
  150: . .  1 .
  151: . .  1 .
  156: . .  1 .
  158: . .  . 1
  163: . .  . 1
  164: . .  . 1
  174: . .  . 1
  175: . .  . 1
  180: . .  . 1
```

From the Betti diagram we see that $\beta_{3,178} = 1$. The monomials of degree 178 are $a^4b^2c^2, a^3cd^3, ab^4d^2, b^3c^3d$. Check that P_{178} is basic and

$$\Delta_{178} = \{\{1,2,3\},\{1,3,4\},\{1,2,4\},\{2,3,4\}\}$$

by computing the monomials of $\mathcal{A}$-degree 178 in the multigraded ring $\mathtt{S}$. See Example 5.2.9. This simplicial complex has spherical homology in degree two.

We will see that all homology fibers of generic toric ideals are basic.

Lemma 5.2.8. *[PS98a, Lemma 2.4] Let $P_{\mathbf{b}}$ be basic and $\mathbf{a} \in P_{\mathbf{b}}$. Then $P_{\mathbf{b}} \backslash \{\mathbf{a}\}$ is again a basic fiber after* $\min\{\mathbf{v} \in P_{\mathbf{b}} \backslash \{\mathbf{a}\}\}$ *has been subtracted from all the elements in $P_{\mathbf{b}} \backslash \{\mathbf{a}\}$.*

Example 5.2.9. By abuse of notation we let $P_{178} = \{a^4b^2c^2, a^3cd^3, ab^4d^2, b^3c^3d\}$.
$P_{178} \backslash \{a^4b^2c^2\}$ gives the basic fiber $P_{147} = \{a^3cd^2, ab^4d, b^3c^3\}$,
$P_{178} \backslash \{a^3cd^3\}$ gives the basic fiber $P_{130} = \{a^4c^2, ab^2d^2, bc^3d\}$,
$P_{178} \backslash \{ab^4d^2\}$ gives the basic fiber $P_{153} = \{a^4b^2c, a^3d^3, b^3c^2d\}$,
$P_{178} \backslash \{b^3c^3d\}$ gives the basic fiber $P_{158} = \{a^3b^2c^2, a^2cd^3, b^4d^2\}$.
Note that the fifth column in the map ϕ_3 corresponds to the second syzygy of degree 178 and it is a combination of first syzygies of degrees $130, 147, 153, 158$. Hence the basic fibers gotten from P_{178} are again homology fibers.

This observation is, in fact, the main theorem of [PS98a].

Definition 5.2.10. [PS98a, Definition 3.1] The *algebraic Scarf complex* of $I_{\mathcal{A}}$ is the complex of free S-modules

$$\mathcal{F}_{\mathcal{L}} = \bigoplus_{P_{\mathbf{b}} \text{ basic}} SE_{\mathbf{b}}$$

where $E_{\mathbf{b}}$ is a basis vector in homological degree $|\mathbf{b}| - 1$. The basis degrees considered are modulo the action of $\mathcal{L}$ by translation.

The differential is

$$\delta(E_{\mathbf{b}}) = \sum_{m \in P_{\mathbf{b}}} \operatorname{sign}(m, P_{\mathbf{b}}) \gcd(P_{\mathbf{b}} \backslash \{m\}) E_{P_{\mathbf{b}} \backslash \{m\}}.$$

Here we have abused notation by identifying monomials with their exponent vectors. The number $\operatorname{sign}(m, P_{\mathbf{b}})$ is $(-1)^{l+1}$ if m is in the lth position in the lexicographic ordering of monomials in $P_{\mathbf{b}}$.

Theorem 5.2.11. *[PS98a, Theorem 4.2] If the toric ideal $I_{\mathcal{A}}$ is generic then the algebraic Scarf complex $\mathcal{F}_{\mathcal{L}}$ is the minimal free resolution of $S/I_{\mathcal{A}}$.*

Example 5.2.12. Continuing our example of the generic toric ideal $I_{\{20,24,25,31\}}$,
$\delta(E_{178}) = (-1)^{1+1}dE_{147} + (-1)^{2+1}b^2E_{130} + (-1)^{3+1}cE_{153} + (-1)^{4+1}aE_{158} =$
$-b^2E_{130} + dE_{147} + cE_{153} - aE_{158}$ which is the syzygy in the fifth column of ϕ_3.

If the toric ideal is not generic, then the Scarf complex is contained in a minimal free resolution of the ideal [PS98a, Theorem 3.2]. The results by Peeva and Sturmfels hold for all lattice ideals although we have stated them here only for toric ideals.

5.3 Tutorial 5

As we saw in Lecture 5 Macaulay 2 can calculate minimal free resolutions of S/I using the command `F = res I` (or equivalently `F = res coker gens I`). You can also display the maps using `F.dd` or the Betti diagram using `betti F`. Sometimes Macaulay 2 does not keep track of the entire calculation when you run the command `res I`. If you are having trouble viewing the maps, try finding the resolution using the command `complete res I` which will force the Macaulay 2 to keep track of all relevant information required in order to display the maps. Also you might want the command `S = ring I` which sets the name of the ring in which I is defined to be S. It is useful whenever your ring was defined inside a function and hence does not have a short name.

1. Using Macaulay 2 find a minimal free resolution and Betti diagram of $S/I_{\mathcal{A}}$ for each of the following matrices. You may use either of the algorithms of Tutorial 3 to compute $I_{\mathcal{A}}$.

   (a) $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 3 & 4 \end{pmatrix}$

2. Let $R = \mathbf{k}[x]/\langle x^2 \rangle$ and let $I = \langle x \rangle \subset R$. What should a minimal free resolution of R/I look like? Now use Macaulay 2 to calculate a minimal resolution of R/I. Is the answer what you expected?

3. Let $S = \mathbf{k}[x_1, x_2, \ldots, x_n]$ and let $I \subset S$ be an ideal. Suppose $\mathcal{G}_{\succ} = \{g_1, g_2, \ldots, g_s\}$ is a Gröbner basis of I with respect to any monomial order $\succ$. Define

$$s_{ij} = \frac{\operatorname{lcm}(\operatorname{in}_{\succ}(g_i), \operatorname{in}_{\succ}(g_j))}{\operatorname{in}_{\succ}(g_i)}\mathbf{e}_i - \frac{\operatorname{lcm}(\operatorname{in}_{\succ}(g_i), \operatorname{in}_{\succ}(g_j))}{\operatorname{in}_{\succ}(g_j)}\mathbf{e}_j - \mathbf{a}_{ij}$$

   where $\mathbf{a}_{ij}$ is the vector of coefficients of the g_k's in the expansion of the S-polynomial S-pair(g_i, g_j) in terms of $\mathcal{G}_{\succ}$ and $\mathbf{e}_i$ is the standard basis vector in $\mathbb{R}^s$.

   For example, consider the ideal I generated by $\mathcal{G} = \{x^2 - x, xz - x, y - z\} \subset \mathbb{Q}[x, y, z]$. Call the polynomials g_1, g_2, and g_3 in the order listed.

The set $\mathcal{G}$ is a Gröbner basis for I with respect to the lexicographic ordering. The S-pair of g_1 and g_2 is

$$z\, g_1 - x\, g_2 = z(x^2 - x) - x(xz - x) = x^2 - xz$$

In order to find $\mathbf{a}_{12}$, we rewrite $x^2 - zx$ in terms of the Gröbner basis elements. We find that $x^2 - zx = g_1 - g_2$, so $\mathbf{a}_{12} = \mathbf{e}_1 - \mathbf{e}_2$. Therefore s_{12} is the vector

$$\frac{\mathrm{lcm}(x^2, xz)}{x^2}\mathbf{e}_1 - \frac{\mathrm{lcm}(x^2, xz)}{xz}\mathbf{e}_2 - (\mathbf{e}_1 - \mathbf{e}_2) = (z-1)\mathbf{e}_1 + (-x+1)\mathbf{e}_2.$$

These vectors are very useful because of the following theorem.

> **Schreyer's Theorem** *[CLO98, Theorem 3.3] With S, I, and $\mathcal{G}_\succ$ as above, the set $\{s_{ij} : 1 \le i, k \le s\}$ generates* $\mathrm{Syz}(I)$*, the syzygies on I, as an S-module.*

[CLO98, p. 238] Consider the following matrices:

$$M = \begin{pmatrix} x^2 - x & xy & y^2 - y \end{pmatrix} \qquad N = \begin{pmatrix} y & 0 \\ -x+1 & y-1 \\ 0 & -x \end{pmatrix}$$

Let I be the ideal generated by the entries of M.

(a) Verify that the matrix product MN equals the 1×2 matrix of all zeros. Explain why this shows that the module generated by the columns of the matrix N is contained in $Syz(I)$.

To show that $Syz(I)$ is, in fact, generated by the columns of N, we can use Schreyer's Theorem.

(b) Check that the generators for I form a Gröbner basis for I with respect to the lexicographic order.

(c) Compute the syzygies s_{12}, s_{13}, s_{23} obtained from the S-polynomials on the generators of I. By Schreyer's Theorem, these three syzygies generate $Syz(I)$.

(d) How are the the columns of N related to the generators s_{12}, s_{13}, s_{23} of S? Why does N only have two columns?

4. The ideal of the twisted cubic is the toric ideal of $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$.

   (a) Compute the minimal graded free resolution of $S/I_{\mathcal{A}}$ using the weight vector $\{1,1,1,1\}$.

   (b) For each degree in which there is a nonzero Betti number, calculate the fiber and then the simplicial complex $\Delta_{\mathbf{b}}$. A helpful Macaulay 2 command is `basis(b,S)` which gives the monomials of S corresponding to the vectors in the fiber $\pi^{-1}(\mathbf{b})$. Do these complexes have the homology of a simplicial sphere?

   Corollary 5.2.2 says that the toric ideal $I_{\mathcal{A}}$ has a minimal generator in degree $\mathbf{b}$ if and only if $\Delta_{\mathbf{b}}$ is disconnected. Verify this statement for the three generators of the twisted cubic.

   (c) Check that all the fibers from part b are basic. Using these basic fibers, write out the algebraic Scarf complex. How is it related to the minimal free resolution?

5.4 Solutions to Tutorial 5

1. Using the Hoşten-Sturmfels algorithm function from Tutorial 3 and a weight vector of all ones, we get the following resolutions and Betti diagrams. The Conti-Traverso algorithm may also be used to get the same result by changing "HSAlg" to "CTAlg" and using the same input. The remainder of the problem is identical.

   (a) We display the differentials in the resolution below.

