Which Schubert varieties are Hessenberg varieties?

joint with Martha Precup and John Shareshian

Flag variety $\mathcal{F}_n := \left\{ (V_1, \ldots, V_n) \;\middle|\; \begin{array}{l} V_i \text{ vector subspace of } \mathbb{C}^n \\ V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n \end{array} \right\}$

$\mathcal{F}_n = G/B$ where $G = SL_n(\mathbb{C})$ and $B =$ upper triangular matrices

Schubert varieties

$B \curvearrowright \mathcal{F}_n$ by $b \cdot MB = (bM)B$

Let $w \in S_n$:
- $\dot{w}B = (\langle e_{w_1} \rangle, \langle e_{w_1}, e_{w_2} \rangle, \ldots) \in \mathcal{F}_n$
- Schubert variety $X_w := \overline{B\dot{w}B} \subseteq \mathcal{F}_n$

Flag description: for $p, q \in [n]$ set $r_{p,q}(w) := |\{ i \in [p] \mid w_i \in [q] \}|$

$(V_1, \ldots, V_n) \in X_w \iff \forall p, q \in [n], \ \dim(V_p \cap \langle e_1, \ldots, e_q \rangle) \geq r_{p,q}(w)$

Facts:
- X_w is B-invariant, irreducible
- $\dim(X_w) = \ell(w)$
- $X_w \subseteq \mathcal{F}_n$ has codimension-one $\iff w = s_i w_0$

Hessenberg varieties

$\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) = \{n \times n \text{ matrices with trace } 0\}$

$G \circlearrowright \mathfrak{g}$ by $M \cdot x = MxM^{-1}$

A Hessenberg space $H \subseteq \mathfrak{g}$ is a B-invariant subspace

Let H be a Hessenberg space and $x \in \mathfrak{g}$:

Hessenberg variety $\mathrm{Hess}(x, H) := \{MB \in \mathcal{F}_n \mid M^{-1}xM \in H\}$

Obs: if $x = 0$ or $H = \mathfrak{g}$ $\Rightarrow$ $\mathrm{Hess}(x, H) = \mathcal{F}_n$

Flag description for some Hessenberg varieties

$\mathfrak{b} := \{\text{upper triangular } x \in \mathfrak{g}\}$

A Hessenberg function is a nondecreasing $h : [n] \longrightarrow [n]$

st $h(i) \geq i$ for all $i \in [n]$.

Bijection: {Hessenberg fcts} $\longrightarrow$ {Hessenberg spaces st $\mathfrak{b} \subseteq H$}

i	1	2	3	4	5
$h(i)$	2	4	4	4	5

$$\longmapsto \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

Assuming $\mathfrak{b} \subseteq H$ and h corresponds to H,

$$\mathrm{Hess}(x, H) = \{ (V_1, \ldots, V_n) \in \mathcal{F}_n \mid \forall i,\ x(V_i) \subseteq V_{h(i)} \}$$

Example: $h(i) = \begin{cases} n-1, & i=1 \\ n, & i>1 \end{cases} \Rightarrow x(V_1) \subseteq V_{n-1}$

$$\mathrm{Hess}(x, h) = \{ [v_1 \cdots v_n] B \mid \det([xv_1\ v_1 \cdots v_{n-1}]) = 0 \}$$

[De Mari-Shayman, De Mari-Procesi-Shayman]:

- $h(i) = \begin{cases} i+1, & i<n \\ n, & i=n \end{cases}$ & x diagonal mtx with n distinct diagonal entries $\Rightarrow \mathrm{Hess}(x,h) =$ TV of braid arrangement

•• For all such x, $\mathrm{Hess}(x,h)$ is smooth & $\chi(\mathrm{Hess}(x,h)) = n!$

(Unless stated, do not assume $\mathfrak{b} \subseteq H$)

Which Schubert varieties are equal to a Hessenberg variety?

$w \in S_n$ avoids 4231 if there do not exist $i_1 < i_2 < i_3 < i_4$ st $w_{i_4} < w_{i_2} < w_{i_3} < w_{i_1}$.

Example: 7654321 avoids 4231

7432651 contains 7351

Theorem [EPS]: let $w \in S_n$. If there exists $x \in \mathfrak{sl}_n(\mathbb{C})$ a Hessenberg space $H \subseteq \mathfrak{sl}_n(\mathbb{C})$ st $X_w = \mathrm{Hess}(x, H)$, then w avoids 4231.

Corollary: for $n \gg 0$, most $X_w \subseteq \mathcal{F}l_n$ are not Hessenberg varieties. Namely,

$$\lim_{n \to \infty} \frac{\left|\{ w \in S_n \mid \exists\, x, H \text{ st } X_w = \mathrm{Hess}(x, H)\}\right|}{n!} = 0$$

Remarks:

- For $n=4$, $\exists\, x, H$ st $X_w = \mathrm{Hess}(x, H)$ if and only if $w \notin \{4231, 4123, 2341, 1423\}$.

