Alexander Duals of Symmetric Simplicial Complexes and Stanley–Reisner Ideals

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January 21, 2023
CAAC Conference, University of Waterloo

Outline

1. Sym-invariant chains of ideals
2. Sym-invariant chains of squarefree monomial ideals
3. Our work

Sym-invariant chains of ideals

Fix a field $\mathbf{k}$, $c \geq 1$. $R_n := \mathbf{k}[x_{i,j} : 1 \leq i \leq c, 1 \leq j \leq n]$.

$\mathrm{Sym}(n)$, the symmetric group, acts on R_n via $\sigma \cdot x_{i,j} := x_{i,\sigma(j)}$.

An ideal I_n of R_n is called Sym(n)-**invariant** if $\mathrm{Sym}(n)(I_n) \subseteq I_n$.

We consider ascending chains

$$I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$$

of ideals with $I_n \subset R_n$ for each n. This is a Sym-**invariant chain** if

- Each I_n is $\mathrm{Sym}(n)$-invariant, and
- $\mathrm{Sym}(m)(I_n) \subseteq I_m$ whenever $m > n$.

Sym-Noetherianity

Set $R = \varinjlim R_n = \mathbf{k}[x_{i,j} : 1 \leq i \leq c, 1 \leq j]$.

$\text{Sym} = \bigcup_{n \geq 1} \text{Sym}(n)$ acts on R.

For a Sym-invariant chain $(I_n)_{n \geq 1}$, let $I = \varinjlim I_n$.

- This is a Sym-**invariant ideal**.

Conversely, a Sym-invariant ideal I of R gives a Sym-invariant chain via $I_n := I \cap R_n$.

Theorem (Cohen '87, Aschenbrenner–Hillar '07, Hillar–Sullivant '12)

*The ring R is Sym-**Noetherian**, meaning that every Sym-invariant ideal I of R is generated by finitely many Sym-orbits of polynomials.*

Motivating question: What can we say about algebraic properties of I_n as $n \to \infty$?

Asymptotic properties of Sym-invariant chains

Nagel–Römer (2017): Defined Hilbert series for Sym-invariant chains and showed rationality $\rightarrow$ $\mathrm{ht}(I_n)$ eventually linear, $\deg(I_n)$ eventually exponential.

Le–Nagel–Nguyen–Römer (2019): $\mathrm{reg}(I_n)$ bounded by linear function; conjectured equality.

Murai (2019): Described Betti tables for monomial Sym-invariant chains when $c = 1$.

Draisma–Eggermont–Farooq (2021): If $(I_n)_{n\geq 1}$ is a Sym-invariant chain of ideals, the number of $\mathrm{Sym}(n)$-orbits of primary components of I_n is eventually a quasi-polynomial in n.

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Theorem (A,B,J-K,N,P 2022)

If $(I_n)_{n\geq 1}$ is a Sym*-invariant chain of* **squarefree monomial ideals**, *the number of* $\mathrm{Sym}(n)$*-orbits of primary components of I_n is eventually a* **polynomial** *in n.*

Stanley–Reisner correspondence

There is a bijection

$$\left\{\begin{matrix}\text{squarefree monomial}\\ \text{ideals of } R_n\end{matrix}\right\} \overset{S-R}{\longleftrightarrow} \left\{\begin{matrix}\text{simplicial complexes on}\\ \text{vertex set } x_{i,j}, 1 \leq i \leq c, 1 \leq j \leq n\end{matrix}\right\}$$

