Generalized chromatic functions

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Generalized chromatic functions

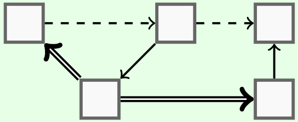
1. **Generalizing** Stanley's chromatic symmetric functions (1995)
2. **Relations** with the Stanley-Stembridge $(3+1)$-Free Conjecture (1993) and the Tree Conjecture (1995)
3. **Answering** Rosas and Sagan's question (2004): Schur functions in noncommuting variables?

Generalized chromatic functions

Edge-coloured digraphs

An edge-coloured digraph is a digraph G with three types of edges $\rightarrow, \Rightarrow, \dashrightarrow$.

Example



Proper colourings of edge-coloured digraphs

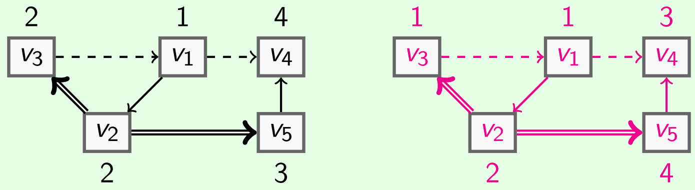
Given an edge-coloured digraph G, a proper colouring of G is a function

$$f : V(G) \to \{1, 2, 3, \dots\}$$

such that

1. If $a \Rightarrow b$, then $f(a) \leq f(b)$.
2. If $a \rightarrow b$, then $f(a) < f(b)$.
3. If $a \dashrightarrow b$, then $f(a) \neq f(b)$.

Example

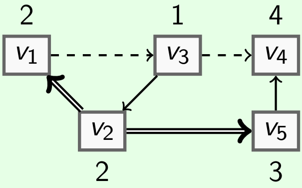


Monomials

Given a proper colouring f of an edge-coloured digraph G on vertices $v_1, v_2, \ldots, v_n$, the monomial corresponding to f in commuting variables $x_1, x_2, x_3, \ldots$ is

$$x_{f(v_1)} x_{f(v_2)} \cdots x_{f(v_n)}.$$

Example



$$x_2 x_2 x_1 x_4 x_3 = x_1 x_2^2 x_3 x_4$$

Generalized chromatic functions (A., Li, van Willigenburg, 2022)

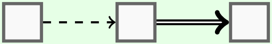
Let G be an edged-coloured digraph with vertices $v_1, v_2, \ldots, v_n$, the generalized chromatic function of G is

$$\mathscr{X}_G = \sum x_{f(v_1)} x_{f(v_2)} \cdots x_{f(v_n)}$$

where the sum is over all proper colourings f of the edge-coloured digraph G.

Example

If G is the following edge-coloured digraph



$$\mathscr{X}_G = \sum_{i \neq j \leq k} x_i x_j x_k = x_1 x_2 x_2 + x_1 x_2 x_3 + \cdots .$$

Stanley's chromatic symmetric function

Proper colourings with infinitely many colours

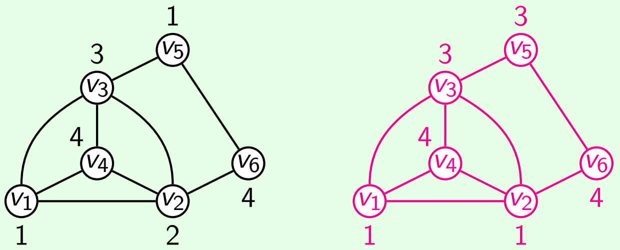
Given a graph H, a proper colouring κ of H is

$$\kappa : V(H) \to \{1, 2, 3, \dots\}$$

so if $v_i, v_j \in V(H)$ are joined by an edge, then

$$\kappa(v_i) \neq \kappa(v_j).$$

Example

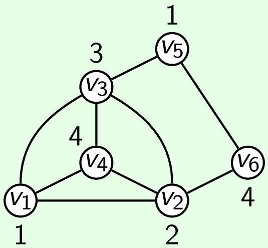


Monomials

Given a proper colouring κ of H on vertices $v_1, v_2, \ldots, v_n$, the monomial corresponding to κ in commuting variables $x_1, x_2, x_3, \ldots$