```
i2 : Ia = ideal HSAlg({{1,2,3}},{1,1,1});
i3 : S = ring Ia;
i4 : (complete res Ia).dd

          1                                2
o4 = 0 : S  <---------------------------- S  : 1
               | x_1^2-x_2 x_1x_2-x_3 |

          2                          1
     1 : S  <---------------------- S  : 2
               {2} | -x_1x_2+x_3 |
               {3} | x_1^2-x_2   |

          1
     2 : S  <----- 0 : 3
               0

o4 : ChainComplexMap

i5 : betti res Ia

o5 = total: 1 2 1
         0: 1 . .
         1: . 1 .
         2: . 1 .
         3: . . 1
```

   (b) In this example the differential matrices are large, and the entire resolution does not fit easily on the page when they are included, so only the modules and arrows are displayed (to achieve this, we used `res Ib` here rather than `(complete res Ib).dd`).

```
i6 : Ib = ideal HSAlg({{1,1,1,1},{0,1,3,4}},{1,1,1,1});
i7 : S = ring Ib;
i8 : res Ib

       1      4      4      1
o8 = S  <-- S  <-- S  <-- S  <-- 0

     0      1      2      3      4

o8 : ChainComplex
i9 : betti res Ib
o9 = total: 1 4 4 1
         0: 1 . . .
         1: . 1 . .
         2: . 3 4 1
```

(c) Again the differentials have not been displayed.

