Twists of Grassmannian Cluster Variables

Moriah Elkin
(joint with Gregg Musiker and Kayla Wright)

University of Minnesota

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The Grassmannian and Plücker coordinates

The Grassmannian Gr(k,n) is the space of all k-dimensional subspaces of an n-dimensional space. Eg k=1 is (n-1) projective space

We can represent its points as rowspans of k-by-n matrices.

Example ($k = 2, n = 4$)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

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Example ($k = 2, n = 4$)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \qquad \Delta_{13} = (13) = 2 - 12 = -10$$

Plücker coordinates are the $k \times k$ minors of a $k \times n$ matrix.

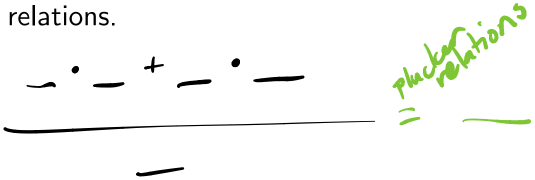
They embed Gr(k, n) into $\left(\binom{n}{k} - 1\right)$-dimensional projective space.

Cluster algebra structure

A cluster algebra is a commutative ring with distinguished generators called cluster variables, which are produced recursively from an initial set of generators via mutation.

Determinants of overlapping sets of columns are related by Plücker relations.

Mutation relations are complicated, but they work well with Plücker relations.



Cluster algebra structure, part 2

Theorem (J. Scott)

The homogeneous coordinate ring of the Grassmannian is a cluster algebra.

Many Grassmannian cluster variables are Plücker coordinates, but not all:

Quadratic differences: Gr(3, 6)

$X = (124)(356) - (123)(456),\ Y = (145)(236) - (123)(456)$

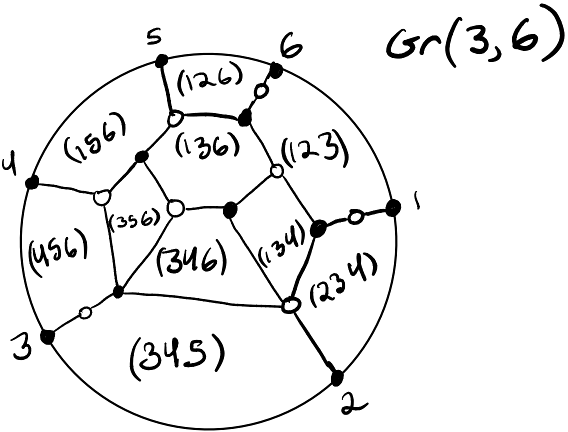
Cubic differences: Gr(3, 8)

$A = (134)(258)(167) - (134)(678)(125) - (158)(234)(167)$

$B = (147)(156)(238) - (123)(178)(456) - (123)(147)(568)$

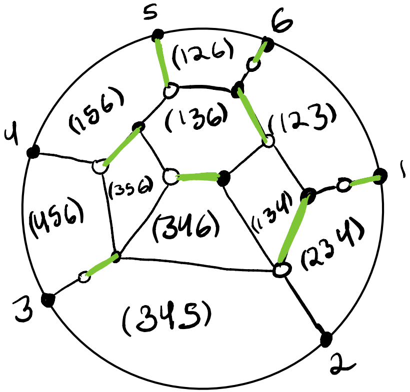
Plabic Graphs

Postnikov introduced plabic graphs, which encode the cluster algebra structure of Gr(k,n).



Dimers

A dimer is a collection of edges that uses each interior vertex exactly once, and some subset of the boundary vertices.



Boundary condition (156)

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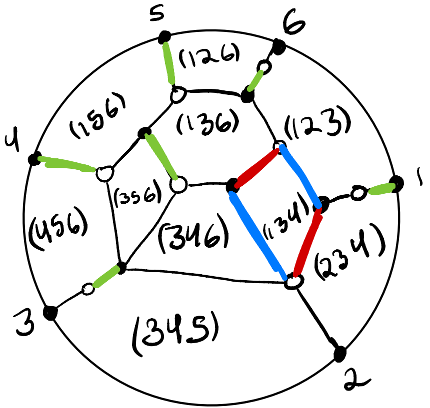

Boundary condition (145)

The twist map

We can assign a weight to each dimer based on the faces it borders.

The sum of the dimer weights is called the twist $\mathcal{T}^*$ of the boundary condition.

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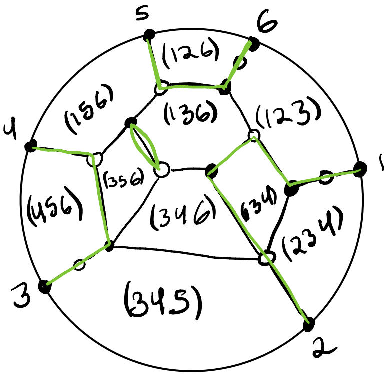
The twist distributes over addition and multiplication:
$\mathcal{T}^*(X) = \mathcal{T}^*((124)(356) - (123)(456))$
$= \mathcal{T}^*((124))\mathcal{T}^*((356)) - \mathcal{T}^*((123))\mathcal{T}^*((456))$.

