KOHNERT POSETS AND POLYNOMIALS OF NORTHEAST DIAGRAMS

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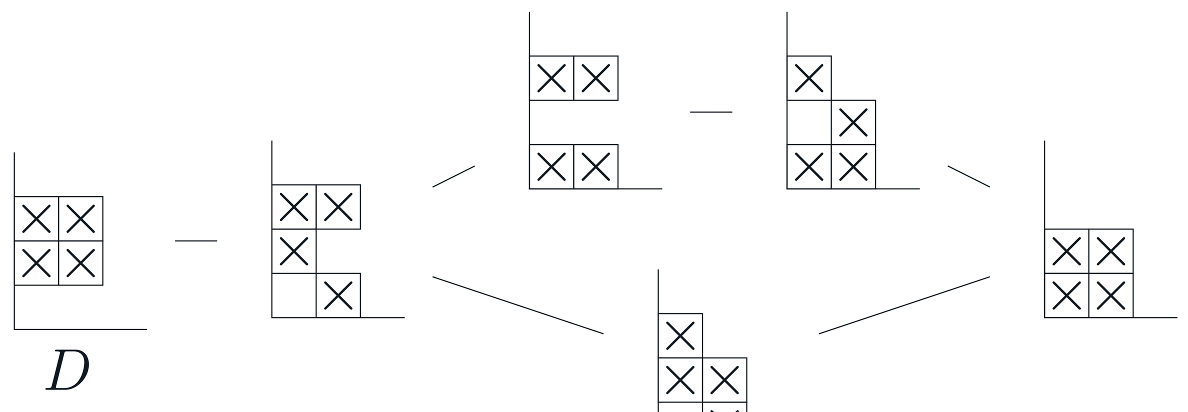
KOHNERT POSETS

Definition 1. *A **diagram** D is a finite subset of $\mathbb{N} \times \mathbb{N}$. A **Kohnert move** on a diagram moves the rightmost cell in a row to the first empty space below it.*

Definition 2. *The **Kohnert poset** of a diagram D, denoted $\mathcal{P}(D)$, is the set of all diagrams that can be obtained from D via a sequence of Kohnert moves, with $D_1 > D_2$ if D_2 can be obtained from D_1 via a sequence of Kohnert moves.*



Example: $\mathcal{P}(D)$ (top on left, bottom on right)

Kohnert moves and posets originated in Kohnert's Ph.D. thesis [5]. Colmenarejo, Hutchins, Mayers, and Phillips initiated the study of boundedness and rankedness of Kohnert posets and classified those of *key diagrams* [2].

KOHNERT POLYNOMIALS

Definition 3 ([1, Definition 2.2]). *The **Kohnert polynomial** of a diagram D is*

$$\mathfrak{K}_D = \sum_{T \in \mathcal{P}(D)} x_1^{\mathrm{rwt}(T)_1} \cdots x_n^{\mathrm{rwt}(T)_n}$$

Example: For D above, $\mathfrak{K}_D = x_2^2x_3^2 + x_1x_2x_3^2 + x_1^2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_2^2$.

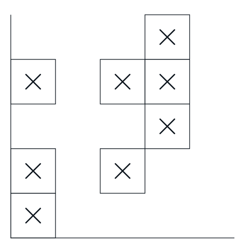
Assaf and Searles showed that if $\{D_\alpha\}$ is any set of diagrams indexed by weak compositions such that $\mathrm{rwt}(D_\alpha) = \alpha$, then $\{\mathfrak{K}_{D_\alpha}\}$ is a basis of the polynomial ring [1]. *Key diagrams* yield *Demazure characters* and *Rothe diagrams* yield *Schubert polynomials.* Criteria for monomial multiplicity-freeness of these families of polynomials were given by Hodges and Yong in [4], and Fink, Mészáros, and St. Dizier in [3], respectively.

NORTHEAST DIAGRAMS

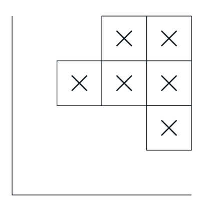
Definition 4. *A diagram D is **northeast** if for all pairs $(r,c),(r',c') \in D$, $(\max(r,r'), \max(c,c')) \in D$ as well.*

Definition 5. *For a weak composition $\alpha = (\alpha_1, \dots, \alpha_n)$, the **lock diagram** $\mathbb{L}(\alpha)$ is the right-justified diagram with exactly α_i cells in row i.*

Lock diagrams are a subclass of northeast diagrams. They are the natural analog of the well-studied, left-justified *key diagrams.* Wang initiated the study of lock polynomials and a crystal structure that intertwines with that of keys [6].



