A new family of bijections between nonnesting partitions and noncrossing partitions

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Nonnesting partitions

A set partition π of $[n]$ is said to be *nonnesting (of type A)* if there is no 4-tuple (i,j,k,l) such that $1 \leqslant i < j < k < l \leqslant n$ and two distinct blocks $A, B \in \pi$ with $i, l \in A$ and $j, k \in B$.

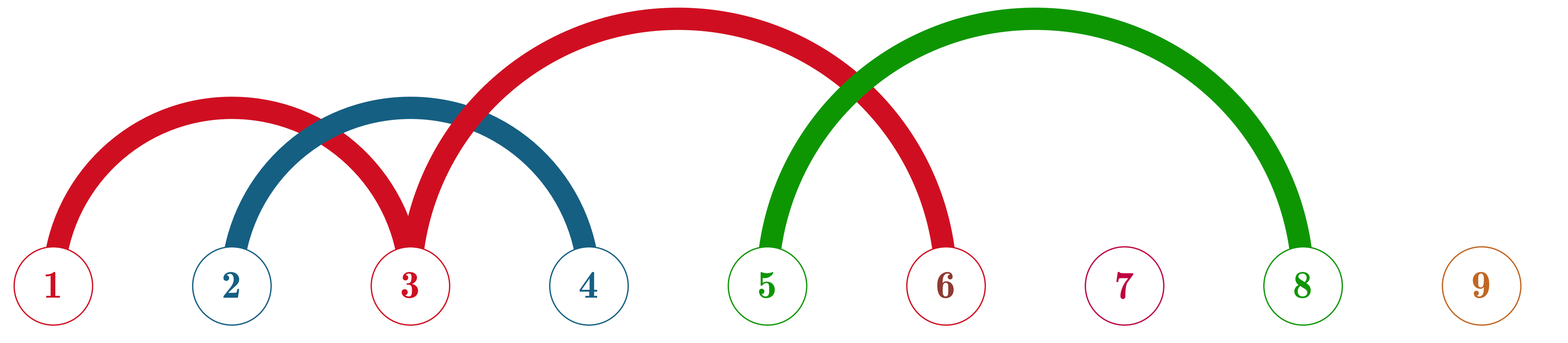

Figure 1. Example of a nonnesting partition of $[9]$.

We will see them throughout the rest of this poster as ideals in the (type A) root poset with which they are in bijection.

