JUMPING NUMBERS OF A SIMPLE COMPLETE IDEAL IN A TWO DIMENSIONAL REGULAR LOCAL RING

Eero Hyry, Tarmo Järvilehto

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Outline

A general set-up for multiplier ideals

- $(A, \mathfrak{m})$ a regular local ring
- $I \subset A$ an ideal
- $f : X \to \operatorname{Spec} A$ a log-resolution of I, i.e., a projective birational morphism with X regular and $I\mathcal{O}_X = \mathcal{O}_X(-D)$ where D is an effective Cartier divisor such that $D + \text{exceptional}(f)$ has simple normal crossing support
- $K_X = \operatorname{div} \mathcal{J}_{X/A}$ the relative canonical divisor

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Multiplier ideals

Definition

Let $c \geq 0$ be a rational number. The multiplier ideal with coefficient c is

$$\mathcal{J}(cI) = \Gamma(X, \mathcal{O}_X(K_X - \lfloor cD \rfloor))$$

where $\lfloor \quad \rfloor$ denotes the round-down of a divisor.

Observation

$$c < c' \Rightarrow \mathcal{J}(c'I) \subset \mathcal{J}(cI)$$

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An elementary lemma

Lemma (Ein–Lazarsfeld–Smith–Varolin)

There exists an inreasing discrete sequence $0 = c_0 < c_1 < c_2 < \dots$ *of rational numbers characterized by the properties that*

$$\mathcal{J}(cI) = \mathcal{J}(c_i I) \quad \text{for } c \in [c_i, c_{i+1}[$$

while

$$\mathcal{J}(c_{i+1} I) \subsetneq \mathcal{J}(c_i I)$$

for all i.

Jumping numbers

Definition

The numbers $c_1, c_2, \ldots$ are called the jumping numbers of I.

Remark

$$\begin{aligned} c_1 &= \sup\{c \in \mathbb{Q} \mid \mathcal{J}(cI) = A\} \\ &= \operatorname{lct}(I) \\ &= \textit{the log-canonical threshold of } I \end{aligned}$$

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Notation and terminology

- $(A, \mathfrak{m}, \Bbbk)$ a two-dimensional regular local ring with $\Bbbk = \bar{\Bbbk}$
- $I \subset A$ a simple complete $\mathfrak{m}$-primary ideal
- C an analytically irreducible plane curve

SIMPLE I is not a product of two proper ideals

COMPLETE I is integrally closed i.e. $I = \overline{I}$, where

$$\overline{I} = \{x \in A \mid x^n + \sum_{i=1}^{n} a_i x^{n-i} = 0, a_i \in I^i\}$$

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PLAIN CURVE $\operatorname{Spec} A/(f)$ where $f \in \mathfrak{m}$

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Simple ideals and plane curves

Zariski's point of view:

Complete ideals are "linear system of curves" satisfying "infinitely near base conditions".

Theorem (Zariski)

A complete ideal is simple iff a general element is analytically irreducible.

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Questions

- What are the jumping numbers of a simple complete ideal?
- How these are related to the jumping numbers of the curve determined by a general element?
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Simple ideals and simple sequences

Fact

simple complete ideals $\leftrightarrow$ finite simple sequences

$$\operatorname{Spec} A = X_1 \leftarrow X_2 \leftarrow \ldots \leftarrow X_n \leftarrow X_{n+1} = X$$

$$X_{i+1} = \text{the blow-up of } X_i \text{ at a closed point } x_i \in E_{i-1}$$
$$E_i = \text{the exceptional divisor of } X_{i+1} \to X_i$$

Now

$$x_i = \text{the unique point on } X_i \text{ where the transform of } I \text{ has positive order}$$

Simple ideals and valuations

Fact

simple complete ideals $\leftrightarrow$ divisorial valuations

$$I \mapsto v := \mathrm{ord}_{E_n} = \mathfrak{m}_{X_n,x_n}\text{-adic valuation}$$

Multiplicity sequence

Definition

The multiplicity sequence of I is $(m_1, \dots, m_n)$ where

$$m_i = \text{the order of the transform of } I \text{ at } x_i$$

Remark

Note that $m_n = 1$, because I is the unique simple complete ideal whose transform at x_n is $\mathfrak{m}_{X_n,x_n}$.

