Cartwright-Sturmfels ideals and multigraded ideals with radical support

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Gröbner bases

fix $<$ a term order

$f \in S$, $\mathrm{in}_<(f) = \max \mathrm{supp}(f)$ is the leading term of f

$\mathrm{in}_<(I) = (\mathrm{in}_<(f) : f \in I)$ is the initial ideal of I

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Facts

- a minimal system of generators is not in general a Gröbner basis,
- a Gröbner basis is not in general universal,
- finite universal Gröbner bases exist, but they tend not to be “natural”.

Example: ideals of maximal minors

$X = (x_{ij})$ $n \times m$ matrix with x_{ij} distinct variables,

$S = k[X] = k[x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m]$

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$X = (x_{ij})$ $n \times m$ matrix with x_{ij} distinct variables, $1 \leq t \leq m \leq n$

$S = k[X] = k[x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m] \supseteq I_t(X) = (t-\text{minors of } X)$

Theorem (Sturmfels)

The t-minors of X are a diagonal Gröbner basis of $I_t(X)$. In particular, their diagonal initial ideals are radical.

A diagonal Gröbner basis is a Gröbner basis wrt an order that selects the products of the elements on the diagonal as a leading term of a minor.

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They are not a universal Gröbner basis of $I_t(X)$ in general.

Theorem (Bernstein, Sturmfels, Zelewinsky)

The m-minors of X are a universal Gröbner basis of $I_m(X)$. In particular, all the initial ideals of $I_m(X)$ are radical.

Multigradings

k field, $S = k[x_{ij} \mid 1 \leq i \leq n, 0 \leq j \leq m_i]$ multigraded,
i.e. $\mathbb{Z}^n$-graded by $\deg(x_{ij}) = e_i = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i}) \in \mathbb{Z}^n$

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$S = \bigoplus_{d \in \mathbb{Z}^n} S_d$, $S_d = \langle \text{monomials of deg } d \in \mathbb{Z}^n \rangle$
$I \subseteq S$ is multigraded if $I = \bigoplus_{d \in \mathbb{Z}^n} (I \cap S_d)$, write $I_d = I \cap S_d$

Throughout the talk: the multigrading is fixed and ideals are multigraded.

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where $z = (z_1, \dots, z_n)$, $z^d = z_1^{d_1} \cdots z_n^{d_n}$, $K_{S/I}(z) \in \mathbb{Z}[z_1, \dots, z_n]$.

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Remark

$\mathrm{in}_<(gI)$ depends on $g \in G$.

Theorem (Galligo, Bayer-Stillmann, Aramova-Crona-De Negri)

If k is infinite, then there is $U \subseteq G$ dense open s.t. $\mathrm{gin}_<(I) := \mathrm{in}_<(gI)$ *is constant for* $g \in U$

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If k is infinite, then there is $U \subseteq G$ dense open s.t. $\mathrm{gin}_<(I) := \mathrm{in}_<(gI)$ is constant for $g \in U$ and $\mathrm{gin}_<(I)$ is Borel-fixed, i.e., fixed by the action of B.

$\mathrm{gin}_<(I)$ is the generic initial ideal of I wrt $<$

$B = B_{m_1+1}(k) \times \ldots \times B_{m_n+1}(k) \subseteq G$ is the Borel subgroup, with $B_m(k) \subseteq GL_m(k)$ the subgroup of upper triangular matrices

Example

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},\ I_{2143} = (x_{11}, \det(X)) \subseteq k[X]$$

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- if $\deg(x_{ij}) = 1$ for all i, j, then $\text{gin}(I) = (x_{11}, x_{12}^3)$
- if $\deg(x_{1j}) = (1, 0, 0)$, $\deg(x_{2j}) = (0, 1, 0)$, $\deg(x_{3j}) = (0, 0, 1)$ for all j, then $\text{gin}(I) = (x_{11}, x_{12}x_{21}x_{31})$

Cartwright-Sturmfels ideals

Definition

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$J, J' \subseteq S$ Borel-fixed with the same multigraded Hilbert series. If J is radical, then $J = J'$.

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Corollary (CDG)

$I, J \subseteq S$ with the same multigraded Hilbert series. If J is radical and Borel-fixed, then $J = \text{gin}(I)$.