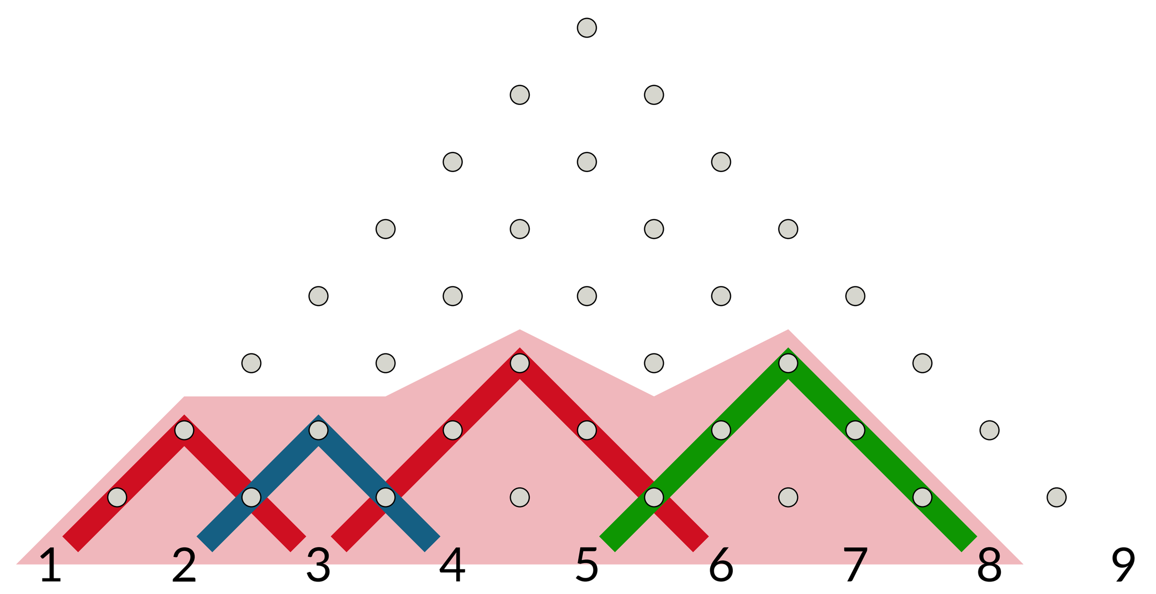

Figure 2. Ideal in the type A root poset associated to the above nonnesting partition

Coxeter element and c-sorting word for $w_\circ$

Consider our type A Weyl group :

$S_n = \langle s_1, \ldots, s_{n-1} \mid s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = 1$ for $|i-j| > 1\rangle$ where $s_i = (i, i+1)$.

A *Coxeter element* is an element $c \in S_n$ which could be written as a product of the s_i's where each s_i appears exactly once. Note also that any of them can be written as one long cycle $(1, w_1, \ldots, w_m, n, w_{m+1}, \ldots, w_{n-2})$ where

$$1 < w_1 < \ldots < w_m < n > w_{m+1} > \ldots > w_{n-2} > 1.$$

$$c = (1,3,4,7,9,8,6,5,2) = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7.$$

Figure 3. Example of a Coxeter element in S_9.

Given a reduced expression $\mathsf{c} = [r_1, \ldots, r_n]$ of a Coxeter element c, we can define a *c-sorting word for $w_\circ$*, the longest element in S_n, to be the leftmost reduced word $\mathsf{w}_\circ(c)$ in c^∞. Note that $\mathsf{w}_\circ(c)$ does not depend on the choice of the reduced word for c. Write $\mathsf{w}_\circ(c) = [t_1, \ldots, t_N]$ where the t_i's are distinct transpositions in S_n and $N = \binom{n}{2}$. For each j, we defined the *inversion*

$$\alpha^{(j)} = t_1 \ldots t_{j-1}(\alpha_{t_j})$$

where α_t is the simple root corresponding to the adjacent transposition t. Since $w_\circ$ has every positive root as an inversion, each positive root appears exactly once in the inversion sequence $\mathrm{inv}(\mathsf{w}_\circ(\mathsf{c})) = [\alpha^{(1)}, \ldots, \alpha^{(N)}]$.

$$\mathsf{w}_\circ(\mathsf{c}) = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 | s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 | s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 | s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 | s_2 s_6 s_5 s_8$$

$$\begin{aligned}\mathrm{inv}(\mathsf{w}_\circ(\mathsf{c})) = [\ &(23),\ (13),\ (24),\ (67),\ (57),\ (27),\ (89),\ (69),\\ &(14),\ (34),\ (17),\ (59),\ (29),\ (19),\ (68),\ (58),\\ &(37),\ (47),\ (39),\ (28),\ (18),\ (38),\ (56),\ (26),\\ &(49),\ (79),\ (48),\ (16),\ (36),\ (46),\ (25),\ (15),\\ &(78),\ (98),\ (76),\ (35),\ (45),\ (72),\ (12)\ ].\end{aligned}$$

Figure 4. Example of a c-sorting word for $w_\circ$ with the reduced expression of c given in the previous figure.

