Hadamard Products and Binomial Ideals

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Abstract

We study the Hadamard product of two varieties V and W, with particular attention to the situation when one or both of V and W is a binomial variety. The main result of this paper shows that when V and W are both binomial varieties, and the binomials that define V and W have the same binomial exponents, then the defining equations of $V \star W$ can be computed explicitly and directly from the defining equations of V and W. This result recovers known results about Hadamard products of binomial hypersurfaces and toric varieties. Moreover, as an application of our main result, we describe a relationship between the Hadamard product of the toric ideal I_G of a graph G and the toric ideal I_H of a subgraph H of G. We also derive results about algebraic invariants of Hadamard products: assuming V and W are binomial with the same exponents, we show that $\deg(V \star W) = \deg(V) = \deg(W)$ and $\dim(V \star W) = \dim(V) = \dim(W)$. Finally, given any (not necessarily binomial) projective variety V and a point $p \in \mathbb{P}^n \setminus \mathbb{V}(x_0x_1\cdots x_n)$, subject to some additional minor hypotheses, we find an explicit binomial variety that describes all the points q that satisfy $p \star V = q \star V$.

Definitions and Preliminaries

Let $\mathbb{P}^n$ denote the projective space over the algebraically closed field k of dimension n, with homogeneous coordinates $[x_0 : x_1 : \cdots : x_n]$, and $X, Y \subseteq \mathbb{P}^n$ be projective varieties. The *Hadamard product* of X and Y is given by

$$X \star Y := \overline{\{p \star q \mid p \in X,\ q \in Y,\ p \star q \text{ is defined}\}} \subseteq \mathbb{P}^n$$

where $p \star q := [p_0q_0 : \cdots : p_nq_n]$ is the point obtained by component-wise multiplication of the points $p = [p_0 : \cdots : p_n]$ and $q = [q_0 : \cdots : q_n]$, and $p \star q$ is defined precisely when there exists at least one index i, $0 \leq i \leq n$, with $p_iq_i \neq 0$ (so that $p \star q = [p_0q_0 : \cdots : p_nq_n]$ is a valid point in $\mathbb{P}^n$).

Two binomial varieties V and W *have the same binomial exponents* if there are two ordered subsets $\{\alpha_1, \ldots, \alpha_s\}$ and $\{\beta_1, \ldots, \beta_s\}$ of $\mathbb{N}^{n+1}$ such that $\alpha_i \neq \beta_i$ for all $i = 1, \ldots, s$, the pairs (α_i, β_i) of exponents are pairwise distinct for all $i = 1, \ldots, s$, and there are nonzero constants $a_1, \ldots, a_s, b_1, \ldots, b_s, c_1, \ldots, c_s, d_1, \ldots, d_s \in k \setminus \{0\}$ such that

$$\mathbb{I}(V) = \langle a_1X^{\alpha_1} - b_1X^{\beta_1}, a_2X^{\alpha_2} - b_2X^{\beta_2}, \ldots, a_sX^{\alpha_s} - b_sX^{\beta_s}\rangle$$

and

$$\mathbb{I}(W) = \langle c_1X^{\alpha_1} - d_1X^{\beta_1}, c_2X^{\alpha_2} - d_2X^{\beta_2}, \ldots, c_sX^{\alpha_s} - d_sX^{\beta_s}\rangle.$$

Let $G = (V, E)$ be a finite simple graph with vertices $V = \{v_1, \ldots, v_n\}$ and edges $E = \{e_1, \ldots, e_q\}$. Consider the ring homomorphism $\varphi \colon k[e_1, \ldots, e_q] \to k[v_1, \ldots, v_n]$ defined by

$$e_i \mapsto \varphi(e_i) := v_{i_1}v_{i_2} \quad \text{for all } e_i = \{v_{i_1}, v_{i_2}\},\ 1 \leq i \leq q.$$

The *toric ideal of* G, denoted I_G, is defined to $\ker(\varphi)$, the kernel of φ.

Given a finite simple graph G and a subgraph H of G, let $E = \{e_1, \ldots, e_q\}$ be the set of edges of G and $E' = \{e_{i_1}, \ldots, e_{i_r}\} \subseteq E$ be the set of edges of H. We have that $I_G \subseteq k[e_1, \ldots, e_q]$ and $I_H \subseteq k[e_{i_1}, \ldots, e_{i_r}]$. There is a natural inclusion Ψ from the ambient ring $k[e_{i_1}, \ldots, e_{i_r}]$ of I_H into $k[e_1, \ldots, e_q]$, so we consider the natural extension I_H^e of I_H to $k[e_1, \ldots, e_q]$, defined by $I_H^e := \langle \Psi(I_H)\rangle$.