A northeast diagram



The lock diagram $\mathbb{L}(0,1,3,2)$

BOUNDEDNESS

Theorem 6. *If D is northeast, then $\mathcal{P}(D)$ is bounded if and only if D does not contain $x_1 = (r_1, c_1), x_2 = (r_2, c_2)$, and $x_3 = (r_3, c_3)$ such that:*

(a) $r_1 \le r_2 < r_3$

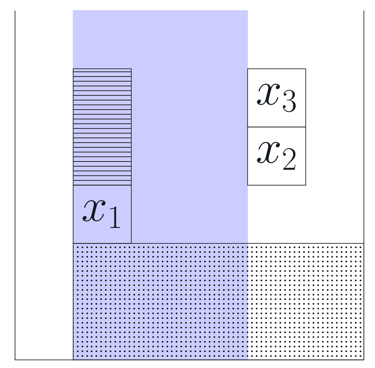
(b) $c_1 < c_2 = c_3$

(c) for all $c_1 \le c < c_2$, $\mathrm{cwt}(D)_c < \mathrm{cwt}(D)_{c_2}$

(d) for each column $c \ge c_1$, there is at least one empty position (r,c) where $r < r_1$

(e) for each $r_1 < r \le r_3$, the cell (r, c_1) is not in D_0

Forbidden configuration:



Corollary 7. *$\mathcal{P}(\mathbb{L}(\alpha))$ is bounded if and only if the nonzero entries of α after the first zero are weakly increasing.*

RANKEDNESS

Theorem 8. *If D is northeast, then $\mathcal{P}(D)$ is ranked if and only if D does not contain $x_1 = (r_1, c_1), x_2 = (r_2, c_2)$, and $x_3 = (r_3, c_3)$ such that:*

(a) $r_1 < r_2 \le r_3$

(b) $c_1 = c_2 < c_3$

(c) for each $c_1 \le c < c_3$ there is at least one empty position (r,c) where $r < r_1$

(d) for each $c \ge c_3$, the number of $r < r_3$ such that $(r,c) \in D$ is less than r_1

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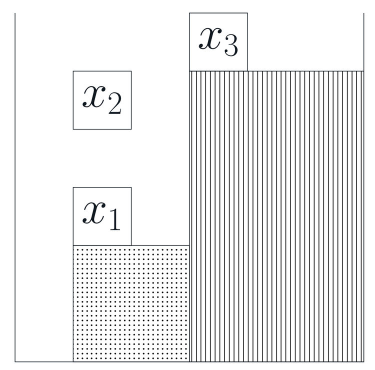
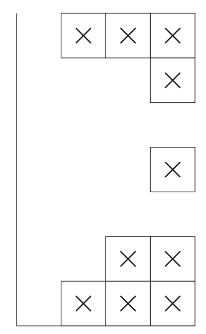


Corollary 9. *$\mathcal{P}(\mathbb{L}(\alpha))$ is ranked if and only if for every pair $\alpha_i, \alpha_{i+k} \ge 2$ with $\alpha_{i+j} \in \{0,1\}$ for all $1 \le j < k$, we have $\#\{j : 1 \le j < k$ and $\alpha_{i+j} = 1\} \ge \#\{j : j < i$ and $\alpha_j = 0\}$.*



$\mathcal{P}(\mathbb{L}(3,2,0,1,0,1,3))$ is ranked and bounded

MONOMIAL MULTIPLICITY-FREENESS

Theorem 10. *If D is a northeast diagram, then the Kohnert polynomial $\mathfrak{K}_D$ is monomial multiplicity-free if and only if D does not contain $x_1 = (r_1, c_1)$ and $x_2 = (r_2, c_2)$ such that:*

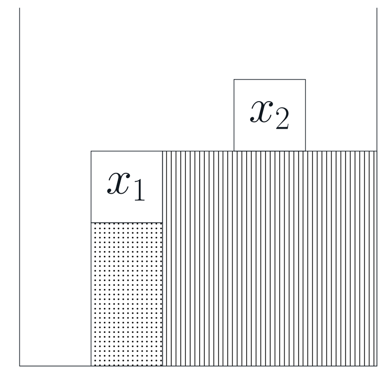
(a) $r_1 < r_2$

(b) $c_1 < c_2$

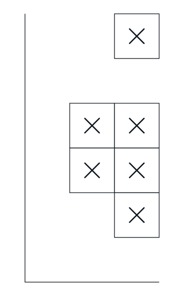
(c) there exists $s_1 < r_1$ such that the position (s_1, c_1) is empty

(d) for each $c > c_1$, there are at least two empty positions (r,c) where $r \le r_1$

Forbidden configuration:



Corollary 11. *The lock polynomial $\mathfrak{K}_{\mathbb{L}(\alpha)}$ is monomial multiplicity-free if and only if α does not contain a subcomposition of the form $(0,0,\alpha_i,\alpha_j)$ for $\alpha_i > 1$ and $\alpha_j > 0$.*



$\mathfrak{K}_{\mathbb{L}(0,1,2,2,0,1)}$ is monomial multiplicity-free

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