Properties of Cartwright-Sturmfels ideals

$S = k[x_{ij} \mid 1 \leq i \leq n, 0 \leq j \leq m_i]$ $\mathbb{Z}^n$-graded by $\deg(x_{ij}) = e_i \in \mathbb{Z}^n$

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- *I and all its initial ideals are radical,*
- $\operatorname{reg}(I), \operatorname{reg}(\operatorname{in}_<(I)) \leq n$ *for any term order $<$,*
- *I has a universal Gröbner basis consisting of polynomials of degree $\leq (1, \ldots, 1) \in \mathbb{Z}^n$, hence of standard degree $\leq n$.*

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$\mathrm{gin}(I) = (x_{11}, x_{12}x_{21}x_{31})$ so I_{2143} is CS. Moreover:

- I and all its initial ideals have regularity 3,
- x_{11} and $-x_{12}(x_{21}x_{33} - x_{23}x_{31}) + x_{13}(x_{21}x_{32} - x_{22}x_{31})$ are minimal generators and a universal Gröbner basis of I,
- every initial ideal of I_{2143} is radical and a complete intersection.

Multiplicity-free and CS ideals

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where $z = (z_1, \ldots, z_n)$, $z^d = z_1^{d_1} \cdots z_n^{d_n}$, $K_{S/I}(z) \in \mathbb{Z}[z_1, \ldots, z_n]$.

The multidegree of S/I is the least degree part of $K_{S/I}(1 - z)$.

The G-multidegree of S/I is the sum with coefficients of the monomials in $K_{S/I}(1 - z)$ which are minimal wrt divisibility.

A polynomial is multiplicity-free if it only has $0, 1$ as coefficients.

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Theorem (Brion, Caminata-Cid Ruiz-Conca)

If $P \subseteq S$ is prime and the multidegree of S/P is multiplicity-free, then P is CS and $\mathrm{gin}(P)$ is Cohen-Macaulay.

Preserving the property of being CS

Corollary

If I is CS and P is an associated prime of I, then P is CS.

Theorem (CDG)

If I is CS, $\ell \in S_{e_i}$, then $I : \ell$, $I + (\ell)$, and $I + (\ell)/(\ell)$ are CS.
If R is a k-subalgebra of S gen'd by variables and I is CS, then $I \cap R$ is CS.

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$I_{\max}(X)$ is CS, hence so is $I_{\max}(L)$ where $L = (\ell_{ij})$ and $\ell_{ij} \in S_{e_i}$

$I_{\max}(L)$ has a universal Gröbner basis which consists of linear combinations of the minors and the minors are a universal Gröbner basis if $m \leq n$

Radicality and degrees of generators

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$\deg(x_{1j}) = (1,0,0)$, $\deg(x_{2j}) = (0,1,0)$, $\deg(x_{3j}) = (0,0,1)$ for all j

$\mathrm{gin}(I) = (x_{11}, x_{12}x_{21}x_{31})$ so I_{2143} is CS

For any $\ell \in S_{(1,0,0)}$ and $f \in S_{(1,1,1)}$, $J = (\ell, f) \subseteq S$ is CS, hence radical.

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Question

Can one conclude that an ideal is radical just by looking at the degrees of its generators?

Multigraded ideals with radical support

Fix $d \in \mathbb{Z}^n$: all $f \in S_d$ are squarefree iff $d \leq (1, \dots, 1) \in \mathbb{Z}^n$.

E.g., $x_{11}^{d_1} \cdots x_{n1}^{d_n}$ is squarefree iff $d \leq (1, \dots, 1)$.

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$A \subseteq \{1, \ldots, n\} \leftrightarrow \sum_{a \in A} e_a \leq (1, \ldots, 1) \in \mathbb{Z}^n$,
$\mathcal{A} = \{A_1, \ldots, A_s\}$ a multiset, where $\emptyset \neq A_i \subseteq \{1, \ldots, n\}$ for all i.
E.g., $\mathcal{A} = \{\{1, 2, 3\}, \{1, 4\}, \{2, 3, 4\}, \{1, 4\}\}$ corresponds to
$d_1 = (1, 1, 1, 0), d_2 = (1, 0, 0, 1), d_3 = (0, 1, 1, 1), d_4 = (1, 0, 0, 1)$.

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$d_1 = (1,1,1,0), d_2 = (1,0,0,1), d_3 = (0,1,1,1), d_4 = (1,0,0,1)$.

Definition

$\mathcal{A}$ is a radical support if for every field k, $m_1, \dots, m_n \in \mathbb{N}$, and $f_1, \dots, f_s \in S = k[x_{ij} \mid 1 \leq i \leq n, 0 \leq j \leq m_i]$ multigraded of $\deg(f_i) = \sum_{a \in A_i} e_a$, $(f_1, \dots, f_s)$ is radical.

