Castelnuovo-Mumford Regularity of Toric Surfaces

AUBURN College of Sciences and Mathematics

Sean Grate

Auburn University

Preliminary Definitions

Let $R = \mathbb{K}[x_0, \dots, x_n]$ and $I \subseteq R$ a homogeneous ideal.

Definition. A *(graded) minimal free resolution* of R/I is an exact sequence of free R-modules

$$F_\bullet : 0 \leftarrow R/I \xleftarrow{\varepsilon} F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} \dots \xleftarrow{d_n} F_n \leftarrow 0$$

with $F_i = \bigoplus_j R(-j)^{\beta_{i,j}}$ and no constant terms in any of the maps.

Definition. The graded Betti numbers of R/I are the integers $\beta_{i,j}$. These values are encoded in a *Betti diagram* $\mathrm{Betti}(R/I)$, where entry $b_{i,j}$ in column i and row j is the value $\beta_{i,i+j}$.

Definition. The *Castelnuovo-Mumford regularity* (or simply *regularity*) of R/I is

$$\mathrm{reg}(R/I) = \max_{i,j}\{j : \beta_{i,i+j} \neq 0\}.$$

Equivalently, it is the index of the bottom row of $\mathrm{Betti}(R/I)$.

Definition. A coherent sheaf $\mathscr{F}$ on $\mathbb{P}^n$ is m-regular if

$$H^i(\mathscr{F}(m-i)) = 0$$

for all $i > 0$. The *Castelnuovo-Mumford regularity* (or simply *regularity*) is

$$\inf\{m : H^i(\mathscr{F}(m-i)) = 0 \text{ for all } i > 0\}.$$

Monomial Curves

Definition. The monomial curve with exponents $a_1 \leq \dots \leq a_{n-1}$ in $\mathbb{P}^n$ is the curve $C \subset \mathbb{P}^n$ of degree $d = a_n$ parameterized by

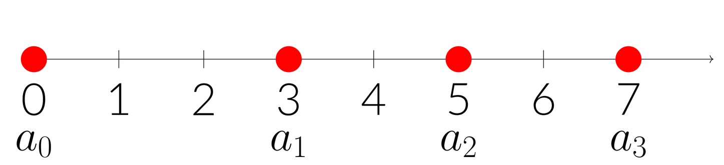
$$\varphi : \mathbb{P}^1 \to \mathbb{P}^n \quad \text{with} \quad (s,t) \mapsto (s^d, s^{d-a_1}t^{a_1}, \dots, s^{d-a_{n-1}}t^{a_{n-1}}, t^d).$$

Theorem [L'vovsky; 1996]. Let $A = (0, a_1, \dots, a_n)$ be a sequence of non-negative integers such that the g.c.d. of the a_j's equals 1, and let C be the corresponding monomial curve. Then C is m-regular, where

$$m = \max_{1 \leq i < j \leq n} \{(a_i - a_{i-1}) + (a_j - a_{j-1})\},$$

i.e., m is the sum of the two largest gaps in A.

Example



$$A = \begin{pmatrix} 0 & 3 & 5 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 3 & 5 & 7 \\ 7 & 4 & 2 & 0 \end{pmatrix}$$

$$\varphi(s,t) = (\underset{x_0}{s^7}, \quad \underset{x_1}{s^3t^4}, \quad \underset{x_2}{s^5t^2}, \quad \underset{x_3}{t^7})$$

$$I_C = \langle x_2^2 - x_1x_3, \quad x_1^3x_2 - x_0^2x_3^2, \quad x_1^4 - x_0^2x_2x_3 \rangle$$

$$0 \leftarrow R/I_C \leftarrow R \leftarrow \begin{matrix} R(-2) \\ \oplus \\ R(-4)^2 \end{matrix} \leftarrow R(-5)^2 \leftarrow 0$$

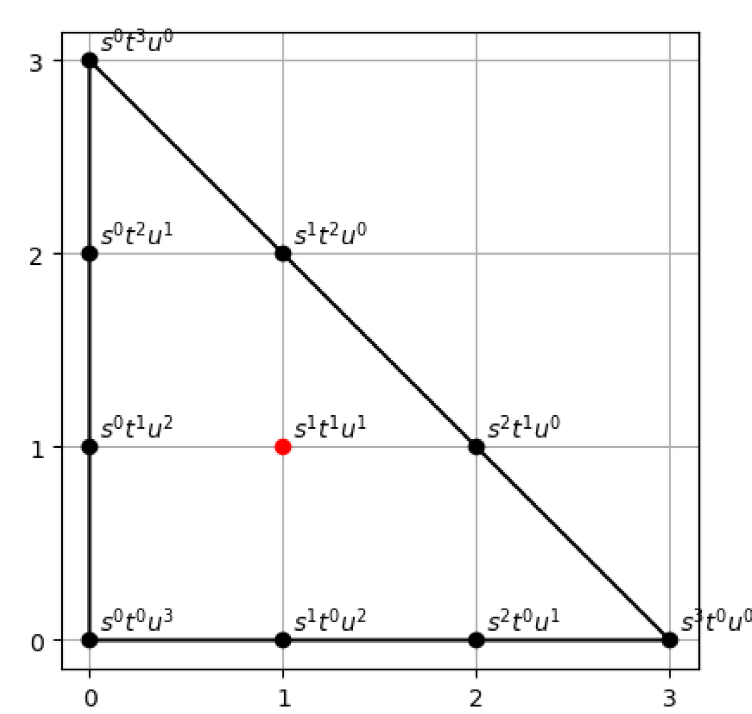
$$\mathrm{reg}(I_C) = 4 \leq (a_1 - a_0) + (a_2 - a_1) = 3 + 2 = 5$$