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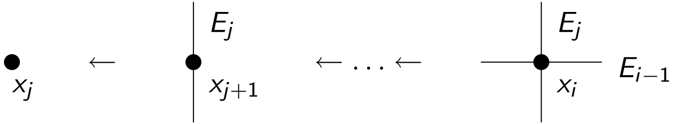
Proximity

Definition

x_i is proximate to x_j iff $x_i \in E_j$ and $j < i$.



- Clearly x_i is proximate to x_{i-1} if $i > 1$.
- If x_i is proximate to x_j with $j < i - 1$, then x_i is satellite to x_j.
- If x_i is a satellite but x_{i+1} not, then x_i is a terminal satellite.

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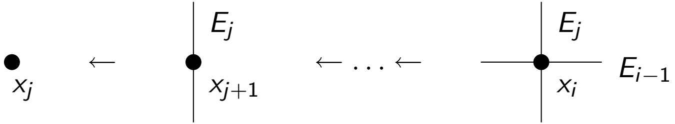

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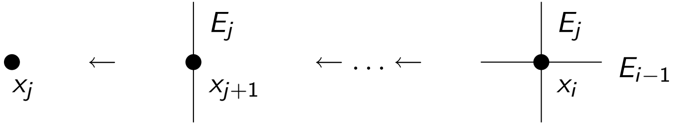

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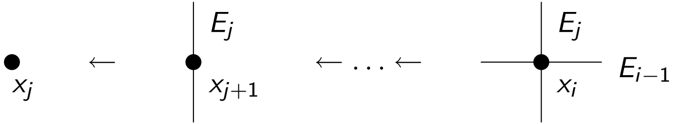

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Terminal satellites

Fact

If $x_{i+1}, \ldots, x_j$ are the points proximate to x_i, then

$$m_i = m_{i+1} + \ldots + m_j.$$

Moreover, $m_{i+1} = \ldots = m_{j-1} \geq m_j$ and $m_i > m_{j-1} = m_j$ iff x_j is a terminal satellite.

- Let $x_{\gamma_1}, \ldots, x_{\gamma_g}$ be the terminal satellites $\neq x_n$.
- Also set $\gamma_0 = 1$ and $\gamma_{g+1} = n$.

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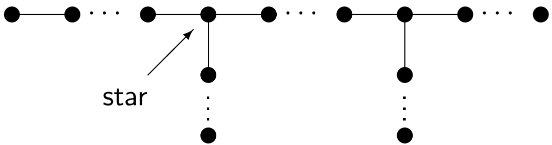
Dual graph

Definition

The dual graph of the resolution $X \to \operatorname{Spec} A$ has the exceptional divisors $E_1, \ldots, E_n \subset X$ as vertices. The vertices E_i and E_j are connected by an edge iff $E_i \cap E_j \neq \emptyset$.

The stars are $E_{\gamma_1}, \ldots, E_{\gamma_g}$:



The main result

Theorem

The jumping numbers of I are

- *for* $\nu = 0, \ldots, g-1$ *and* $s, t, k \in \mathbb{N}$

$$c + \frac{k}{m_{\gamma_{\nu+1}}}$$

where

$$c = \frac{s+1}{m_{\gamma_\nu}} + \frac{t+1}{(m_1^2 + \ldots + m_{\gamma_{\nu+1}}^2) : m_{\gamma_\nu}} \leq \frac{1}{m_{\gamma_{\nu+1}}}$$

- *for* $s, t \in \mathbb{N}$

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Remarks

- $(m_1^2 + \ldots + m_{\gamma_{\nu+1}}^2) : m_{\gamma_\nu} \in \mathbb{N}$
- Jumping numbers are periodic with period one (a general fact).
- $1 + \dfrac{1}{v(I)}$ is the least jumping greater than one (note that $v(I) = m_1^2 + \ldots + m_n^2$).
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The monomial case

Example

Suppose that $g = 0$ (i.e. I is monomial with respect to some regular parameters). The jumping numbers of I are then

$$\frac{s+1}{\operatorname{ord}(I)} + (t+1)\frac{\operatorname{ord}(I)}{v(I)} \quad (s, t \in \mathbb{N}).$$