$$I \longleftrightarrow \{F : x^F \notin I\}$$

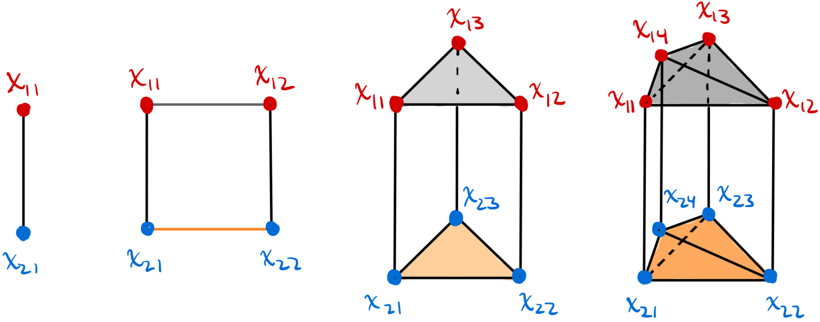
$$\langle x^F : F \notin \Delta\rangle \longleftrightarrow \Delta$$

Key fact: I_n a Sym(n)-invariant squarefree monomial ideal $\implies$

$\Delta(I_n)$ a Sym(n)-invariant simplicial complex.

A cute example

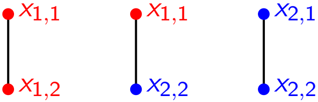
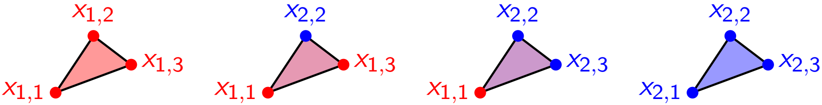
Let $c = 2$, $I_n = \langle \mathrm{Sym}(n) \cdot x_{1,1}x_{2,2} \rangle = \langle x_{1,i}x_{2,j} : i \neq j \rangle \subset R_n$ for $n \geq 2$.



For all $n \geq 2$, there are **3** $\mathrm{Sym}(n)$-orbits of facets in $\Delta(I_n)$.

A more complicated example

Let $c = 2$, $I_n = \langle \mathrm{Sym}(n) \cdot x_{1,1}x_{2,1} \rangle = \langle x_{1,i}x_{2,i} : 1 \leq i \leq n \rangle$ for all $n \geq 1$.

n	Facets of $\Delta(I_n)$ up to symmetry
1	
2	
3	
n	$x_{1,1} \cdots x_{1,k} x_{2,k+1} \cdots x_{2,n}, \quad 0 \leq k \leq n$

The complex $\Delta(I_n)$ has $\mathbf{n+1}$ $\mathrm{Sym}(n)$-orbits of facets for all $n \geq 1$.

Alexander duality

What is the algebraic interpretation of facets? The **Alexander dual** $I_n^\vee$ is another squarefree monomial ideal such that

$$\left\{\begin{matrix}\text{minimal}\\ \text{generators of } I_n^\vee\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{facets of}\\ \Delta(I_n)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{primary}\\ \text{components of } I_n\end{matrix}\right\}$$

Intuitively, monomials in $I_n^\vee$ **share a factor** with all generators of I_n.

In our example with $I_n = \langle \mathrm{Sym}(n) \cdot x_{1,1}x_{2,2} \rangle$:

n	$I_n^\vee$
1	0
2	$\langle x_{1,1}x_{1,2},\ x_{1,1}x_{2,1},\ x_{2,1}x_{2,2} \rangle$
3	$\langle x_{1,1}x_{1,2}x_{1,3},\ \mathrm{Sym}(3) \cdot x_{1,1}x_{1,2}x_{2,1}x_{2,2},\ x_{2,1}x_{2,2}x_{2,3} \rangle$
4	$\langle x_{1,1}x_{1,2}x_{1,3}x_{1,4},\ \mathrm{Sym}(4) \cdot x_{1,1}x_{1,2}x_{1,3}x_{2,1}x_{2,2}x_{2,3},\ x_{2,1}x_{2,2}x_{2,3}x_{2,4} \rangle$

Facet counts up to symmetry

Theorem (A,B,J-K,N,P 2022)

Let $(I_n)_{n\geq 1}$ be a Sym*-invariant chain of squarefree monomial ideals. Then the number of* $\mathrm{Sym}(n)$*-orbits of facets of $\Delta(I_n)$ grows eventually polynomially in n.*
Moreover, the degree of this polynomial is at most $\binom{c}{\lfloor \frac{c}{2} \rfloor} - 1$.

