# Building Monomial Ideals with Fixed Betti Numbers


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## Background

**Simplicial complex**

Let $V$ be a set. The simplicial complex $\Delta$ over $V$ is a set of subsets of $V$, such that if $F \in \Delta$ and $G \subseteq F$ , then $G \in \Delta$. An element of $\Delta$ is called a **face** of $\Delta$. If a face is contained in exactly one maximal face, then we call it a **free face**.

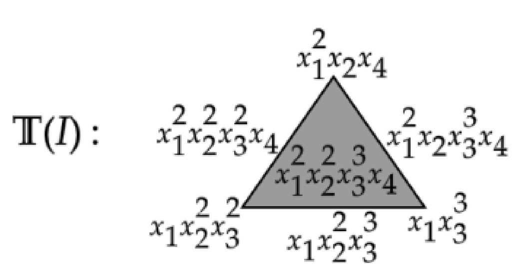
**(Minimal) free resolution of homogeneous ideal $I$ in polynomial rings**

$$0 \longrightarrow S^{\beta_p} \xrightarrow{\phi_p} \ldots\ldots \xrightarrow{\phi_2} S^{\beta_1} \xrightarrow{\phi_1 = I} S^{\beta_0 = 0} \longrightarrow S/I \longrightarrow 0$$

1. $\phi_i$ S-module homomorphism
2. $\phi_i$ are exact i.e. $\ker \phi_i = \mathrm{im}\phi_{i+1}$
3. Minimal condition: $\beta_i$ are the smallest possible. They are uniquely determined and called **Betti numbers**.

## Taylor resolution and multigraded Betti numbers

A **Taylor complex** is a simplicial complex with exactly one maximal face and vertices labeled by the generators of a monomial ideal. For example:




$\mathbb{T}(I)$ has exactly one maximal face (a full triangle) and is labeled by the monomial ideal $I = (x_1^2x_2x_4, x_1x_2^2x_3^2, x_1x_3^3)$.

**Support a free resolution**

A subcomplex $\Delta$ of $\mathbb{T}(I)$ supports a free resolution if and only if

$$\Delta_{\leq u} = \{\text{faces of } \Delta \text{ whose labels divide u}\}$$

is empty or acyclic for all monomials $u$.

We check that $\mathbb{T}(I)$ supports a free resolution of $I$. So, the **Taylor resolution** is

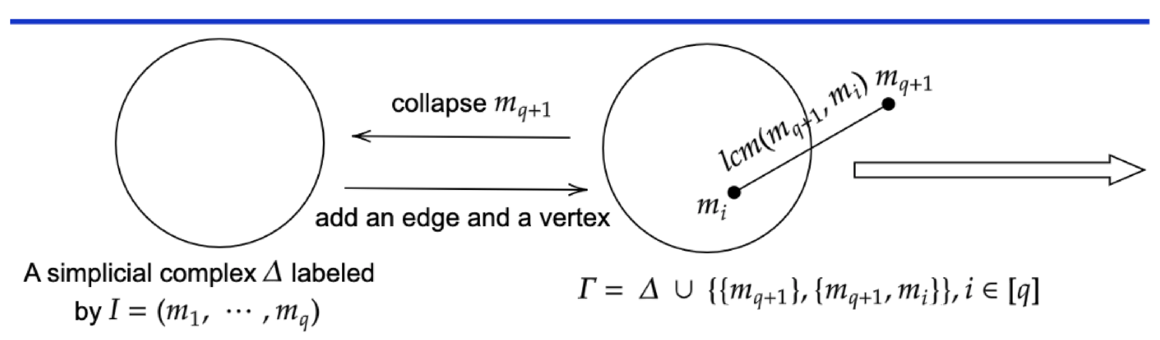
$$0 \longrightarrow S(x_1^2x_2^2x_3^3x_4) \longrightarrow \begin{matrix} S(x_1^2x_2^2x_3^2x_4) \\ \oplus \\ S(x_1^2x_2x_3^3x_4) \\ \oplus \\ S(x_1x_2^2x_3^3) \end{matrix} \longrightarrow \begin{matrix} S(x_1x_2^2x_3^2) \\ \oplus \\ S(x_1x_3^3) \\ \oplus \\ S(x_1^2x_2x_4) \end{matrix} \longrightarrow S \longrightarrow 0$$

The $i$-th **multigraded Betti numbers** of $S/I$ is

$$\beta_{i,m}(S/I) = \text{ number of copies of } S(m) \text{ in } i\text{-th homological degree.}$$