```
i10 : Ic=ideal HSAlg({{1,2,2,3},{0,1,3,4}},{1,1,1,1});
i11 : S = ring Ic;
i12 : res Ic

       1      5      6      2
o12 = S  <-- S  <-- S  <-- S  <-- 0

      0      1      2      3      4

o12 : ChainComplex
i13 : betti res Ic
o13 = total: 1 5 6 2
          0: 1 . . .
          3: . 1 . .
          5: . 1 . .
          6: . 1 1 .
          7: . 1 2 .
          8: . 1 2 1
          9: . . 1 1
```

2. In a polynomial ring an ideal with one generator has no syzygies, however, over $R = \mathbf{k}[x]/\langle x^2\rangle$, multiplication by x sends the element x to x^2 which is

zero in R. Now for the syzygies on the syzygies, we again want syzygies on x, so again we get x. Repeating this process we get the infinite repeating resolution:

$$0 \longleftarrow R/I \longleftarrow R \xleftarrow{(x)} R \xleftarrow{(x)} R \xleftarrow{(x)} R \longleftarrow \cdots .$$

Putting this ring and ideal into Macaulay 2, we find that the resolution stops after two steps. The computer cannot handle the infinite resolution.

```
i14 : R=(QQ[x])/ideal(x^2);
i15 : res ideal(x)


       1      1      1
o15 = R  <-- R  <-- R

      0      1      2

o15 : ChainComplex
```

3. (a) Since the generators of I are the three entries of the matrix M, any column vector $\mathbf{v}$ such that $M\mathbf{v} = 0$ is a syzygy on the generators of I. The matrix MN contains only zeros, so each column of N is a syzygy on the generators of I, and hence the module the columns generate must be contained in the syzygy module $\mathrm{Syz}(I)$.

   (b) To check that these polynomials form a Gröbner basis, we calculate the S-pairs. Let $g_1 = x^2 - x$, $g_2 = xy$ and $g_3 = y^2 - y$ and $G = \{g_1, g_2, g_3\}$. We will use the notation $h \equiv 0 \mod G$ to mean that the remainder obtained by dividing the polynomial h by the ordered list of polynomials of G is 0.

$$\begin{aligned}
\text{S-pair}(g_1, g_2) &= \frac{x^2y}{x^2}(x^2 - x) - \frac{x^2y}{xy}(xy) = -xy = -g_2 \\
&\equiv 0 \mod G \\
\text{S-pair}(g_1, g_3) &= \frac{x^2y^2}{x^2}(x^2 - x) - \frac{x^2y^2}{y^2}(y^2 - y) = -xy^2 + x^2y \\
&= (xy)(x - y) = g_2(x - y) \equiv 0 \mod G \\
\text{S-pair}(g_2, g_3) &= \frac{xy^2}{xy}(xy) - \frac{xy^2}{y^2}(y^2 - y) = xy = g_2 \\
&\equiv 0 \mod G
\end{aligned}$$

   Since all three S-pairs reduce to 0, G is a Gröbner basis for I.

(c) From our calculations above, we see

$$\begin{aligned}
s_{12} &= \frac{x^2y}{x^2}\mathbf{e}_1 - \frac{x^2y}{xy}\mathbf{e}_2 - (-\mathbf{e}_2) = y\mathbf{e}_1 + (-x+1)\mathbf{e}_2 \\
s_{13} &= \frac{x^2y^2}{x^2}\mathbf{e}_1 - \frac{x^2y^2}{y^2}\mathbf{e}_3 - (x-y)\mathbf{e}_2 = y^2\mathbf{e}_1 - (x-y)\mathbf{e}_2 - x^2\mathbf{e}_3 \\
s_{23} &= \frac{xy^2}{xy}\mathbf{e}_2 - \frac{xy^2}{y^2}\mathbf{e}_3 - \mathbf{e}_2 = (y-1)\mathbf{e}_2 - x\mathbf{e}_3
\end{aligned}$$

(d) The syzygies s_{12} and s_{23} are the columns of N. The remaining syzygy $s_{13} = (y^2, -(x-y), -x^2) = ys_{12} + xs_{23}$ so it is in the module generated by s_{12} and s_{23}. Hence it is unnecessary to include it in a minimal generating set for $\mathrm{S}yz(I)$.

4. (a) Use Macaulay 2 and the Hoşten-Sturmfels algorithm to find the ideal I and the resolution of S/I.

```
i16 : I = ideal HSAlg({{1,1,1,1},{0,1,2,3}},{1,1,1,1});

i18 : (complete res I).dd


           1
o18 = 0 : S  <---
                                                     3
       ------------------------------------------- S  : 1
       | x_2^2-x_1x_3 x_2x_3-x_1x_4 x_3^2-x_2x_4 |

           3                          2
      1 : S  <------------------------ S  : 2
              {2,2} | -x_3 x_4  |
              {2,3} | x_2  -x_3 |
              {2,4} | -x_1 x_2  |

           2
      2 : S  <----- 0 : 3
              0

o18 : ChainComplexMap
```

(b) Looking at the resolution, we see that there are generators in multidegrees $(2,2)$, $(2,3)$, and $(2,4)$. We add the multidegree of any monomial from the first column of the differential map from S^2 to

S^3 to the multidegree of its row, and we find $(3,4)$ is the multidegree of one of the first syzygies. Similarly, using the second column, we find the other first syzygy has multidegree $(3,5)$.

The fiber in multidegree $\mathbf{b}$ can be calculated in Macaulay 2 by finding a basis for $S_{\mathbf{b}}$ using the command `basis`. For example if $\mathbf{b} = (2,2)$:

```
i19 : basis({2,2},S)

