Counting lattice points that appear as algebraic invariants of Cameron Walker Graphs

Sara Faridi, Iresha Madduwe Hewalage

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Many commutative algebraists are interested in studying the properties of finite simple graphs through monomial ideals.

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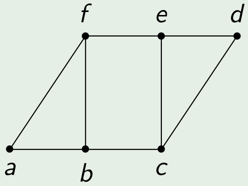

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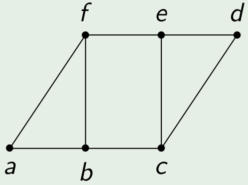

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So, our goal is to study the algebraic invariants of these edge ideals of finite simple graphs through combinatorial tools of those graphs.

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dim(R/I)**:**
Let $R = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring and K be the field. Then the **dimension** of R/I is the length of the longest chain of prime ideals in R/I.

For a graph G :

$$\dim R/I(G) = \max \{|S| \mid S \text{ is an independent set of } G\}$$

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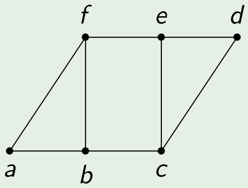

- $\dim R/I(G) = |\{a, d\}| = 2.$

Let I be a monomial ideal in $K[x_1, \dots, x_n]$, K field. A mininmal free resolution of R/I is an exact sequence of free R-modules (or K- vector spaces):

$$0 \to R^{\beta_p} \to R^{\beta_{p-1}} \to \dots \to R^{\beta_0}$$

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i.e. if G is a Cameron Walker Graph then these homological invariants of R/I do not depend on the characteristic K.

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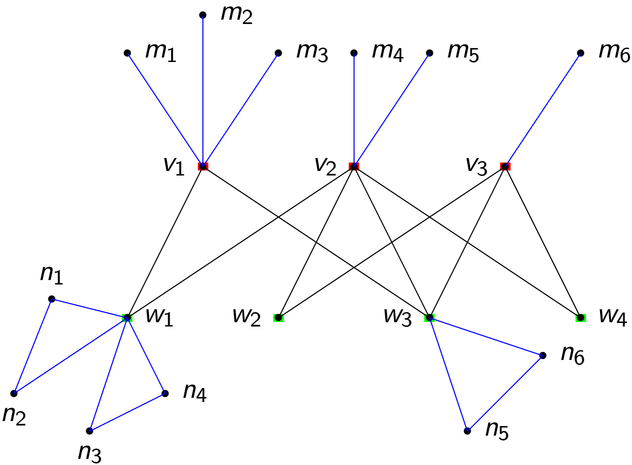

Figure: A Cameron Walker Graph

In 2022, Hibi, Kimura, Matsuda, and Van Tuyl, studied for the possible tuples

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for all graphs G on a fixed number of vertices.

They found that such lattice points lie between two convex lattice polytopes.

Moreover, it was shown that if we restricted to the class of Cameron Walker graphs the set of all such lattice points form a convex lattice polytope.

Inspired by this result, Hibi, Kanno, Kimura, Matsuda and Van Tuyl, studied the pair

$$(\mathrm{depth}(R/I(G)), \dim(R/I(G)))$$

for all the finite simple graphs G with a fixed number of vertices and they showed the set of all such points always lie between two convex lattice polytopes.

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for all the finite simple graphs G with a fixed number of vertices and they showed the set of all such points always lie between two convex lattice polytopes.

However, when they restricted to the family of Cameron Walker graphs the set of all possible lattice points did not form a convex lattice polytope.

Interestingly, within this special class of graphs, they were able to give a full characterization for the pair

$$(\mathrm{depth}(R/I(G)), \dim(R/I(G))):$$

Let $\mathrm{CW}_{\mathrm{depth,dim}}(\mathrm{n}) = \{(\mathrm{depth}\ R/I(G), \dim\ R/I(G)) \mid G \in CW(n)\}$.

Theorem (Hibi, Kanno, Kimura, Matsuda, Van Tuyl, 2021)

Let $n \geq 5$ be an integer. Then $CW_{depth,dim}(\mathrm{n})$ is

$$CW_{2,dim}(n) \cup \left\{(b, b) \in \mathbb{N}^2 \mid \frac{n}{3} < b < \frac{n}{2}\right\} \cup C$$

where

$$CW_{2,dim}(n) = \begin{cases} \{(2, n-2),\ (2, n-3)\}, & \textit{if } n \textit{ is even} \\ \{\{(2, n-2),\ (2, n-3),\ \left(2, \frac{n-1}{2}\right)\} & \textit{if } n \textit{ is odd} \end{cases}$$

and

$$C = \left\{(a, b) \in \mathbb{N}^2 \mid 3 \leq a \leq \left\lfloor \frac{n-1}{2} \right\rfloor,\ max\left\{a, \frac{n-a}{2}\right\} < b \leq n - a\right\}.$$

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Moving from this point they found all the tuples

$$(\text{depth}(R/I(G)), \text{reg}(R/I(G)), \dim(R/I(G)), \deg \text{h}(R/I(G)))$$

of Cameron Walker graphs.