Kroweras complement [DFISTW22]

Let $c \in S_n$ be a Coxeter element. The *c-Kroweras complement* is an action on nonnesting partitions, seen as an ideal in the root poset, defined by a sequence of toggles determined by the inversion sequence: if $\mathrm{inv}(\mathsf{w}_\circ(\mathsf{c})) = [\alpha^{(1)}, \ldots, \alpha^{(N)}]$, then

$$\mathrm{Krow}_c = \mathrm{tog}_{\alpha^{(N)}} \circ \ldots \circ \mathrm{tog}_{\alpha^{(1)}}$$

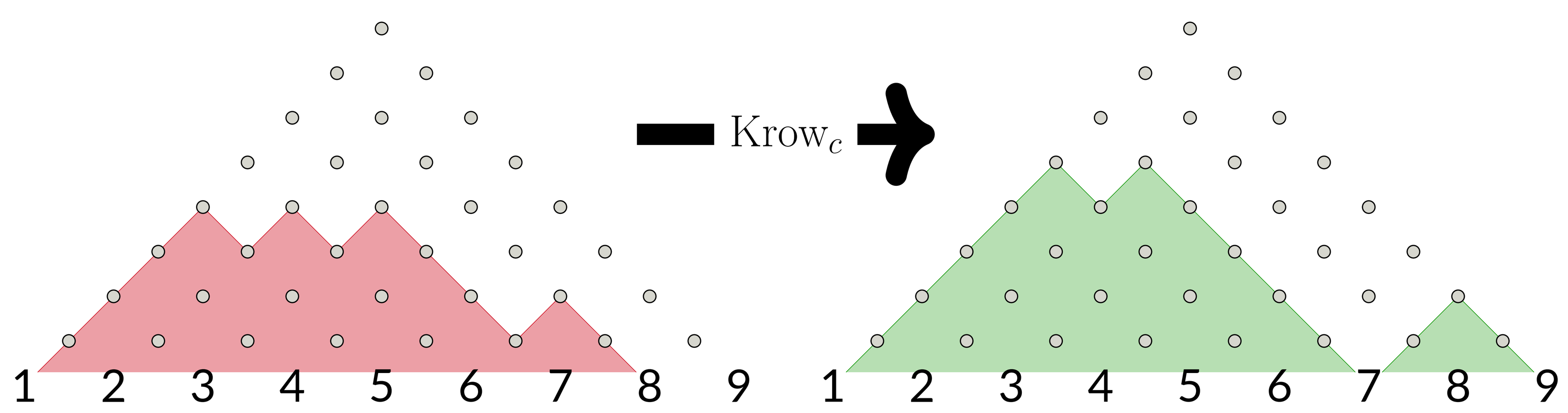

Figure 5. Example of application of Krow_c for $c = (1,3,4,7,9,8,6,5,2)$.

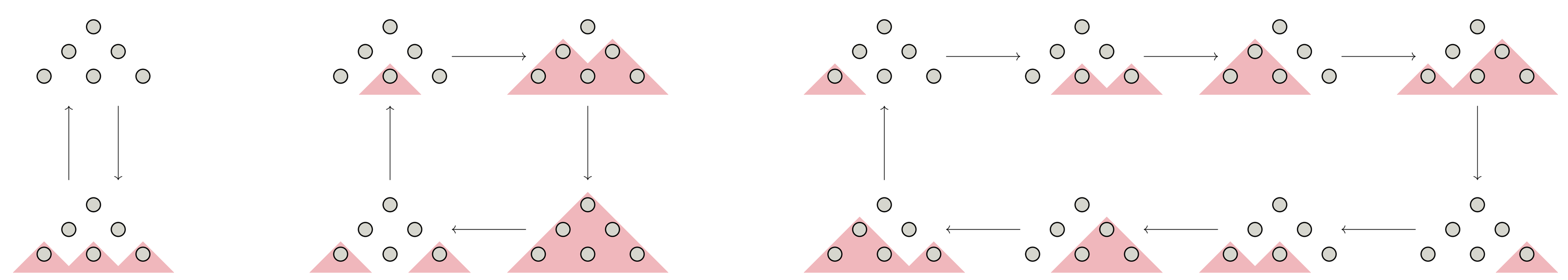

Figure 6. The orbits of Krow_c for $c = (1,2,4,3) = s_1 s_3 s_2$, where the c-Kroweras complement is given by $\mathrm{inv}(\mathsf{w}_\circ(\mathsf{c})) = [(34), (12), (14), (13), (24), (23)]$