The structure of the set $\mathcal{H}(I)$

Notation

- Set $\mathcal{H}(I) = \{\text{jumping numbers of } I\}$
- Set $\xi_\nu := \frac{1}{m_{\gamma_\nu}} + \frac{1}{(m_1^2 + \ldots + m_{\gamma_{\nu+1}}^2) : m_{\gamma_\nu}} \quad (\nu = 0, \ldots, g)$

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- $\xi_\nu = \min\{\xi \in \mathcal{H}(I) \mid \xi \geq \frac{1}{m_{\gamma_\nu}}\} \quad (\nu = 0, \ldots, g)$
- $\frac{1}{m_{\gamma_0}} < \xi_0 < \frac{1}{m_{\gamma_1}} < \xi_1 < \ldots < \frac{1}{m_{\gamma_g}} < \xi_g$
- $\frac{1}{m_{\gamma_\nu}} \notin \mathcal{H}(I)$ $(\nu = 0, \ldots, g+1)$

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Jumping numbers of plane curves

Theorem

Let C be an analytically irreducible plane curve whose strict transform intersects transversally at x_n every exceptional divisor going through x_n. The set of jumping numbers of C is then

$$\mathcal{H}(C) = \{\xi + k \mid \xi \in \mathcal{H}(I) \cup \{1\}, 0 < \xi \leq 1, k \in \mathbb{N}\}.$$

Conversely,

$$\mathcal{H}(I) = (\mathcal{H}(C) \setminus \{1\}) \cup \{1 + \frac{k+1}{v(I)} \mid k \in \mathbb{N}\}.$$

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Remarks

- This applies to a curve determined by a general element of I.
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Jumping numbers determine the order

Theorem

Let $\xi < \psi < \zeta$ be the three smallest jumping numbers. Then

$$\operatorname{ord}(I) = \begin{cases} \dfrac{5}{3\xi} & \text{if } 6\xi = 10\psi - 5\zeta; \\ \dfrac{1}{2\xi - \psi} & \text{if } 6\xi \neq 10\psi - 5\zeta. \end{cases}$$

Moreover,

- $\xi > 1$ *iff* $\operatorname{ord}(I) = 1$
- $\xi < 1 < \zeta$ *implies* $\operatorname{ord}(I) = 2$

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Example

- If the multiplicity sequence $= (2,1,1)$, then $\xi = \frac{5}{6}$, $\psi = \frac{7}{6}$ and $\zeta = \frac{8}{6}$ so that
$$6\xi = 10\psi - 5\zeta.$$
- If the multiplicity sequence $= (3,1,1,1)$, then $\xi = \frac{7}{12}$, $\psi = \frac{10}{12}$ and $\zeta = \frac{11}{12}$ so that
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Jumping numbers determine multiplicities

Corollary

The jumping numbers and the multiplicity sequence are equivalent data. In particular, for an analytically irreducible plane curve, the equisingularity class is determined by the jumping numbers.

Remark

In fact, already $\xi_0, \ldots, \xi_g$ determine the rest of the jumping numbers.

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The starting point of the proof

Definition

Let $J \subset A$ be any ideal. The log-canonical threshold of I w.r.t. J is

$$\operatorname{lct}(I; J) = \inf\{c > 0 \mid \mathcal{J}(cI) \not\supseteq J\}.$$

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Jumping numbers as lc–thresholds

Notation

For $R = (r_1, \ldots, r_n) \in \mathbb{N}^n$, set

$$J_R = \prod_{i=1}^{n} p_i^{r_i}$$

where $I = p_n \subset p_{n-1} \subset \ldots \subset p_1 = \mathfrak{m}$ denote the simple v-ideals.

Lemma

$$\mathcal{H}(I) = \{\mathrm{lct}(I; J_R) \mid R \in \mathbb{N}^n\}$$

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  the log-canonical threshold
- Vaquié (1994):
  Jumping numbers $\Leftrightarrow$ the Hodge spectrum
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- Vaquié (1994):
  Jumping numbers $\Leftrightarrow$ the Hodge spectrum
- Saito (2000): the Hodge spectrum