Let V be any projective variety in $\mathbb{P}^n$ and let $p = [p_0 : \cdots : p_n] \in \mathbb{P}^n$ denote a fixed point in $\mathbb{P}^n$ such that $p_0 \cdots p_n \neq 0$. We call $p \star V$ the *Hadamard transformation of* V *by* p. We define the set

$$\psi(V, p) := \{q \in \mathbb{P}^n \mid q \star V = p \star V\} \subseteq \mathbb{P}^n$$

which is the set of points in $\mathbb{P}^n$ which yield the same Hadamard product with V as for p.

Results

Theorem 1: Let V and W be binomial varieties of $\mathbb{P}^n$. Assume that V and W have the same binomial exponents. In addition, suppose that V or W contains a point $p = [p_0 : \cdots : p_n]$ with $p_0 \cdots p_n \neq 0$. Then $V \star W$ is also a binomial variety that has the same binomial exponents as V and W. More precisely, if

$$\mathbb{I}(V) = \langle a_1X^{\alpha_1} - b_1X^{\beta_1}, a_2X^{\alpha_2} - b_2X^{\beta_2}, \ldots, a_sX^{\alpha_s} - b_sX^{\beta_s}\rangle$$

and

$$\mathbb{I}(W) = \langle c_1X^{\alpha_1} - d_1X^{\beta_1}, c_2X^{\alpha_2} - d_2X^{\beta_2}, \ldots, c_sX^{\alpha_s} - d_sX^{\beta_s}\rangle,$$

then

$$\mathbb{I}(V \star W) = \langle a_1c_1X^{\alpha_1} - b_1d_1X^{\beta_1}, a_2c_2X^{\alpha_2} - b_2d_2X^{\beta_2}, \ldots, a_sc_sX^{\alpha_s} - b_sd_sX^{\beta_s}\rangle.$$

Example 2: Let $R = k[x, y, z]$ be the associated coordinate ring, and suppose that

$$I = \langle x^3 - 2y^2z\rangle \text{ and } J = \langle x^3 - 2yz^2\rangle.$$

Note that the exponents that appear in the binomial generators of I and J are not the same. We can compute $I \star J$ using *Macaulay2* [1] to find that $I \star J = \langle 0\rangle$. If $V = \mathbb{V}(I)$ and $W = \mathbb{V}(J)$, we thus have

$$V \star W = \mathbb{V}(I) \star \mathbb{V}(J) = \mathbb{V}(I \star J) = \mathbb{P}^2.$$

Note that I and J do not have the same exponents, so Theorem 1 does not apply. On the other hand, certainly the ideal I has the same binomial exponents as itself. If we compute $I \star I$ using *Macaulay2*, we get $I \star I = \langle x^3 - 4y^2z\rangle$. This agrees with the conclusion of Theorem 1.

Corollary 3: Let V and W be binomial varieties of $\mathbb{P}^n$. Assume that V and W have the same binomial exponents. Let $V' \subseteq V$ be any subvariety. If V' contains a point $p = [p_0 : \cdots : p_n]$ with $p_0 \cdots p_n \neq 0$, then

$$p \star W = V' \star W = V \star W.$$

Theorem 4: Let G be a finite simple graph with edge set $E = \{e_1, \ldots, e_q\}$ and suppose that H is a subgraph of G with edge set $E' = \{e_{i_1}, \ldots, e_{i_r}\}$. Let $I_G \subseteq k[e_1, \ldots, e_q]$ be the toric ideal of G, and let $I_H \subseteq k[e_{i_1}, \ldots, e_{i_r}]$ denote the toric ideal of H. If I_H^e is the extension of I_H defined above, then $I_G \star I_H^e = I_H^e$.

Theorem 5: Let $V \subseteq \mathbb{P}^n$ be a nonempty projective variety. Suppose that $\mathbb{I}(V) : \langle x_0x_1 \cdots x_n\rangle = \mathbb{I}(V)$. Let $p = [p_0 : \cdots : p_n] \in \mathbb{P}^n$ be a point with $p_0 \cdots p_n \neq 0$. Let $<$ be a monomial order on $k[x_0, \ldots, x_n]$ and let $\mathcal{G} = \{f_1, \ldots, f_m\}$ denote a reduced Gröbner basis for $\mathbb{I}(V)$ with respect to $<$. Assume that every element of the Gröbner basis is of degree ≥ 2. For each $i = 1, \ldots, m$, write

$$f_i = X^{\alpha_{1,i}} - a_{2,i}X^{\alpha_{2,i}} - \cdots - a_{k_i,i}X^{\alpha_{k_i,i}} \text{ where } LT(f_i) = X^{\alpha_{1,i}}$$

where we assume $X^{\alpha_{k_i,i}} < \cdots < X^{\alpha_{1,i}}$ with respect to $<$ so that $X^{\alpha_{1,i}}$ is the leading term of f_i. Let

$$J := \langle b_{1,i}X^{\alpha_{1,i}} - b_{\ell,i}X^{\alpha_{\ell,i}} \mid \text{for each } i = 1, \ldots, m \text{ and } 1 < \ell \leq k_i\rangle$$

where the constants $b_{j,i}$ are chosen so that they satisfy the equations $b_{1,i}p^{\alpha_{1,i}} = b_{\ell,i}p^{\alpha_{\ell,i}}$ for all $1 \leq i \leq m$. If $\mathbb{I}(\mathbb{V}(J)) : \langle x_0x_1 \cdots x_n\rangle = J$, then $J = \sqrt{J} = \mathbb{I}(\psi(V, p))$. In particular, under the hypotheses above, the ideal $\mathbb{I}(\psi(V, p))$ is a binomial ideal, and a set of generators of $\mathbb{I}(\psi(V, p))$ can be computed via a reduced Gröbner basis of $\mathbb{I}(V)$.