Noncrossing partitions

Let $c \in S_n$ be a Coxeter element. A set partition χ of $[n]$ is said to be *c−noncrossing (of A-type)* if there are no integer 4-tuples (i,j,k,l) such that $0 \leqslant i < j < k < l \leqslant n-1$ and two distinct blocks $A, B \in \chi$ with $c^i(1), c^k(1) \in A$ and $c^j(1), c^l(1) \in B$

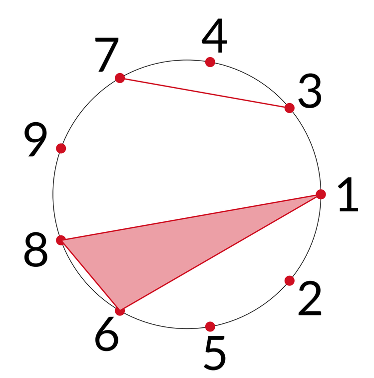

Figure 7. A geometric (and prettier) way to represent c-noncrossing partition for $c = (1,3,4,7,9,8,6,5,2)$.

Note that a c-noncrossing partition can also be defined as an element of $[1;c]_T$ in the poset S_n ordered by the absolute length.

Kreweras complement

We illustrate in the following figure the way we define the c-Kreweras complement for c-noncrossing partition.

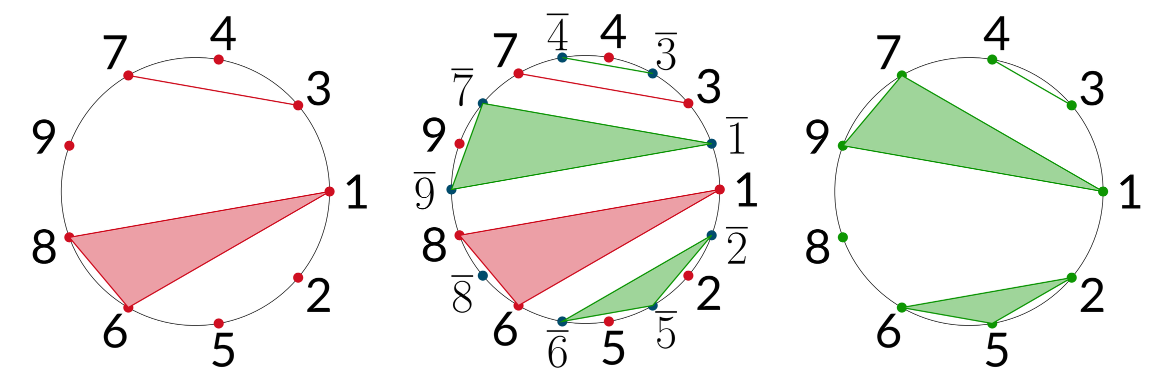

Figure 8. Calculation of the c-Kreweras complement of the previous c-noncrossing partition.

We can also compactly define it as $w \longmapsto w^{-1}c$ for $w \in [1;c]_T$.

Main result [DFISTW22]

There exists a unique family of bijections $(\mathrm{Charm}_c : \mathsf{NC}(S_n, c) \longrightarrow \mathsf{NN}(S_n))_c$ indexed by Coxeter elements of S_n such that:

- $\mathrm{Charm}_c \circ \mathrm{Krew}_c = \mathrm{Krow}_c \circ \mathrm{Charm}_c$
- $\mathrm{Supp}_{\mathsf{NC}} = \mathrm{Supp}_{\mathsf{NN}} \circ \mathrm{Charm}_c$.