o19 = | x_1x_3 x_2^2 |

                     1       2
o19 : Matrix S  <--- S
```

So the maximal faces of the simplicial complex $\Delta_{(2,2)}$ are $\{1,3\}$ and $\{2\}$, which means $\Delta_{(2,2)}$ consists of three vertices and one edge connecting two of the vertices. In other words, $\Delta_{(2,2)}$ is contractible to a 0-sphere (see Figure 5.1), and hence has the same homology as a simplicial sphere.

For $\mathbf{b} = (2,3)$, the fiber is $\{x_1x_4, x_2x_3\}$, and for $\mathbf{b} = (2,4)$, the fiber is $\{x_2x_4, x_3^2\}$. The simplicial complexes $\Delta_{(2,3)}$ and $\Delta_{(2,4)}$ are both contractible to 0-spheres (see Figure 5.1) which we expected because they are generators of the ideal.

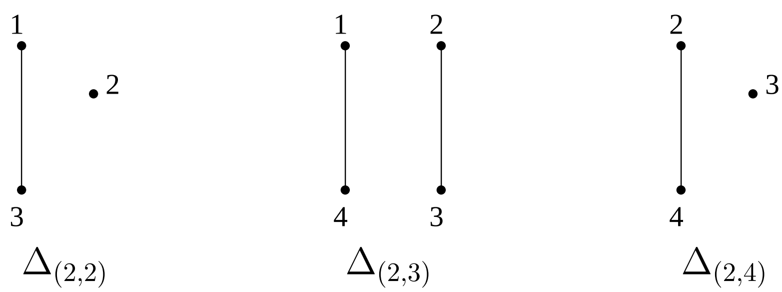


Figure 5.1: Some simplicial complexes

Now we check the statement for the first syzygies. For $\mathbf{b} = (3,4)$, the fiber is $\{x_1x_2x_4, x_1x_3^2, x_2^2x_3\}$. The maximal faces of the simplicial complex are $\{1,2,4\}$, $\{1,3\}$, and $\{2,3\}$ which contracts to a simplicial 1-sphere (see Figure 5.2).

Similarly for $\mathbf{b} = (3,5)$, the fiber is $\{x_1x_3x_4, x_2^2x_4, x_2x_3^2\}$. Again we

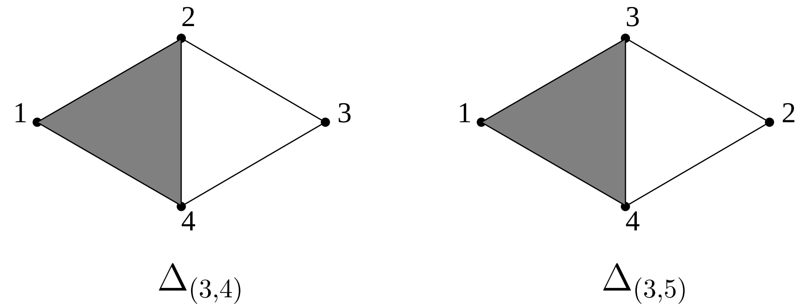


Figure 5.2: Some simplicial complexes

see that the associated simplicial complex contracts to a simplicial 1-sphere (see Figure 5.2).

Looking back at the simplicial complexes associated to the minimal generators, it is clear that each is indeed disconnected, while the simplicial complexes associated to higher syzygies are connected. This verifies Corollary 5.2.2.

(c) The fiber for $\mathbf{b} = (2, 2)$ is basic since the greatest common divisor of x_1x_3 and x_2^2 is 1. Similarly for $\mathbf{b} = (2, 3)$ and $\mathbf{b} = (2, 4)$ it is clear that the fiber is basic since the gcd of each pair of generators is 1.

For $\mathbf{b} = (3, 4)$ we can see that $\gcd(x_1x_2x_4, x_1x_3^2, x_2^2x_3) = 1$. Removing one element at a time, we see that $\gcd(x_1x_3^2, x_2^2x_3) = x_3$, $\gcd(x_1x_2x_4, x_2^2x_3) = x_2$, and $\gcd(x_1x_2x_4, x_1x_3^2) = x_1$. Hence this fiber is basic. Similarly for $\mathbf{b} = (3, 5)$ calculations show the fiber is basic.

All five fibers are basic so the Scarf complex will be exactly the same as the minimal free resolution. Notice that even though for generic toric ideals all fibers are basic, the reverse is not true since the twisted cubic is not generic.

Chapter 6

Connections to Algebraic Geometry

6.1 Introduction

In this chapter we will explain the connections between the theory we have covered in the first five lectures and toric varieties as they occur in algebraic geometry. We also make additional connections between commutative algebra and toric geometry. More familiarity with the basics of modern algebraic geometry is assumed here. For simplicity we will always work over the field $\mathbb{C}$.

6.2 A brief introduction to toric varieties

To an algebraic geometer, a toric variety is a variety defined by the combinatorial data of a polyhedral fan. The name "toric" comes from the fact that a toric variety is precisely a d-dimensional normal variety X that contains a dense copy of the algebraic torus $(\mathbb{C}^*)^d$ and with an action of $(\mathbb{C}^*)^d$ on X extending the action of $(\mathbb{C}^*)^d$ on itself by multiplication. For example, $\mathbb{C}^d$ is a toric variety, with $(t \cdot v)_i = t_i v_i$ for $t \in (\mathbb{C}^*)^d, v \in \mathbb{C}^d$.

We begin by defining an affine toric variety. Let $C \subseteq \mathbb{R}^d$ be a rational polyhedral cone. The word *rational* here means that every one-dimensional ray of C contains a lattice point (a point with all integer coordinates). Equivalently, it means that all the facet inequalities for C can be written using rational numbers. We will also assume that C is *pointed*, which means that there is a vector $\mathbf{v}$ with $\mathbf{v} \cdot \mathbf{x} \geq 0$ for all $\mathbf{x} \in C$. Let $D = C \cap \mathbb{Z}^d$ be the semigroup of lattice points inside C, and let $R = \mathbb{C}[D]$ be the corresponding semigroup

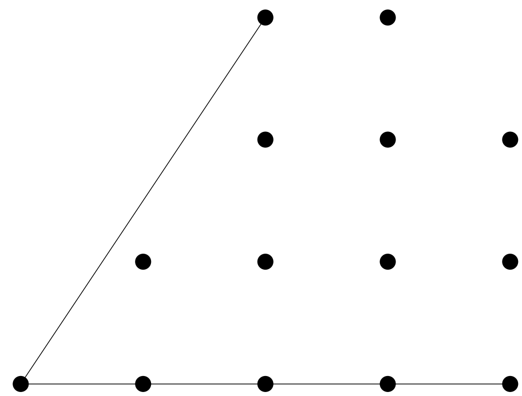


Figure 6.1:

algebra. The algebra R has as a vector-space basis the elements $t^{\mathbf{v}}$ for all $\mathbf{v} \in D$, with multiplication given by $\mathbf{t}^{\mathbf{v}}\mathbf{t}^{\mathbf{v}'} = \mathbf{t}^{\mathbf{v}+\mathbf{v}'}$. The *affine toric variety* X_C is defined to be $\mathrm{Spec}(R)$.

Example 6.2.1. Let C be the cone in $\mathbb{R}^2$ generated by $(1,0)$ and $(2,3)$. This is illustrated in Figure 6.1. The semigroup D is generated by $(1,0), (1,1)$, and $(2,3)$, so $R = \mathbb{C}[x, xy, x^2y^3]$.

Recall from Definition 3.4.4 of Chapter 3 that a *Hilbert basis* of the cone C is a finite generating set for $C \cap \mathbb{Z}^d$. Since C is pointed, its Hilbert basis is unique. Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be the Hilbert basis for C, and let $S = \mathbb{C}[x_1, \ldots, x_n]$. Then $R = S/I_{\mathcal{A}}$, where $I_{\mathcal{A}}$ is a toric ideal in the sense of Lecture 3.

Example 6.2.2. Let C be the cone of Example 6.2.1. Then $R = \mathbb{C}[x, y, z]/I_{\mathcal{A}}$, where $\mathcal{A} = \{(1, 0,), (1, 1), (2, 3)\}$.