Example

$\mathcal{A} = \{\{1\}, \{1,2,3\}\}$ is a radical support.

Supports of regular sequences

$\mathcal{A} = \{A_1, \ldots, A_s\}$ a multiset, where $\emptyset \neq A_i \subseteq \{1, \ldots, n\}$ for all i. TFAE:

- there exist $f_1, \ldots, f_s \in S = k[x_{ij} \mid 1 \leq i \leq n, 0 \leq j \leq m_i]$ multigraded of $\deg(f_i) = \sum_{a \in A_i} e_a$ s.t. $f_1, \ldots, f_s$ is a regular sequence,
- there exist $f_1, \ldots, f_s \in S = k[x_{ij} \mid 1 \leq i \leq n, 0 \leq j \leq m_i]$ multigraded of $\deg(f_i) = \sum_{a \in A_i} e_a$ s.t. $f_1, \ldots, f_s$ is a regular sequence and $(f_1, \ldots, f_s)$ is radical,

Supports of regular sequences

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Example

$\mathcal{A} = \{\{1,2,3\}, \{1,4\}, \{2,5\}, \{1,6\}\}$ is the support of the regular sequence $f_1 = x_{10}x_{20}x_{30}$, $f_2 = x_{11}x_{40}$, $f_3 = x_{21}x_{50}$, $f_4 = x_{12}x_{60}$.
f_1, f_2, f_3, f_4 generate a radical monomial ideal, if $m_1 \geq 2$, $m_2 \geq 1$, $m_3, m_4, m_5, m_6 \geq 0$.

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Cartwright-Sturmfels supports

Definition

$\mathcal{A}$ is a Cartwright-Sturmfels support if for every field k, $m_1, \ldots, m_n \in \mathbb{N}$, and $f_1, \ldots, f_s \in S = k[x_{ij} \mid 1 \leq i \leq n, 0 \leq j \leq m_i]$ multigraded of $\deg(f_i) = \sum_{a \in A_i} e_a$, $(f_1, \ldots, f_s)$ is Cartwright-Sturmfels.

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- Each Cartwright-Sturmfels support is a radical support.
- If we have a regular sequence with support $\mathcal{A}$ which generates a Cartwright-Sturmfels ideal, then $\mathcal{A}$ is a Cartwright-Sturmfels support.

Example

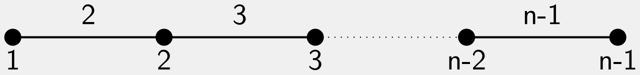
$f_1 = y_{10}y_{20}, f_2 = y_{11}y_{30}$ is a regular sequence s.t. (f_1, f_2) is CS
$g_1, g_2 \in S = k[x_{ij}]$ of degrees $(1,1,0), (1,0,1)$, then
$(g_1, g_2) = (f_1 + g_1, f_2 + g_2) + (y_{10}, y_{11})/(y_{10}, y_{11})$ is CS.

The graph associated to a support

Associate a graph $G(\mathcal{A})$ to a multiset $\mathcal{A} = \{A_1, \dots, A_s\}$ as follows: the graph has s vertices labelled $1, \dots, s$. Distinct vertices $i, j \in \{1, \dots, s\}$ are connected by an edge labelled by a if and only if $a \in A_i \cap A_j$.

Example

$\mathcal{A} = \{\{1,2\},\{2,3\},\dots,\{n-1,n\}\} \subseteq 2^{\{1,\dots,n\}}$ corresponds to



A characterization of radical supports

Theorem (CDG)

$\mathcal{A} = \{A_1, \ldots, A_s\}$ a multiset, $G(\mathcal{A})$ the associated graph. TFAE:

- *$\mathcal{A}$ is a radical support,*
- *$\mathcal{A}$ is a Cartwright-Sturmfels support,*
- *there exists a field k, $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, and a regular sequence $f_1, \ldots, f_s \in S$ with $\deg(f_i) = \sum_{a \in A_i} e_a$ for all i s.t. the ideal $(f_1, \ldots, f_s)$ is Cartwright-Sturmfels,*
- *for every field k, $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ with $m_i \geq |\{j : i \in A_j\}|$, and every regular sequence $f_1, \ldots, f_s \in S$ with $\deg(f_i) = \sum_{a \in A_i} e_a$ for all i, the ideal $(f_1, \ldots, f_s)$ is Cartwright-Sturmfels,*
- *every cycle of $G(\mathcal{A})$ has constant edge labels.*

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Thank you for your attention!