$$x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}.$$

Example



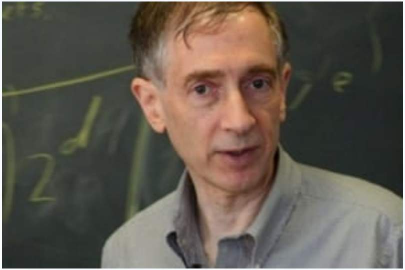
$$x_1 x_2 x_3 x_4 x_1 x_4 = x_1^2 x_2 x_3 x_4^2$$

Chromatic symmetric functions: Stanley 1995

Given a graph H with vertices $v_1, v_2, \ldots, v_n$ the chromatic symmetric function of H is

$$X_H = \sum x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$$

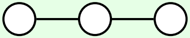
where the sum is over all proper colourings κ of H.



Example

Example

Let H be the following graph.



Then the proper colourings of H are

and so on.

$$X_H = 6x_1x_2x_3 + x_1^2x_2 + \cdots$$

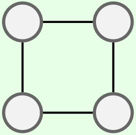
Every chromatic symmetric function is a generalized chromatic function

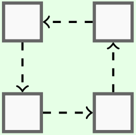
Let H be a graph. Then

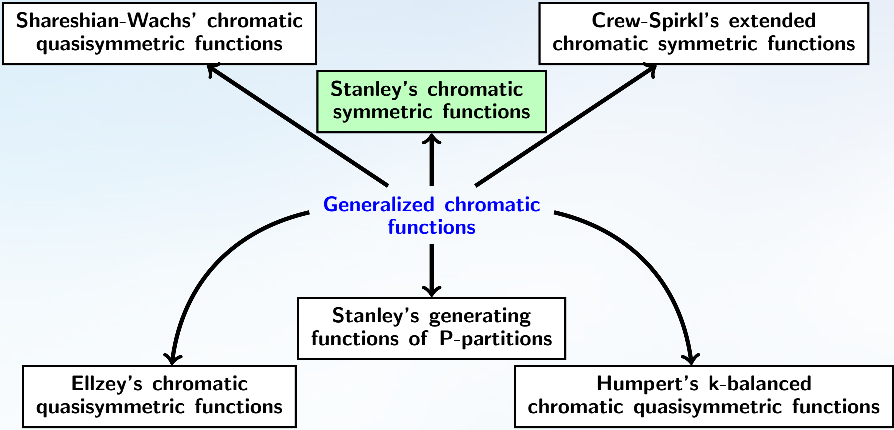
$$X_H = \mathscr{X}_{\overset{\dashrightarrow}{H}}$$

where $\overset{\dashrightarrow}{H}$ is an edge-coloured digraph obtained by replacing the edges of H by dashed edges.

Example

$H =$ 

$\overset{\dashrightarrow}{H} =$ 


Shareshian-Wachs' chromatic quasisymmetric functions
Stanley's chromatic symmetric functions
Crew-Spirkl's extended chromatic symmetric functions
Generalized chromatic functions
Stanley's generating functions of P-partitions
Ellzey's chromatic quasisymmetric functions
Humpert's k-balanced chromatic quasisymmetric functions

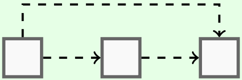
Stanley-Stembridge $(3+1)$-Free Conjecture and Tree Conjecture

Useful edge-coloured graphs

Notation	Expression
P_n	The directed path with n vertices and solid edges
K_n	A tournament with n vertices and dashed edges

Example

P_3



K_3

Elementary symmetric functions

An integer partition λ of n, denoted $\lambda \vdash n$, is a list $\lambda_1 \lambda_2 \cdots \lambda_{\ell(\lambda)}$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)}$ and their sum is n.