A general toric variety is defined by a polyhedral fan Σ. For each cone $\sigma \in \Sigma$, we assign the affine chart $A_\sigma = X_{\sigma^\vee}$, where $\sigma^\vee = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{y} \geq 0 \text{ for all } \mathbf{y} \in \sigma\}$ is the polar cone of σ. If σ and σ' share a face $\tau \in \Sigma$, then A_τ is an open subvariety of A_σ and of $A_{\sigma'}$. We then glue together A_σ and $A_{\sigma'}$ along the subvariety A_τ. The resulting variety X_Σ is the toric variety corresponding to the fan Δ.

Example 6.2.3. Let Σ be the fan in Figure 6.2, and let $\sigma_1 = \mathrm{pos}((1,0),(0,1))$, $\sigma_2 = \mathrm{pos}((0,1),(-1,-1))$, and $\sigma_3 = \mathrm{pos}((-1,-1),(1,0))$ be its three maximal cones. The dual cones are $\sigma_1^\vee = \mathrm{pos}((1,0),(0,1))$, $\sigma_2^\vee = \mathrm{pos}((-1,0),(-1,1))$, and $\sigma_3^\vee = \mathrm{pos}((0,-1),(1,-1))$. These are shown in Figure 6.3.

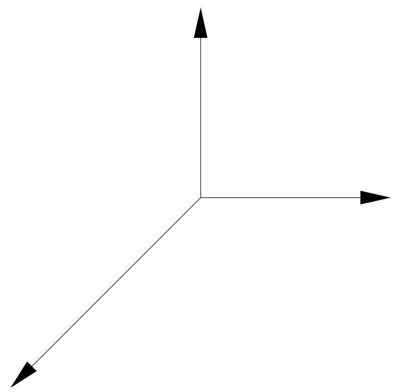


Figure 6.2:

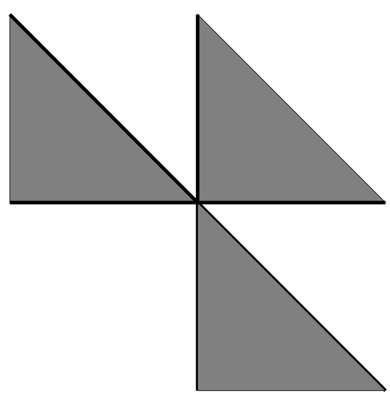


Figure 6.3:

Our three coordinate patches are now $A_i = \mathrm{Spec}(R_i)$, $i = 1, 2, 3$, where $R_1 = \mathbb{C}[x, y]$, $R_2 = \mathbb{C}[x^{-1}, y/x]$, $R_3 = \mathbb{C}[x/y, y^{-1}]$. These correspond, in order, to the three coordinate patches $z \neq 0$, $x \neq 0$, and $y \neq 0$ of $\mathbb{P}^2 = \mathrm{Proj}(\mathbb{C}[x, y, z])$. Since the overlaps are also consistent, we conclude that $X_\Sigma = \mathbb{P}^2$.

The construction of $\mathbb{P}^n$ as a toric variety is similar, with the rays of the fan Σ in $\mathbb{R}^n$ generated by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and $-\sum_{i=1}^n \mathbf{e}_i$. The n-dimensional cones in Σ are the positive hull of all but one of the above rays. There are thus $n+1$ of these cones. Each cone gives one of the standard affine patches for $\mathbb{P}^n$.

If our fan Σ is the normal fan of an integral polytope P, then X_Σ is a *projective* toric variety. A map of X_Σ into some projective space is obtained by placing P at height one in $\mathbb{R}^{d+1}$ and letting $\mathcal{A}$ be the Hilbert basis for $\mathrm{pos}(P \times \{1\})$. If P is sufficiently large we then have $\mathrm{Proj}(S/I_\mathcal{A}) \cong X_\Sigma$. There are many different choices of a polytope P with normal fan Σ. Each choice corresponds to the choice of an ample divisor on X_Σ. Demanding that P be sufficiently large is ensuring that this ample divisor is very ample.

6.3 The Cox homogeneous coordinate ring

One of the most important themes in algebraic geometry and commutative algebra is that algebro-geometric questions about projective space $\mathbb{P}^n$ can be translated into commutative algebra questions about the graded polynomial ring in $n+1$ variables. This section introduces an analogous object for toric varieties, the Cox homogeneous coordinate ring. We shall see that much of what is "nice" about projective space carries over to this setting.

6.3.1 Projective space

We will start by reviewing well-known facts about projective space and graded polynomial rings.

1. The homogeneous coordinate ring of $\mathbb{P}^n$ is the graded polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$. There is a distinguished ideal in S, $\mathfrak{m} = \langle x_0, \ldots, x_n \rangle$, which is called the *irrelevant ideal*. The *degree* of a monomial in S is the total degree in the variables x_i.

2. The variety $\mathbb{P}^n$ is the quotient of $\mathbb{C}^{n+1} - (0, \ldots, 0)$ by the diagonal action of $\mathbb{C}^*$ on the coordinates. This construction is just the identification of $\mathbb{P}^n$ with the set of lines through the origin in $\mathbb{C}^{n+1}$. It justifies the names

"coordinate ring" and "irrelevant ideal"; a point in projective space has coordinates $(x_0 : \cdots : x_n)$ except that the vanishing locus of $\mathfrak{m}$, namely the origin $(0, \ldots, 0)$, is disallowed.

3. (Nullstellensatz) A graded ideal $I \subset S$ gives rise to a subvariety $V(I)$ of $\mathbb{P}^n$. The variety $V(I)$ is empty if and only if $\mathfrak{m}^k \subset I$ for some integer k.

4. (Ideal variety correspondence) The map $I \to V(I)$ is a 1-1 correspondence from graded radical ideals I contained in $\mathfrak{m}$ to subvarieties of $\mathbb{P}^n$.

   More generally in algebraic geometry, subschemes (possibly not reduced) are given by ideal sheaves. The correspondence above generalizes when we consider general saturated ideals I.

5. A finitely generated graded module M over S gives rise to a coherent sheaf $\widetilde{M}$. Moreover, every coherent sheaf comes this way from some M.

6.3.2 Construction of homogeneous coordinate ring

Our goal will be to generalize all of the properties of the last section to general toric varieties. Recall that a toric variety X_Σ was defined as a variety arising from a rational polyhedral fan $\Sigma \subset \mathbb{R}^n$. Let $\Sigma(1)$ denote the set of one-dimensional cones (rays) of Σ. Given a ray $\rho \in \Sigma(1)$, let $\mathbf{n}_\rho$ be the first lattice point on ρ.