Theorem (Hibi, Kanno, Kimura, Matsuda, Van Tuyl, 2021)

Let $n \geq 5$ be an integer. Then $CW_{depth,reg,dim,deg\ h}(\mathrm{n})$ is

$$CW_{2,reg,dim,deg\ h}(\mathrm{n}) \cup A \cup B \cup C$$

where

$$CW_{2,reg,dim,deg\ h}(\mathrm{n}) = \begin{cases} \{(2,2,n-2,n-2),\ (2,2,n-3,n-3)\}, & \text{if } n \text{ is even} \\ \{(2,2,n-2,n-2),\ (2,2,n-3,n-3),\ (2,\frac{n-1}{2},\ \frac{n-1}{2},\ \frac{n-1}{2})\} & \text{if } n \text{ is odd}, \end{cases}$$

$$A = \left\{(a,d,d,d) \in \mathbb{N}^4 \;\middle|\; 3 \leq a \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor,\ n < a + 2d \right\},$$

$$B = \{(a,a,d,d) \in \mathbb{N}^4 \mid 3 \leq a < d \leq n - a,\ n \leq 2a + d - 1\},$$

and

$$C = \{(a,r,d,d) \in \mathbb{N}^4 \mid 3 \leq a < r < d < n - r,\ n + 2 \leq a + r + d\}.$$

In 2022, Hibi, Kimura, Matsuda and Van Tuyl, gave a precise description for the size of the set

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of all the Cameron Walker graphs.

Inspired from their results, our goal is to find the size of the sets $\mathrm{CW}_{\mathrm{depth},\mathrm{dim}}(\mathrm{n})$ and $\mathrm{CW}_{\mathrm{depth},\mathrm{reg},\mathrm{dim},\deg \mathrm{h}}(\mathrm{n})$ based on the description of the elements.

Theorem (Faridi, Madduwe Hewalage)

The size of $CW_{depth,dim}(\mathrm{n})$ *for any given integer* n,

$$|CW_{depth,dim}(\mathrm{n})| = \begin{cases} 0 & \text{if } n < 5 \\ 1 & \text{if } n = 5 \\ \frac{1}{6}(n-3)^2 + \frac{1}{2}, & \text{if } n = 6k \\ \frac{1}{6}(n-3)^2 + \frac{7}{3}, & \text{if } n = 6k+1 \text{ or } n = 6k+5 \\ \frac{1}{6}(n-3)^2 + \frac{5}{6}, & \text{if } n = 6k+2 \text{ or } n = 6k+4 \\ \frac{1}{6}(n-3)^2 + 2, & \text{if } n = 6k+3 \end{cases}$$

Next, we determined the size of the set $\mathrm{CW}_{\text{depth,reg,dim,deg h}}(\mathrm{n})$;

Theorem (Faridi, Madduwe Hewalage)

Let $n \geq 5$ be an integer. Then $|CW_{depth,reg,dim,deg\ h}(\mathrm{n})|$ is

$$
\begin{aligned}
&= \sum_{a=3}^{\lfloor \frac{n}{3} \rfloor} \left\lfloor \frac{n-1}{2} \right\rfloor - \left(\frac{n-a}{2} \right) + \sum_{a=\lfloor \frac{n}{3} \rfloor + 1}^{\lfloor \frac{n-1}{2} \rfloor} \left\lfloor \frac{n-1}{2} \right\rfloor - (a-1) \\
&+ \sum_{a=3}^{\lceil \frac{n+1}{3} \rceil - 1} a + \sum_{a=\lceil \frac{n+1}{3} \rceil}^{\lceil \frac{n}{2} \rceil - 1} n - 2a \\
&+ \sum_{a=3}^{\lfloor \frac{n-1}{2} \rfloor - 2} \left(\sum_{r=a+1}^{\lfloor \frac{n+2-a}{2} \rfloor} (a-2) + \sum_{r=\lfloor \frac{n+2-a}{2} \rfloor + 1}^{\lfloor \frac{n-1}{2} \rfloor} (n-2r-1) \right) \\
&+ k
\end{aligned}
$$

where $k = \begin{cases} 2 & \text{if } n = 5 \text{ or even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$

Further Work

It would be nice to compare the number of integer points in $\mathsf{CW}_{\mathsf{depth},\mathsf{dim}}(\mathrm{n})$ to the number of integer points in $\mathsf{Graph}_{\mathsf{depth},\mathsf{dim}}(\mathrm{n})$ similar to the comparison described for

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between the finite simple graphs and the Cameron Walker graphs by Hibi, Kimura, Matsuda and Van Tuyl in 2022.

Then one might be able to find the percentage of lattice points recognized by the Cameron Walker graphs.

Question: What is the value of

$$\lim_{n\to\infty} \frac{|\mathsf{CW}_{\mathsf{depth},\mathsf{dim}}(\mathrm{n})|}{|\mathsf{Graph}_{\mathsf{depth},\mathsf{dim}}(\mathrm{n})|}?$$

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However, computational evidence shows that this is true.

Theorem (Faridi, Madduwe Hewalage)

Suppose $n > 5$ *and* $\lim_{n\to\infty} \frac{|CW_{depth,dim}(\mathrm{n})|}{|Graph_{depth,dim}(\mathrm{n})|}$ *exists. Then*

$$\frac{1}{3} \leq \lim_{n\to\infty} \frac{|CW_{depth,dim}(\mathrm{n})|}{|Graph_{depth,dim}(\mathrm{n})|} \leq \frac{4}{9}.$$

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Thank You!