$$3221 \vdash 8$$

The elementary symmetric function for $\lambda = \lambda_1 \lambda_2 \cdots \lambda_{\ell(\lambda)} \vdash n$ is

$$e_\lambda = \mathscr{X}_{P_{\lambda_1}} \mathscr{X}_{P_{\lambda_2}} \cdots \mathscr{X}_{P_{\lambda_{\ell(\lambda)}}}.$$

Symmetric functions

Let

$$\mathrm{Sym}_n = \mathbb{Q}\text{-span}\{e_\lambda : \lambda \vdash n\}.$$

Then

$$\mathrm{Sym} = \bigoplus_{n \geq 0} \mathrm{Sym}_n.$$

Proposition

Sym is a graded subalgebra of bounded degree power series, and $\{e_\lambda\}$ is a basis for it.

$$X_G \in \mathrm{Sym}$$

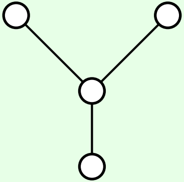
Chromatic symmetric functions in terms of e-basis

Example

If G is the path

$$X_G = 3e_3 + e_{21}$$

but if G is a claw,



$$X_G = e_{211} - 2e_{22} + 5e_{31} + 4e_4.$$

Which chromatic symmetric functions are e-positive?

Unit interval graphs

Consider a set $\{I_1, I_2, \ldots, I_n\}$ and assign a unit interval to each of them

$$I_1 \overset{f}{\mapsto} [a_1, b_1], I_2 \overset{f}{\mapsto} [a_2, b_2], \ldots, I_n \overset{f}{\mapsto} [a_n, b_n]$$

such that $a_1 \leq a_2 \leq \cdots \leq a_n$.

A unit interval graph is a graph with vertex set $\{I_1, I_2, \ldots, I_n\}$ such that I_i is adjacent to I_j if $f(I_i) \cap f(I_j) \neq \emptyset$.

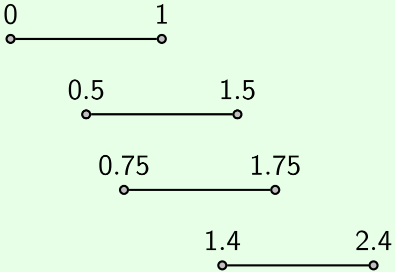
Example

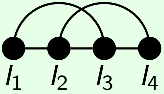
$I_1 =$

$I_2 =$

$I_3 =$

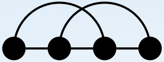
$I_4 =$





First open problem: Stanley-Stembridge, Guay-Paquet

If G is the following unit interval graph



then $X_G = 2e_{31} + 16e_4$.

Stanley-Stembridge $(3+1)$-Free Conjecture

Every unit interval graph is e-positive.

Open problems

For which edge-coloured digraphs G is $\mathscr{X}_G$ symmetric?
For which edge-coloured digraphs G is $\mathscr{X}_G$ e-positive?

Second open problem

Stanley 1995:

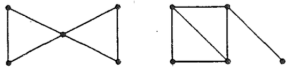


FIG. 1. Graphs G and H with $X_G = X_H$.

"We do not know whether X_G distinguishes trees."

Heil and Ji 2018: The chromatic symmetric function distinguishes all trees up to 29 vertices.
(There are 5469566585 nonisomorphic trees on 29 vertices!)

Tree Conjecture

Let T and T' be trees. $X_T = X_{T'}$ if and only if $T \cong T'$.

Open problem

For which edge-coloured trees T does $\mathscr{X}_T$ distinguish T?

Thank you

Thank you very much for listening!

1. F. Aliniaeifard, S. Li, and S. van Willigenburg, **Schur functions in noncommuting variables**, Adv. Math. 406, 108536 (2022) [37 pages].

2. F. Aliniaeifard, S. Li, and S. van Willigenburg, **Generalized chromatic functions**, (2022) [33 pages] arXiv:2208.08458.