The various ρ correspond exactly to the $(n-1)$-dimensional orbits of the torus action on X_Σ. Their closures D_ρ are the codimension-one irreducible subvarieties of X_Σ. We will consider the group, isomorphic to $\mathbb{Z}^{\Sigma(1)}$, of formal integer linear combinations of the D_ρ. This is the group of torus invariant (Weil) divisors. Following this terminology we will call a linear combination $D = \sum a_\rho D_\rho$ a divisor, where the sum is over all $\rho \in \Sigma(1)$.

There is a natural map $\mathbb{Z}^n \to \mathbb{Z}^{|\Sigma(1)|}$ sending $\mathbf{m} \in \mathbb{Z}^n$ to $\sum(\mathbf{m} \cdot \mathbf{n}_\rho) D_\rho$. Two divisors are said to be *linearly equivalent* if their difference is in the image of this map. The cokernel of this map is a group $\mathrm{Cl}(X)$, called the *divisor class group* of X. Given a divisor D, let $[D]$ denote its image in $\mathrm{Cl}(X)$.

With all of this terminology straight we can finally define the homogeneous coordinate ring S_X of a toric variety X.

Definition 6.3.1. Let $S_X = \mathbb{C}[x_\rho : \rho \in \Sigma(1)]$. A monomial $\mathbf{x}^D := \prod x_\rho^{a_\rho}$ corresponds to a divisor $D = \sum a_\rho D_\rho$. The *degree* of $\mathbf{x}^D$ is defined to be $[D] \in \mathrm{Cl}(X)$.

Thus S_X is multigraded by $\mathrm{Cl}(X)$, which can be any abelian group. We shall see that this S_X satisfies many of the properties of the coordinate ring for projective space. The next step is to find the appropriate notion of irrelevant ideal. Up until now we have only used the one-dimensional cones. For each cone $\sigma \in \Sigma$ let $\mathbf{x}^{\widehat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$. Here $\sigma(1)$ is the set of one-dimensional generators of σ.

Definition 6.3.2. The *irrelevant ideal* $B \subset S_X$ is

$$B = \langle \mathbf{x}^{\widehat{\sigma}} \ : \ \sigma \in \Sigma \rangle.$$

It is easy to see that B is generated by the $\mathbf{x}^{\widehat{\sigma}}$ for the maximal cones $\sigma \in \Sigma$.

Example 6.3.3. (Affine space)
Affine space $\mathbb{C}^n$ is a toric variety generated by one maximal cone spanned by the standard basis vectors $\mathbf{e}_i = (0, \dots, 1, \dots, 0)$ with a 1 in the ith coordinate. The coordinate ring is now $\mathbb{C}[x_1, \dots, x_n]$. In this case the map $\mathbb{Z}^n \to \mathbb{Z}^{|\Sigma(1)|}$ is an isomorphism and $\mathrm{Cl}(X) = \{0\}$. Hence the ring is entirely in degree 0 and can be considered ungraded.

Example 6.3.4. (Projective space)
Projective space $\mathbb{P}^n$ is a toric variety with $\Sigma(1) = \{\mathbf{e}_1, \dots, \mathbf{e}_n, -\mathbf{e}_1 - \cdots - \mathbf{e}_n\}$. Here we recover the homogeneous coordinate ring in $n+1$ variables with the usual grading, since $\mathrm{Cl}(\mathbb{P}^n) = \mathbb{Z}$. Every set of n rays span a maximal cone, hence the irrelevant ideal is $\langle x_0, \dots, x_n \rangle$ as expected.

Example 6.3.5. (Weighted projective space)
Consider the complete fan in $\mathbb{R}^2$ with one-dimensional generators the set $\{(1,0), (0,1), (-a,-b)\}$. The coordinate ring is $\mathbb{C}[x_0, x_1, x_2]$, is graded by $\mathrm{Cl}(X) = \mathbb{Z}$, and the irrelevant ideal is $\langle x_0, x_1, x_2 \rangle$ just as in the case of $\mathbb{P}^2$. However, the variables x_0 and x_1 have degrees a and b respectively while x_2 has degree 1. This is called the *weighted projective space* of type $(a, b, 1)$.

Example 6.3.6. ($\mathbb{P}^r \times \mathbb{P}^s$)
One can realize the product of two projective spaces as a toric variety by taking the product of the associated complete fans. There are now $r + s + 2$ one-dimensional cones. The coordinate ring is $\mathbb{C}[x_0, \dots, x_r, y_0, \dots, y_s]$. The irrelevant ideal is generated by the monomials $x_i y_j$, and the grading is by $\mathrm{Cl}(X) = \mathbb{Z}^2$ where the x variables have degree $(1,0)$ while the y variables have degree $(0,1)$.

6.3.3 Quotient construction

We now emulate the quotient construction of projective space. Consider

$$Z = V(B) = \{t \in \mathbb{C}^{\Sigma(1)} \ : \ \prod_{\rho \notin \sigma} t_\rho = 0 \ \text{ for all } \sigma \in \Sigma\}$$

We will construct X as a quotient of $\mathbb{C}^{|\Sigma(1)|} - Z$. For any toric variety X of dimension n, the group $\mathrm{Cl}(X)$ is finitely generated of rank $s - n$ where $s = |\Sigma(1)|$. The corresponding complex multiplicative group $G = \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{C}^*)$ is the product of $(\mathbb{C}^*)^{s-n}$ and a finite group. G acts on $\mathbb{C}^{|\Sigma(1)|}$ by $g \cdot t = (g([D_\rho])t_\rho)$. Recall from Definition 2.3.9 that the n-dimensional fan Σ is simplicial if every maximal cone of Σ has n generators.

Theorem 6.3.7. *Let X be the toric variety determined by the fan Σ, and let $Z = V(B) \subset \mathbb{C}^{|\Sigma(1)|}$ be as above.*

(i) The set $\mathbb{C}^{|\Sigma(1)|} - Z$ is invariant under the action of the group G.

(ii) X is naturally isomorphic to the categorical quotient of $\mathbb{C}^{|\Sigma(1)|} - Z$ by G.