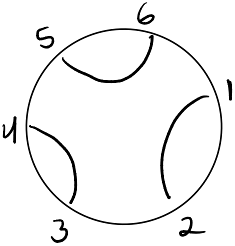
But can we compute the twists of quadratic and cubic differences directly, by viewing them as "boundary conditions"?

Twisting quadratic differences

$$\begin{cases} X = (124)(356) - (123)(456) \\ Y = (145)(236) - (123)(456) \end{cases}$$

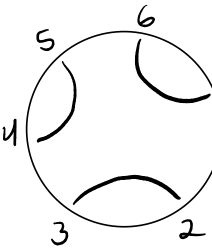
To twist a quadratic difference, we need a double dimer: a collection of edges that uses each vertex twice. It looks like a collection of paths, cycles, and doubled edges.



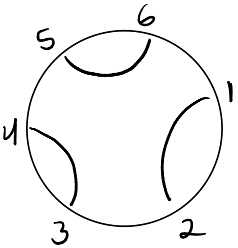


But which double dimers?

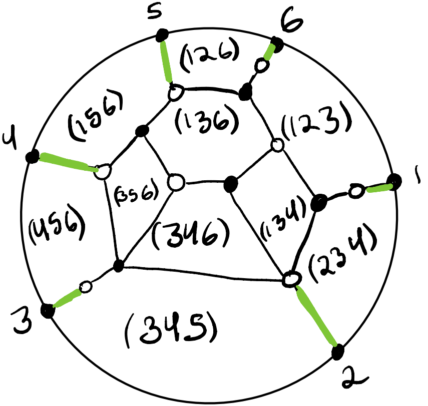
Connectivity for X

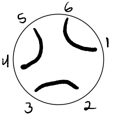


Connectivity for Y



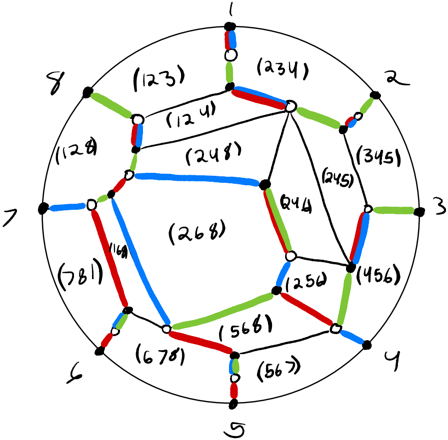
Twisting X

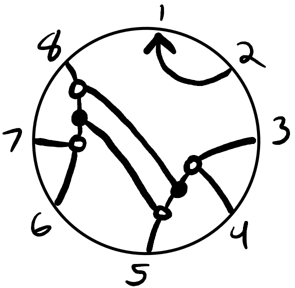


Boundary condition X = 

Triple dimers and webs

To twist a cubic difference, we need a triple dimer: a collection of edges that uses each vertex three times.



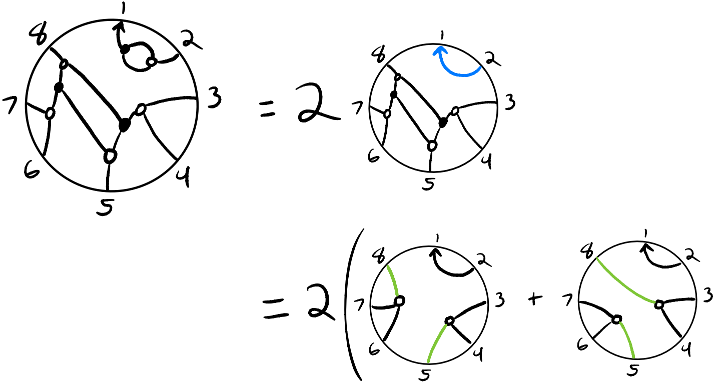


It looks like a web: a collection of paths, cycles, and components with interior trivalent vertices.

Decomposing webs

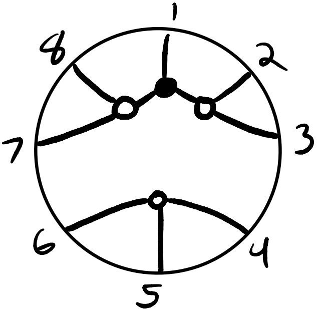
A nonelliptic web is a web without squares, bigons, or cycles.

