The Determinantal Matroid

Lisa Nicklasson (Mälardalen University)
joint work with Manolis Tsakiris (Chinese Academy of Science)

Matrix completion

$$\begin{pmatrix} ? & \checkmark & ? & \checkmark & \checkmark & \checkmark & \checkmark & ? \\ \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark \\ \checkmark & \checkmark & ? & \checkmark & \checkmark & ? & ? & \checkmark \\ ? & ? & \checkmark & \checkmark & \checkmark & ? & ? & \checkmark \\ \checkmark & \checkmark & \checkmark & ? & \checkmark & ? & ? & ? \end{pmatrix}$$

Given an $m \times n$-matrix of rank r, with some entries missing.

Can we recover the missing entries ?

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$$\begin{pmatrix} 1 & 2 & 3 & s \\ 0 & 1 & -1 & 7-s \\ 1 & 3 & 2 & 7 \end{pmatrix}$$

infinitely many completions

Determinantal Matroid

Given numbers m, n, r

$$Z = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & \\ \vdots & & \ddots & \\ z_{m1} & & & z_{mn} \end{pmatrix}$$

$\mathbb{K}[Z]$ polynomial ring

$I_{r+1}(Z)$ ideal of $(r+1)$-minors

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Subset $\Omega \subseteq Z$ $\rightsquigarrow$ Two subideals J_Ω and E_Ω of $J_{r+1}(Z)$

J_Ω = ideal generated by the $(r+1)$-minors supported on Ω

$E_\Omega = J_{r+1}(Z) \cap \mathbb{K}[\Omega]$

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$$J_\Omega \subseteq E_\Omega$$

Determinantal Matroid $\mathcal{M}(r, m \times n)$

Ground set Z

independent sets: Ω s.t. $E_\Omega = 0$

dependent sets: Ω s.t. $E_\Omega \neq 0$

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Problem: Describe the independent sets / dependent sets / bases of this matroid.

What is known?

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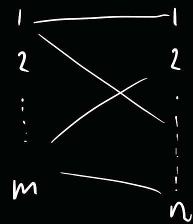
$\Omega \subseteq Z$ ($m \times n$ matrix of variables) $\longleftrightarrow$ Bipartite graph G_Ω



$i - j$ edge if $z_{ij} \in \Omega$

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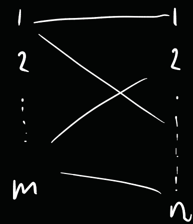

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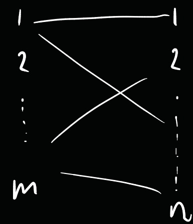

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<u>$r=2$</u> Bernstein 2017

<u>$r = \min(m,n)-2$</u> Tsakiris 2024

Detecting dependent sets using commutative algebra

Theorem 1. Ω is a dependent set of the matroid $\mathcal{M}(r, m\times n)$ if there is another set T such that

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Example. $m = n = 5 \quad r = 2$

$$\Omega = \begin{pmatrix} z_{11} & z_{12} & z_{13} & & \\ z_{21} & z_{22} & z_{23} & & \\ z_{31} & z_{32} & & z_{34} & z_{35} \\ & & z_{43} & z_{44} & z_{45} \\ & & z_{53} & z_{54} & z_{55} \end{pmatrix} \qquad T = \begin{pmatrix} z_{11} & z_{12} & z_{13} & & \\ z_{21} & z_{22} & z_{23} & & \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} \\ & & z_{43} & z_{44} & z_{45} \\ & & z_{53} & z_{54} & z_{55} \end{pmatrix}$$

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$|T \setminus \Omega| = 1 \quad \rightsquigarrow \quad \Omega$ is dependent

Detecting dependent sets using matroid theory

Theorem (Rado 1957, Asche 1966).

Let $A_1, \ldots, A_s$ be dependent sets in a matroid on ground set X, such that $A_i \cap \left(\bigcup_{j<i} A_j\right)$ is independent for $i = 2, \ldots, s$.

Then $\bigcup_{i=1}^{s} A_i \setminus B$ is dependent for any $B \subseteq X$, $|B| < s$.

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$$A_1 = \begin{pmatrix} z_{11} & z_{12} & z_{13} & & \\ z_{21} & z_{22} & z_{23} & & \\ z_{31} & z_{32} & z_{33} & & \\ & & & & \\ & & & & \end{pmatrix} \qquad A_2 = \begin{pmatrix} & & & & \\ & & & & \\ & & z_{33} & z_{34} & z_{35} \\ & & z_{43} & z_{44} & z_{45} \\ & & z_{53} & z_{54} & z_{55} \end{pmatrix}$$

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dependent sets

$(A_1 \cup A_2) \setminus \{z_{33}\} = \Omega$ dependent

Detecting dependent sets using matroid theory

Theorem 2.

For $i=1,\dots,s$ let T_i be a $m_i \times n_i$ submatrix of Z with $m_i, n_i > r$, such that

$$T_i \cap \Big(\bigcup_{j<i} T_j \Big) \subseteq \text{submatrix with} \le r \text{ rows or} \le r \text{ columns}.$$

If $\quad \left| \bigcup_{i=1}^{s} T_i \setminus \Omega \right| < \sum_{i=1}^{s} (m_i - r)(n_i - r)$

then Ω is dependent.

Detecting dependent sets using matroid theory

Example $\qquad r=4$

$$\Omega = \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} & & & \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & z_{26} & & & \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} & z_{36} & & & \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} & z_{46} & & & \\ z_{51} & z_{52} & z_{53} & z_{54} & & & z_{57} & z_{58} & z_{59} \\ z_{61} & z_{62} & z_{63} & z_{64} & & & z_{67} & z_{68} & z_{69} \\ & & & & z_{75} & z_{76} & z_{77} & z_{78} & z_{79} \\ & & & & z_{85} & z_{86} & z_{87} & z_{88} & z_{89} \\ & & & & z_{95} & z_{96} & z_{97} & z_{98} & z_{99} \end{pmatrix}$$

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T_1 = upper left 6×6 block

T_2 = lower right 5×5 block

$$\left.\begin{aligned} &|(T_1 \cup T_2)\setminus \Omega| = 4 \\ &(6-r)(6-r) + (5-r)(5-r) = 4+1 = 5 \end{aligned}\right\} \longrightarrow \Omega \text{ dependent}$$

Comparing the two theorems

Inequality from Theorem 1: $|T \setminus \Omega| < \text{height}(J_T)$

Inequality from Theorem 2:

$$\left| \bigcup_{i=1}^{s} T_i \setminus \Omega \right| < \sum_{i=1}^{s} (m_i - r)(n_i - r)$$

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Inequality from Theorem 1: $|T \setminus \Omega| < \operatorname{height}(J_T)$

Inequality from Theorem 2:

$$\left| \bigcup_{i=1}^{s} T_i \setminus \Omega \right| < \sum_{i=1}^{s} (m_i - r)(n_i - r) = \sum_{i=1}^{s} \operatorname{height}(J_{T_i})$$

Do our theorems detect all dependent sets?

Yes, if $r = 1$ or $r \geq \min(m, n) - 2$.

For $1 < r < \min(m, n) - 2$, Open problem!

Thank you!