Corollary (A,B,J-K,N,P 2022)

For each fixed i, the number of $\mathrm{Sym}(n)$*-orbits of i-faces of $\Delta(I_n)$ grows eventually polynomially.*

Ingredients in the proof of the theorem:

- Generators of $I_n^\vee$ can be constructed with **upper order ideals** of $2^{[c]}$.
- Minimal generators of $I_n^\vee$ correspond to **integer points in a convex polyhedron**, which can be counted with Ehrhart theory.

Monomials to matrices

We can encode a squarefree monomial in R_n with a $c \times n$ $0-1$ matrix:

$$x_{1,1}x_{2,2} \text{ in } R_2 \longleftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_{1,1}x_{2,2} \text{ in } R_3 \longleftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Alexander dual membership can be checked by looking at these support matrices, e.g.

$$\mathbf{x}^{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]}, \mathbf{x}^{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]} \in \langle x_{1,1}x_{2,2} \rangle^{\vee}$$

$$\mathbf{x}^{\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]}, \mathbf{x}^{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]} \notin \langle x_{1,1}x_{2,2} \rangle^{\vee}.$$

Introducing symmetry

$$\begin{pmatrix}\text{Sym(2)-orbit} \\ \text{of } x_{1,1}x_{2,2}\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{all column} \\ \text{permutations of}\end{pmatrix.\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\Bigg)$$

The Sym(n)-orbit of a squarefree monomial $\mathbf{x}^A$ is determined by its multiset of **column supports**, which are subsets of $[c]$.

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To each Sym(n)-orbit of a squarefree monomial $\mathbf{x}^A$, we assign a vector

$$(z_T : T \in 2^{[c]})$$

such that the matrix A has z_T columns with support exactly equal to T.

The Sym(2)-orbit of $x_{2,1}x_{2,2}$ has $z_{\{2\}} = 2$, $z_\varnothing = z_{\{1\}} = z_{\{1,2\}} = 0$.

Description of the minimal generators for one orbit

Let $I_n = \langle \mathrm{Sym}(n) \cdot \mathbf{x}^A \rangle \subset R_n$ for $n \gg 0$.

For each antichain $\mathcal{C}$ in $2^{[c]}$, there is a set of minimal generators $M\mathcal{G}_{\mathcal{C}}(n)$ with its elements satisfying:

- their column supports lie exactly in $\mathcal{C}$,
- the number of nonzero columns grows with n,
- lower bounds on column supports; one for each subset of $\mathcal{C}$ (not depending on n).

Theorem (A,B,J-K,N,P 2022)

The union $\bigcup_{\mathcal{C}} M\mathcal{G}_{\mathcal{C}}(n)$ *minimally generates* $I_n^{\vee}$ *for* $n \gg 0$.

Minimal generating set example

Let $I_n = \langle \mathrm{Sym}(n) \cdot x_{1,1}x_{2,2} \rangle = \left\langle \mathrm{Sym}(n) \cdot \mathbf{x}^{\left[\begin{smallmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{smallmatrix}\right]} \right\rangle$ for $n \geq 2$.

$\mathcal{C}$	$M\mathcal{G}_{\mathcal{C}}(n)$
$\{\{1\}\}$	$\left\{ \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\}$
$\{\{2\}\}$	$\left\{ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \right\}$
$\{\{1\}, \{2\}\}$	$\varnothing$
$\{\{1, 2\}\}$	$\left\{ \begin{bmatrix} 1 & \cdots & 1 & 0 \\ 1 & \cdots & 1 & 0 \end{bmatrix} \right\}$

These matrices correspond to the same 3 $\mathrm{Sym}(n)$-orbits we saw earlier.