Every web may be expressed as a sum of nonelliptic webs via Skein relations:

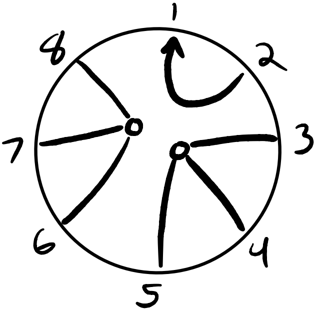


Webs for A and B

Web for A: the batwing

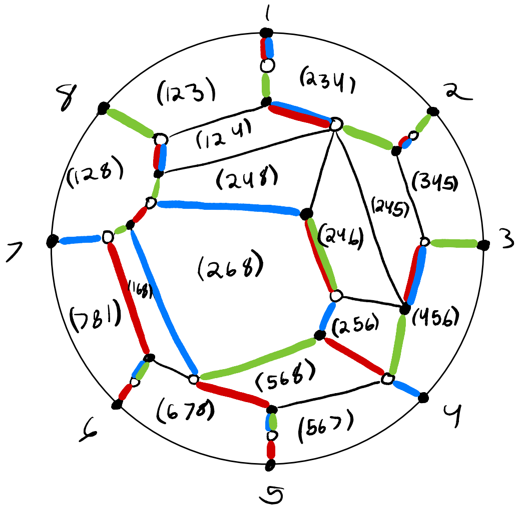


Web for B: the octopus



Twisting *A* and *B*

1. Take a triple dimer.



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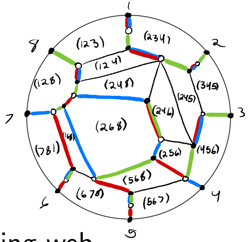
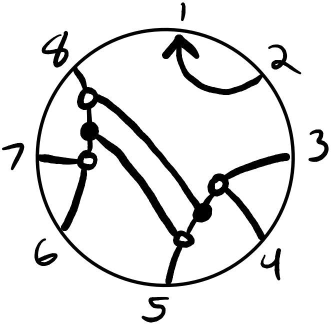


2. Find the corresponding web.



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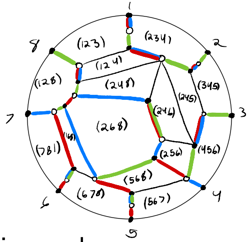

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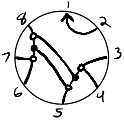
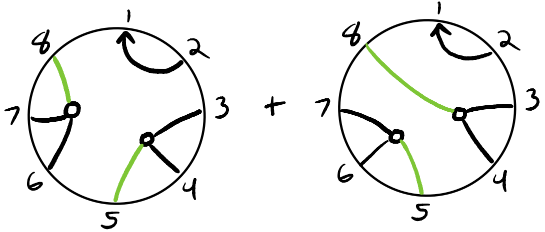


3. Decompose it into nonelliptic summands.



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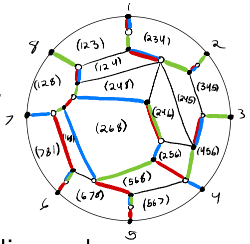

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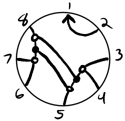

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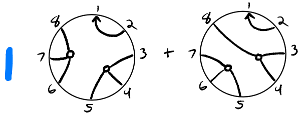

4. The coefficient of the octopus should be the coefficient of the dimer's weight in the twist of B.

Twisting A and B

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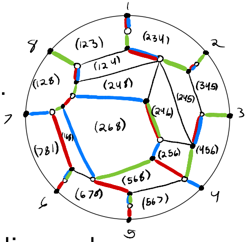

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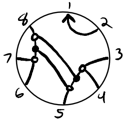

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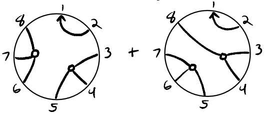
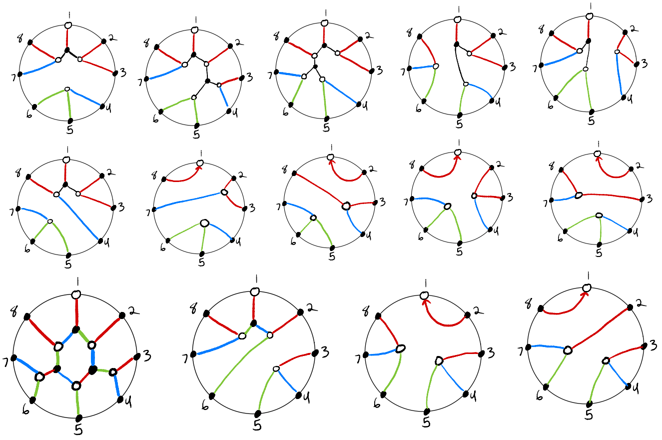


4. The coefficient of the octopus should be the coefficient of the dimer's weight in the twist of B. (And the coefficient of the batwing should be the coefficient of the dimer's weight in the twist of A.)



Thank you!

elkin048@umn.edu

Proof idea: twisting quadratic differences

$$(124)(356) - (123)(456) = X$$