- • $\nexists\, x, H$ st $X_{14235} = \mathrm{Hess}(x, H)$.

Which codimension-one Schubert varieties are isomorphic to a codimension-one Hessenberg variety?

X, Y isomorphic $\Rightarrow$ both irreducible or both not irreducible.

$$\chi(X) = \chi(Y)$$

$\mathrm{Sing}(X)$ isomorphic to $\mathrm{Sing}(Y)$

Theorem [EPS]: Let $s_i w_0 \in S_n$. There exists $\mathrm{Hess}(x, H) \subseteq \mathcal{F}l_n$ isomorphic to $X_{s_i w_0}$ if and only if $i \in \{1, n\}$.

Proposition [EPS]: if $\mathrm{Hess}(x, H) \subseteq \mathcal{F}l_n$ has codimension-one, then either

$$\begin{cases} x \text{ has an eigenspace of dimension } n-1, \text{ or} \\ H = \begin{bmatrix} & * & \\ 0 & & \end{bmatrix}. \end{cases}$$

If $\mathrm{Hess}(x,H)$ is also irreducible, then $\chi(\mathrm{Hess}(x,H))$ is divisible by $(n-2)!$

On the other hand, $\chi(X_{s_i w_0}) = |\{v \in S_n \mid v \leq s_i w_0\}|$

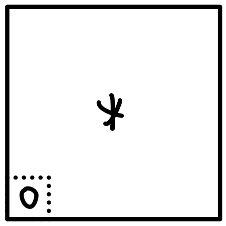
$$= n! - i!(n-i)!$$

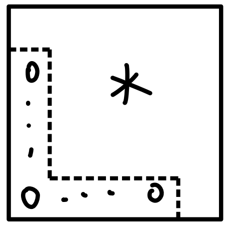
Consequence: If $i \neq 1, 2, n-1, n-2$ and $(n,i) \notin \{(8,3),(8,5)\}$, then there is no irreducible $\mathrm{Hess}(x,H)$ of codimension-one st $\chi(X_w) = \chi(\mathrm{Hess}(x,H))$.

Case $i \in \{1, n-1\}$

Lemma [EPS]: if x has an eigenspace of dimension $n-1$ and $\mathrm{Hess}(x,H)$ is irreducible of codimension-one, then
$$\begin{cases} \mathrm{Hess}(x,H) \cong X_{s_1 w_0} & \text{or} \\ \mathrm{Hess}(x,H) \cong X_{s_{n-1} w_0} \end{cases}$$

Case $i \in \{2, n-2\}$

Theorem [EPS]: If $\mathrm{Hess}(x, H)$ is of codimension-one with x nilpotent (so that $H =$ ), then

$$\mathrm{Sing}(\mathrm{Hess}(x,H)) = \mathrm{Hess}(x, H'), \quad H' = $$  .

Theorem [EPS]: if $\boxed{\mathrm{Hess}(x,H)}$ has codimension-one and the same Betti numbers as $\boxed{X_{s_2 w_0}}$ (and $X_{s_{n-2} w_0}$), then

$$\chi(\mathrm{Sing}(\mathrm{Hess}(x,H))) \neq \chi(\mathrm{Sing}(X_{s_2 w_0})) \neq \chi(\mathrm{Sing}(X_{s_{n-2} w_0})).$$

Betti numbers of $\mathrm{Sing}(\mathrm{Hess}(x,H)) \neq \mathrm{Sing}(X_{s_2 w_0})$

Case $n = 8$, $i \in \{3, 5\}$

settled using formulas for Betti numbers from [Precup].

More general Hessenberg varieties

[Goresky-Kottwitz-MacPherson]

$\psi : G \longrightarrow GL(V)$ finite-dimensional, rational representation

A Hessenberg space $H \subseteq V$ is a B-invariant subspace.

Let H be a Hessenberg space and $x \in V$, the Hessenberg variety $\mathrm{Hess}_\psi(x, H) := \{ MB \in G/B \mid \psi(M^{-1}) x \in H \}$

Obs: $\mathrm{Hess}(x, H) = \mathrm{Hess}_\psi(x, H)$, ψ adjoint representation

$(V = \mathfrak{g}, \quad \psi(M^{-1})x = M^{-1} x M.)$

Theorem [EPS]:

λ a strictly dominant weight for G, e.g. $\lambda = (n, n-1, \dots, 1)$

$\psi : G \longrightarrow GL(V(\lambda))$ associated highest weight representation

v_λ highest weight vector

For every X_w there exists a Hessenberg space

$H_w \subseteq V(\lambda)$ such that $X_w = \mathrm{Hess}_\varphi(v_\lambda, H_w)$.

H_w = Demazure module.

Thank you!