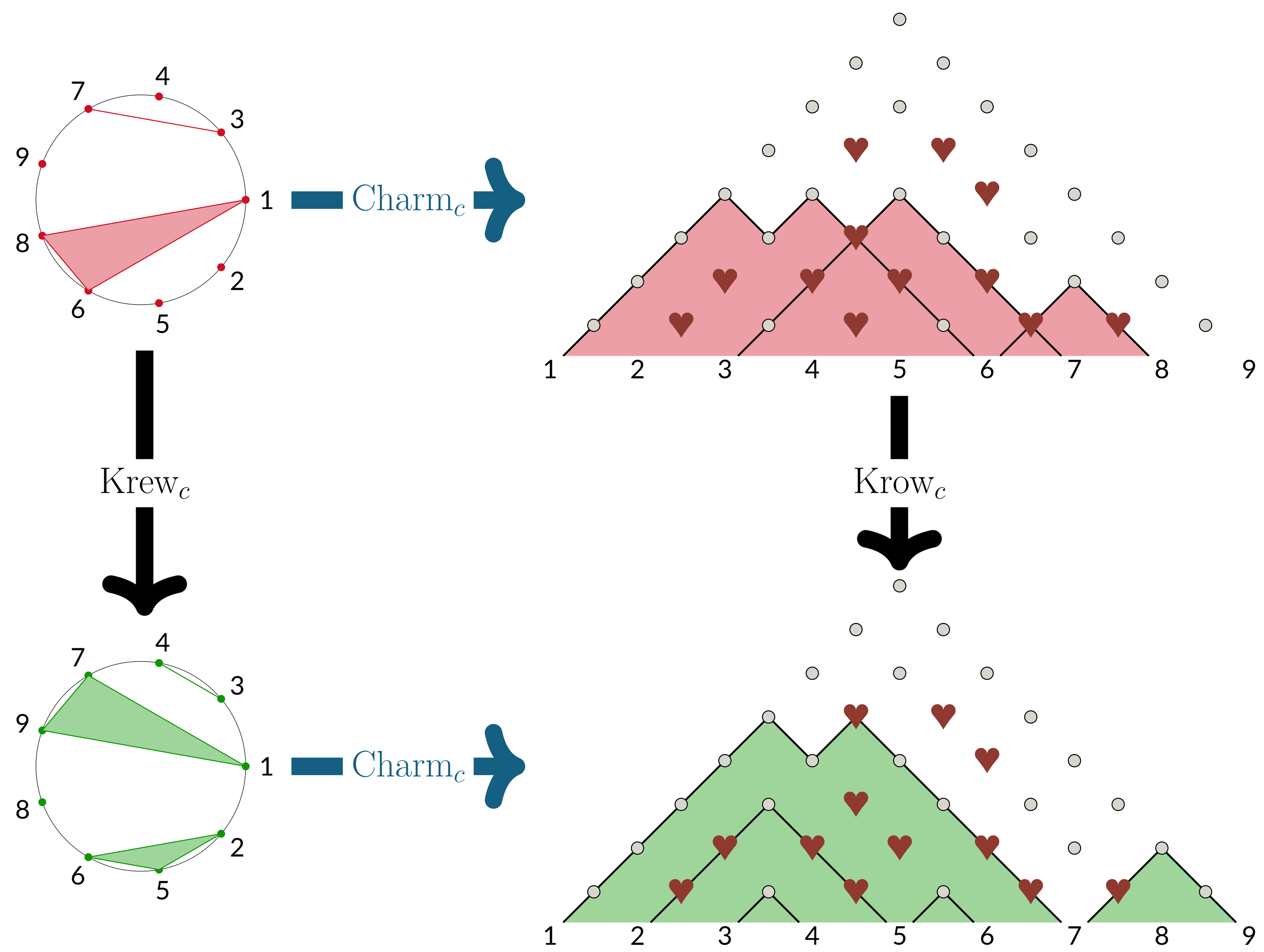

Figure 9. A summary of the commutative square obtained with the bijection Charm_c on a example. The construction of Charm_c is based on certain families of lattice paths on the root poset and the notion of charmed roots (the roots labelled by ♥).

Our article : `https://arxiv.org/abs/2212.14831`