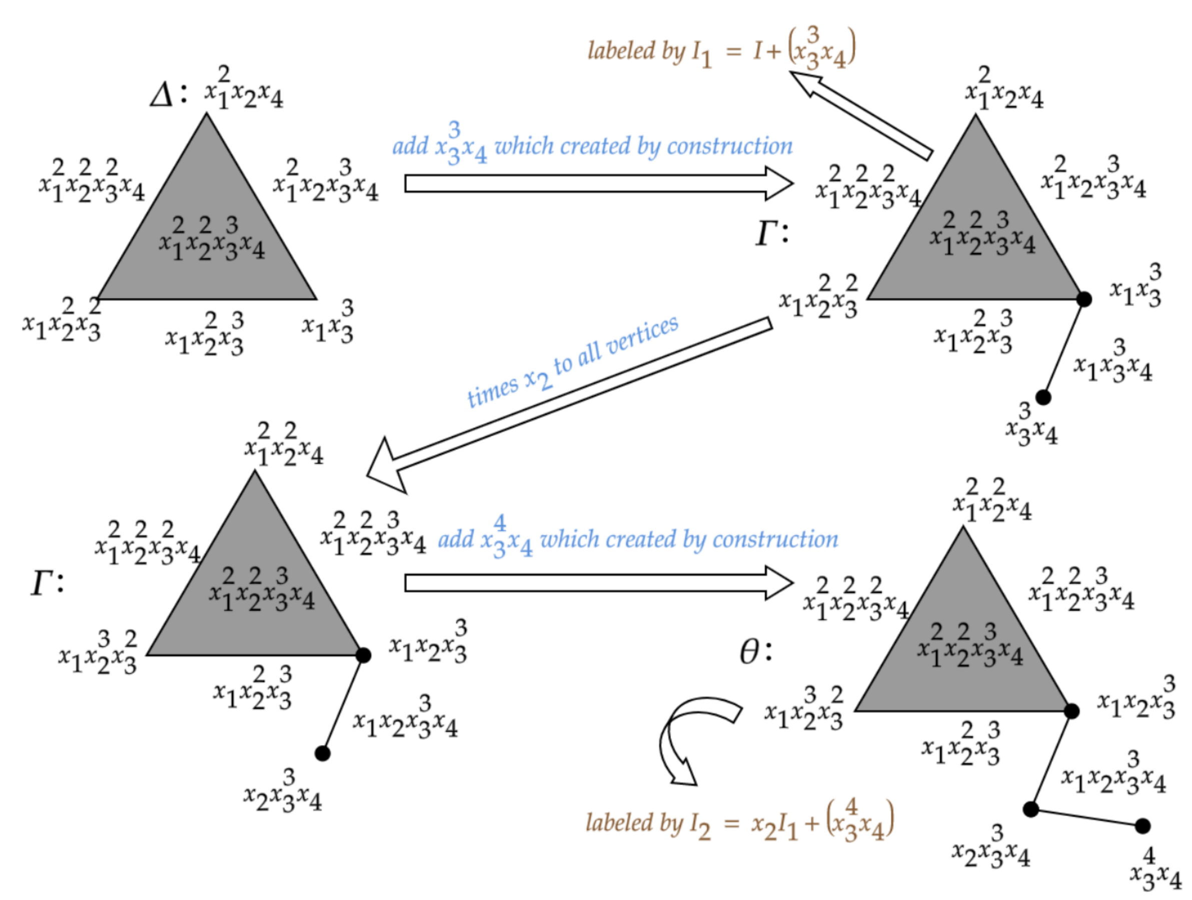
Suppose $\Delta$ supports a free resolution of $I$. When $i \geq 1$, Bayer and Sturmfels showed that

$$\beta_{i,m}(S/I) = \begin{cases} \dim \tilde{H}_{i-2}(\Delta_{<m}; k) & \text{if } \Delta_{\leq m} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$




This is the application of simplicial collapsing corresponding to the right side.

## Example




We observe that $\theta$ collapse to $\Gamma$, and $\Gamma$ collapse to $\Delta$. Therefore, those three simplicial complexes have isomorphic homology groups. So by Bayer and Sturmfels' result, we can predict the Betti numbers of the corresponding monomial ideals. This is the basic idea of Theorem 1 and Theorem 2.

With Macaulay 2, we obtain the Betti numbers of $S/I$

$$\beta_0 = 1, \beta_1 = 3, \beta_2 = 3, \beta_3 = 1.$$

By theorem 2, we can directly obtain the Betti numbers of $S/I_1$

$$\beta_0 = 1, \beta_1 = 4, \beta_2 = 4, \beta_3 = 1.$$

Similarly, the Betti numbers of $S/I_2$

$$\beta_0 = 1, \beta_1 = 5, \beta_2 = 5, \beta_3 = 1.$$

We can apply Theorem 2 as many times as we want (see Cor. 3). With an arbitrary ideal, **we can create an "infinite" large table of monomial ideals with fixed Betti numbers.**

$(1,3,3,1)$ means $\beta_3 = 1, \beta_2 = 3, \beta_1 = 3, \beta_0 = 1$.