(iii) If Σ is simplicial then X is called a simplicial toric variety. If X is simplicial then X is the actual geometric quotient.

This theorem implies that a point in X is always an equivalence class of points in $\mathbb{C}^{|\Sigma(1)|}$. If X is simplicial then the only disallowed points are those in the irrelevant set Z. In the nonsimplicial case we have to disallow more points, namely those for which the G orbits are not closed.

Proof. The basic idea behind the proof is to notice that if we set S_σ to be the localization of S at $\mathbf{x}^{\hat{\sigma}}$, then $\mathbb{C}^{|\Sigma(1)|} - Z$ is covered by the affine open sets $U_\sigma = \mathrm{Spec}(S_\sigma)$. The connection with the affine cover of the toric variety is that $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n] = (S_\sigma)_0 = S_\sigma^G$. A standard result in invariant theory tells us that the quotient U_σ/G is equal to $\mathrm{Spec}(S_\sigma^G)$. It remains to check the gluing. Showing that this is actually a geometric quotient in the simplicial case is harder. $\square$

6.3.4 Homogeneous coordinate ring and algebraic geometry

Our next result generalizes Conditions 3 and 4 of Section 6.3.1 when X is a simplicial toric variety. For any graded ideal $I \subset S$ the set $V(I) - Z$ determines a closed subset of X, $V_X(I)$ via the above quotient. By abuse of notation we can write

$$V_X(I) = \{t \in X \ : \ f(t) = 0 \text{ for all } f \in I\},$$

where t is taken in homogeneous coordinates on X.

Theorem 6.3.8. *Let X be a simplicial toric variety and $B = \langle x^{\hat{\sigma}} \ : \ \sigma \in \Sigma \rangle$ the irrelevant ideal.*

(i) (The Toric Nullstellensatz) For any graded ideal $I \subset S$, $V_X(I) = 0$ if and only if $B^k \subset I$ for some integer k.

(ii) (The Toric Ideal-Variety Correspondence) The map $I \to V_X(I)$ induces a one to one correspondence between radical graded ideals of S contained in B and subvarieties of X.

Finally we can consider more general graded modules over S. A module M is said to be $\mathrm{Cl}(X)$-*graded* if there is a direct sum decomposition:

$$M = \sum_{\alpha \in \mathrm{Cl}(X)} M_\alpha$$

which respects the multiplication, so $S_\alpha M_\beta \subseteq M_{\alpha+\beta}$.

As in the projective space case, we can construct a sheaf $\widetilde{M}$ on X from a graded module M.

Theorem 6.3.9. *For any toric variety X, every (quasi)-coherent sheaf on X is of the form $\widetilde{M}$ for some graded S-module M.*

6.4 Open questions on normality

The toric ideals for which $\mathcal{A}$ is the Hilbert basis of a rational cone are an important subclass of all toric ideals. We saw in this chapter that geometrically they are the defining ideals of affine toric varieties. Algebraically, they are the set of normal toric ideals, which are the toric ideals $I_\mathcal{A}$ such that $S/I_\mathcal{A}$ is integrally closed. If $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subseteq \mathbb{Z}^d$ then $\mathcal{A}$ is called *normal* if $I_\mathcal{A}$ is normal. This is the case if the semigroup $\mathbb{N}\mathcal{A} = \{\sum_i \lambda_i \mathbf{a}_i : \lambda_i \in \mathbb{N}\}$ is normal, in the sense that if $\mathbf{b} \in \mathbb{Z}^d$ with $l\mathbf{b} \in \mathbb{N}\mathcal{A}$ for some $l \in \mathbb{N}$, $l > 0$, then $\mathbf{b} \in \mathbb{N}\mathcal{A}$. We list here some open questions about normal toric ideals.

A configuration $\mathcal{A}$ is called *graded* if there is a $\mathbf{c} \in \mathbb{Z}^d$ with $\mathbf{a}_i \cdot \mathbf{c} = 1$ for all $\mathbf{a}_i \in \mathcal{A}$.

6.4.1 Degrees of Gröbner bases

It is well-known (Sturmfels refers to it as a "folklore result" [Stu96, Theorem 13.14]) that if $\mathcal{A} \subseteq \mathbb{Z}^d$ is graded, and $\mathbb{N}\mathcal{A}$ is normal, then $I_\mathcal{A}$ is generated in degree at most d. The following question, however, is still open.

Question 6.4.1. *Let $\mathcal{A} \subseteq \mathbb{Z}^d$ be a graded configuration for which $\mathbb{N}\mathcal{A}$ is normal. Is there a Gröbner basis for $I_\mathcal{A}$ consisting of binomials of degree at most d?*

There are some results on this topic due to Hoşten, O'Shea, and Thomas [HT03], [OT05]. In [OT05] it is shown that such a Gröbner basis exists for the Δ-*normal* configurations defined in [HT03].

Definition 6.4.2. A configuration $\mathcal{A} \subseteq \mathbb{Z}^d$ is Δ-*normal* if there is a triangulation Δ of $\mathcal{A}$ for which if σ is a maximal cone of Δ then the Hilbert basis of $C_\sigma = \mathrm{pos}(\mathbf{a}_i : i \in \sigma)$ is $\mathcal{A} \cap C_\sigma$.

We note that the term order inducing such a Gröbner basis cannot always be chosen to be a lexicographic or reverse-lexicographic order.

6.4.2 Projective normality

In the next two sections we consider the case where X_σ is a smooth projective variety.

Let X_Σ be the projective toric variety determined by a polytope $P \subseteq \mathbb{R}^d$. The variety X_σ is smooth if every cone in Σ is simplicial and *unimodular* (the first lattice points on the extreme rays generate the semigroup of lattice points in the cone). In this case we also call the corresponding polytope P smooth.