Counting minimal generators

$M\mathcal{G}_\mathcal{C}(n)$ is defined by **inequalities** $\to$ we can make a **polyhedron** $\mathcal{P}_\mathcal{C}$.

(Integer points $\mathbf{z} \in \mathcal{P}_\mathcal{C}$) $\longleftrightarrow$ (elements of $M\mathcal{G}_\mathcal{C}(n)$ for some n).

Intersecting $\mathcal{P}_\mathcal{C}$ with a hyperplane and taking integer points gives elements of $M\mathcal{G}_\mathcal{C}(n)$.

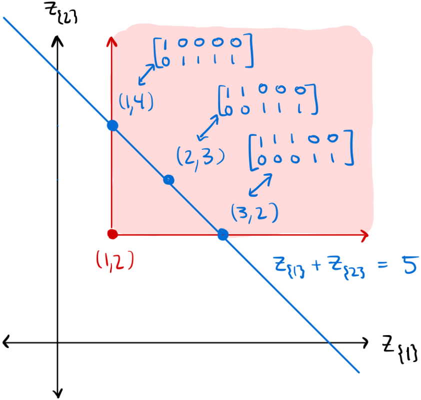
Theorem (A,B,J-K,N,P 2022)

If $\mathcal{P}_\mathcal{C}$ is nonempty, then there exist disjoint, pointed, rational cones $C_1, \ldots, C_t \in \mathbb{R}^{|\mathcal{C}|}$ with integral apices such that

$$\mathcal{P}_\mathcal{C} \cap \mathbb{Z}^{|\mathcal{C}|} = \bigsqcup_{i=1}^{t} \left(C_i \cap \mathbb{Z}^{|\mathcal{C}|} \right)$$

Polyhedron example

Consider $I_n = \left\langle \mathrm{Sym}(n) \cdot \mathbf{x}^{\left[\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right]} \right\rangle$, $\mathcal{C} = \{\{1\},\{2\}\}$, and $n = 5$.



Inequalities defining $\mathcal{P}_{\mathcal{C}}$:

$$z_{\{1\}} \geq 1$$
$$z_{\{2\}} \geq 2$$

Hyperplane to get $M\mathcal{G}_{\mathcal{C}}(n)$:

$$z_{\{1\}} + z_{\{2\}} = n$$

Putting the ideas together

The sets $M\mathcal{G}_{\mathcal{C}}(n)$ are disjoint for each n.

Using the cone decomposition, the integer points of $\mathcal{P}_{\mathcal{C}} \cap$ (hyperplane) can be counted with the Ehrhart polynomial.

The degree of each Ehrhart polynomial is $|\mathcal{C}| - 1 \leq \binom{c}{\lfloor \frac{c}{2} \rfloor} - 1$.

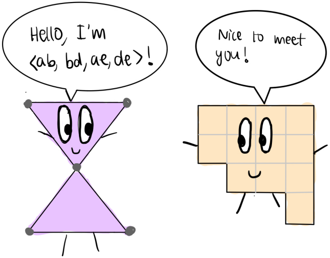
For $I_n = \langle \mathrm{Sym}(n) \cdot \mathbf{x}^{A_1}, \ldots, \mathbf{x}^{A_s} \rangle$:

- Minimal generators are indexed by pairs $(\mathcal{C}, \mathcal{F})$ of antichains $\mathcal{C}$ and solutions $\mathcal{F}$ to a system of inequalities associated to an s-tuple of order ideals $(J_1, \ldots, J_s)$.
- Polyhedron $\mathcal{P}_{\mathcal{C},\mathcal{F}}$ is defined by additional inequalities (bounding above).
- We use inclusion-exclusion to count minimal generators.

Thank you!

arXiv: 2209.14132

Special thanks to Alessio D'Alì, Mariel Supina, and Lorenzo Venturello for organizing the REACT 2021 online workshop, and to Aida Maraj for TA'ing our minicourse!



Combinatorial algebra meeting algebraic combinatorics