Building monomial ideals with fixed Betti numbers by Corollary 3

| (1,3,3,1) | (1,4,4,1) | (1,5,5,1) | (1,6,6,1) |
|---|---|---|---|
| $(a^2b, b^2c, c^2d)$ | $(a^3b, ab^2c, ac^2d, bc^2d)$ | $(a^4b, a^2b^2c, a^2c^2d, abc^2d, b^2c^2d)$ | $\cdots$ |
| $(a^2b^2, b^3c, c^2d)$ | $(a^2b^3, b^4c, bc^2d, ac^2d)$ | $(a^2b^4, b^5c, b^2c^2d, abc^2d, a^2c^2d)$ | $\cdots$ |
| $(a^2b, b^2c^2, c^3d)$ | $(a^2bc, b^2c^2, c^3d, a^3b)$ | $(a^2bc^2, b^2c^3, c^4d, a^3bc, a^4b)$ | $\cdots$ |
| $(a^3b^2, ab^3c, c^2d)$ | $(a^3b^2, ab^3c, abc^2d, c^3d)$ | $(a^4b^3, a^2b^4c, a^2b^2c^2d, abc^3d, c^4d)$ | $\cdots$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

## Simplicial collapsing

Let $\Delta$ be a simplicial complex, $\tau \in (\Delta)$ and $\sigma \subsetneq \tau$ a free face of $\Delta$. A **collapse** of $\Delta$ along $\sigma$ is the simplicial complex

$$\Delta_{\searrow\sigma} = \Delta \setminus \{\gamma : \sigma \subseteq \gamma\} = \{F \in \Delta : \sigma \nsubseteq F\}.$$

If $dim(\tau) = dim(\sigma) + 1$, then the collapse is called an **elementary collapse**.

If a series of elementary collapsing from $\Delta$ leads to $\Gamma$, then we say $\Delta$ collapses to $\Gamma$. If $\Delta$ collapses to $\Gamma$, then $\Delta$ and $\Gamma$ are homotopy equivalent, in particular $\dim \tilde{H}_i(\Delta; k) = \dim \tilde{H}_i(\Gamma; k)$.

## Main thoerems

- Let $m_1, ..., m_q$ monomials in $S = k[x_1, ..., x_n], n > 1$,
- $I$ ideal generated by $m_1, ..., m_q$,
- $\Delta$ simplicial complex supporting a free resolution of $I$.

**Theorem 1**

Suppose $\gcd(m_1, ..., m_q) \neq 1$ and $\Delta$ supports a free resolution of $I$, then there is an infinite set of monomials $C(I)$ such that

$$\Gamma = \Delta \cup \{\{m\}, \{m, m_i\}\}$$

supports a free resolution of $I + (m)$ for all $m \in C(I)$.

**Remark:** Each $m \in C(I)$ is constructed from some $m_i$ for $i \in [q]$.

**Theorem 2**

Let $\alpha$ be a monomial and $m \in C(\alpha I)$ ($\alpha$ could be 1). Then for some $i \in [q]$, all monomials $u$ and all $j \geq 0$, we have

$$\beta_{j,u}(S/(\alpha I + (m))) = \begin{cases} 1 & \text{if } j = 1, u = m \\ 0 & \text{if } j = 1, u = \alpha m_i, m \mid \alpha m_i \\ \beta_{j,u/\alpha}(S/I) + 1 & \text{if } j = 2, u = \mathrm{lcm}(m, \alpha m_i), m \nmid \alpha m_i \\ \beta_{j,u/\alpha}(S/I) & \text{otherwise,} \end{cases}$$

$$\beta_j(S/(\alpha I + (m))) = \begin{cases} \beta_j(S/I) + 1 & \text{if } j \in \{1, 2\}, m \nmid \alpha m_i \\ \beta_j(S/I) & \text{otherwise.} \end{cases}$$

**Corollary 3**

Let $I$ be minimally generated. Construct

$$I_s = u_1 \cdots u_s I + (u_1 \cdots u_{s-1} v_1) + \cdots + (u_1 \cdots u_{s-j} v_j) + \cdots + (u_1 v_{s-1}) + (v_s)$$

where $u_j$ are any monomials and $v_j \in C(I_{j-1})$ for all $j \in [s]$. Then

$$\beta_j(S/I_s) = \begin{cases} \beta_j(S/I) + w & \text{if } j \in \{1, 2\} \\ \beta_j(S/I) & \text{otherwise} \end{cases}$$

where $w$ is the number of $u_j \cdots u_{j-s} v_j$, which do not divide any generator of $I_{j-1}$ for $j \in [s]$.

## References


[1] D. Bayer, I. Peeva, and B. Sturmfels. "Monomial resolutions". *Mathematical Research Letters* **5** (1998), 31–46.

[2] D. Bayer and B. Sturmfels. "Cellular resolutions of monomial modules". *Journal für die reine und angewandte Mathematik* **502** (1998), pp. 123–140.

[3] S. Faridi and P. Li. "Building monomial ideals with fixed Betti numbers". In progess.

[4] D. Kozlov. *Combinatorial algebraic topology*. Algorithms and Computation in Mathematics, 21. Berlin: Springer, 2008.

[5] I. Peeva. *Graded Syzygies*. Algebra and Applications, 14. London: Springer-Verlag London, Ltd., 2011.

[6] D. Taylor. "Ideals Generated by Monomials in an R-Sequence". PhD thesis. University of Chicago, 1966.