For a general P if we take the embedding of X_Σ into projective space as $\mathrm{Proj}(S/I_\mathcal{A})$ where $\mathcal{A}$ is the configuration of lattice points inside the polytope, the resulting ideal $I_\mathcal{A}$ will not be normal. However there is some evidence that this is never the case when X_Σ is smooth.

Question 6.4.3. *If X_Σ is a smooth projective toric variety, defined by a polytope P, and $\mathcal{A}$ is the configuration of lattice points inside the polytope $P \times \{1\} \subseteq \mathbb{R}^{d+1}$, is $\mathbb{N}\mathcal{A}$ a normal semigroup?*

This question is asking whether the corresponding embedding into projective space is projectively normal. When P is two-dimensional, the answer to the question is yes for any polytope (smooth or not). This follows from the fact that every polygon in the plane has a unimodular triangulation.

This is a special case of a question raised by Oda [Oda97]. In polytope language Oda's question is the following.

Question 6.4.4. *Let P be a smooth lattice polytope in $\mathbb{R}^d$, and let Q be a lattice polytope whose normal fan is a coarsening of that of P. Is every lattice point in $P+Q$ the sum of a lattice point in P and a lattice point in Q?*

Question 6.4.3 asks this question for $Q = kP$. In algebro-geometric language Question 6.4.4 asks whether the map

$$H^0(X_\Sigma, P) \times H^0(X_\Sigma, Q) \to H^0(X_\Sigma, P+Q)$$

is surjective, where $P, Q, P+Q$ in this statement denote the corresponding ample or nef divisors. Question 6.4.4 has been answered affirmatively by Fakhruddin [Fak02] when P and Q are two-dimensional.

6.4.3 Quadratic generation

Another question one can ask about smooth projective toric varieties is about the degrees of generators of their defining ideals.

Question 6.4.5. *Let X_Σ be a smooth toric variety, defined by a polytope P, and let $\mathcal{A}$ be the configuration of lattice points inside the polytope $P \times \{1\} \subseteq \mathbb{Z}^{d+1}$. Is $I_\mathcal{A}$ generated by quadrics? In other words, is the maximal degree of a minimal generator of $I_\mathcal{A}$ equal to two?*

This question is still open, and suspected to have a negative answer, even in the case where $\mathcal{A}$ defines a normal semigroup.

6.4.4 Higher syzygies

Questions about projective normality and quadratic generation are special cases of questions about whether the toric variety has the property N_p.

Definition 6.4.6. Let P be a lattice polytope in $\mathbb{Z}^d$, corresponding to the toric variety X_Σ. Let $\mathcal{A}$ be the set of lattice points in $P \times \{1\}$, let $C = \mathrm{pos}(\mathcal{A})$, and let D be the semigroup $C \cap \mathbb{Z}^{d+1}$. As before, we set S to be the polynomial ring with one generator for every lattice point in $P \times \{1\}$, and R to be the S-module $\mathbb{C}[D]$, with $x_i \cdot \mathbf{t}^\mathbf{v} = \mathbf{t}^{\mathbf{a}_i + \mathbf{v}}$.

Let

$$0 \longrightarrow F_k \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R \longrightarrow 0$$

be the minimal free resolution of R as an S-module, so each F_i is the direct sum of shifts of S.

We say the line bundle corresponding to P has property N_p if $F_0 = S$ and $F_i = \oplus S[-i-1]$ for $1 \leq i \leq p$.

Note that the condition N_0 means that the embedding into projective space defined by P is projectively normal ($\mathbb{N}\mathcal{A}$ is a normal semigroup, as all Hilbert basis elements of C are at height one). This means that $R = S/I_{\mathcal{A}}$. The condition N_1 means that $I_{\mathcal{A}}$ is quadratically generated.

Hering, Schenck, and Smith [HSS05] have recently proved the following theorem about when a toric variety has property N_p.

Theorem 6.4.7. *Let P be a lattice polytope in $\mathbb{Z}^d$, corresponding to the toric variety X_Σ. Then the line bundle corresponding to $(d-1+p)P$ has property N_p.*

Their proof, however, is not at all combinatorial. This leads to the following question.

Question 6.4.8. *Give a combinatorial proof of Theorem 6.4.7.*

This question has been answered in the case $p = 0, 1$. The result that the semigroup generated by the lattice points in $(d-1)P$ is normal (N_0) is a result of Ewald and Wessels [EW91]. Another proof is given in [LTZ93]. The fact that the when $\mathcal{A}$ is the set of lattice points in dP the ideal $I_{\mathcal{A}}$ is quadratically generated appears in [BGT97]. These proofs are all combinatorial, and inspired the work in [HSS05].

6.5 Further reading

One of the canonical introductions to toric varieties in their algebro-geometric formulation is Fulton's book [Ful93]. Another good reference, which does not assume any algebraic geometry background is Ewald [Ewa96]. The Cox homogeneous coordinate ring was introduced for simplicial toric varieties by Cox in [Cox95]. It was generalized to all toric varieties by Mustaţa [Mus02]. More about the connection between toric ideals and the toric varieties discussed here can be found in Chapter 13 of [Stu96], and [Stu97]. The latter contains some other (still) open questions about toric ideals.

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