Example 6: Let $V = \mathbb{V}(x^2 - xy - yz) \subseteq \mathbb{P}^3$, and $p = [1 : 2 : 3 : 4]$. Thus, $\mathbb{I}(V) = \langle x^2 - xy - yz\rangle$, a principal ideal of $R = k[x, y, z, w]$. Let $>$ be the lexicographical monomial order given by $x > y > z > w$. Since $\mathbb{I}(V)$ is principal, and the leading coefficient of $f = x^2 - xy - yz$ is 1, we can conclude that $\mathcal{G} = \{f\}$ is a reduced Gröbner basis for $\mathbb{I}(V)$. Furthermore, one can verify by using *Macaulay2* that $\mathbb{I}(V) : \langle xyzw\rangle = \mathbb{I}(V)$, so the above theorem applies. As per Theorem 5, the binomials which generate $\mathbb{I}(\psi(V, p))$ are of the form $g_1 = a_1x^2 - b_1xy$ and $g_2 = a_2x^2 - b_2yz$. We solve for the coefficients by substituting $x = 1$, $y = 2$, $z = 3$, and $w = 4$. We have $2b_1 = a_1$ and $6b_2 = a_2$. Therefore, $\mathbb{I}(\psi(V, p)) = \langle x^2 - (1/2)xy, x^2 - (1/6)yz\rangle$.

Example: Hadamard Product of Toric Ideals

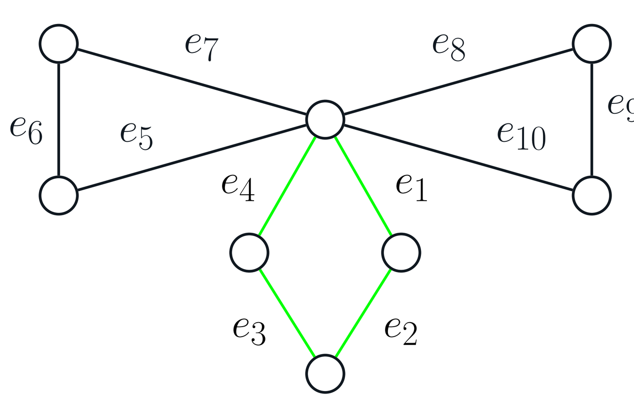
A *walk* on a finite simple graph G is simply a sequence $(e_1, \ldots, e_n)$ of adjacent edges of G. A walk is called *even* if n is even, and called *closed* if the second vertex of e_n and the first vertex of e_1 coincide (we view each edge in the walk as an *ordered* pair of vertices). Villarreal [2] showed that the generators of the toric ideal I_G correspond to the closed even walks of G:

Theorem 7 (Villarreal): Let $\Gamma = (e_{i_1}, \ldots, e_{i_{2m}})$ be a closed even walk of a finite simple graph G. Define the binomial

$$f_\Gamma = \prod_{2 \nmid j} e_{i_j} - \prod_{2 \mid j} e_{i_j}.$$

Then I_G is generated by all the binomials f_Γ, where Γ is a closed even walk of G.

Example 8: Consider the graph G below and the subgraph H of G highlighed in green.



By the previous theorem, we have

$$I_G = \langle e_1e_3 - e_2e_4, e_1e_3e_5e_7e_9 - e_2e_4e_6e_8e_{10}, e_5e_7e_9 - e_6e_8e_{10}\rangle$$

and

$$I_H^e = \langle e_1e_3 - e_2e_4\rangle.$$

Using *Macaulay2*, we find that

$$I_G \star I_H^e = \langle e_1e_3 - e_2e_4\rangle = I_H^e,$$

which agrees with the conclusion of Theorem 4.

Remarks

The Hadamard product of two varieties V and W is well know and easily computable when one of the varieties is a single point $p = [x_0 : \cdots : x_n] = V$ with no zero homogeneous coordinates [3]. Many of our results come from identifying when all points in a projective variety give the same Hadamard transformation, i.e., showing that $p \star V = q \star V$ for all $p, q \in W \setminus \mathbb{V}(x_0 \ldots x_n)$.

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References

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The full list of references for our work can be found in our preprint, *Hadamard products and binomial ideals*, available at `arXiv:2211.14210`.