Goal

Extend L'vovsky's result to toric surfaces, i.e., find a combinatorial bound on the regularity of toric surfaces.

- Look at toric surfaces which are defined by an incomplete linear series.
- Include all points on the boundary of a polygon and exclude all its interior points.
- These are usually not normal, but may be smooth.

Setup



$$A = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}$$

$\downarrow$ $\mathrm{Conv}(A) \setminus \mathrm{Int}(A)$

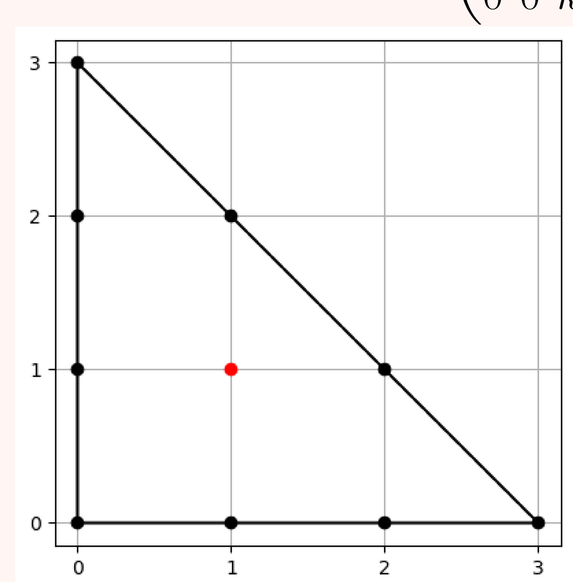
$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 3 & 2 & 1 \end{pmatrix}$$

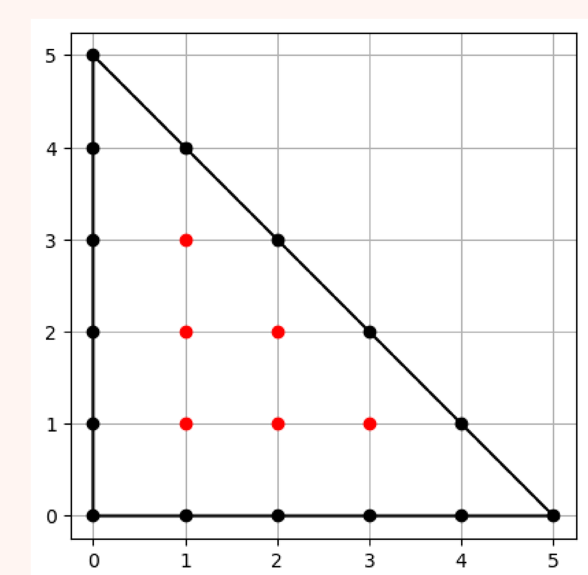
$\downarrow$ homogenize

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 3 & 2 & 1 \\ 3 & 2 & 1 & \cdots & 0 & 1 & 2 \end{pmatrix}$$

Hollow Polygons

Definition. Suppose $A = \begin{pmatrix} 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$. The *hollow triangle* of length k is $\triangle^k \coloneqq \tilde{A}$.





Betti Tables for $\triangle^k$

$\mathrm{Betti}(\triangle^3) =$

	0	1	2	3	4	5	6	7	8
total	1	17	53	91	108	83	37	9	1
0	1	.	.	.	.	.	.	.	.
1	.	17	43	36	8	.	.	.	.
2	.	.	10	55	100	83	37	9	1

$\mathrm{Betti}(\triangle^5) =$

	0	1	2	3	4	5	6	7	8	9	...
total	1	54	389	2028	7845	18957	30393	34672	29106	18162	...
0	1	.	.	.	.	.	.	.	.	.	...
1	.	54	266	462	174	15	.	.	.	.	...
2	.	.	123	1566	7671	18942	30393	34972	29106	18162	...

Results

Lemma. For all $d \geq 2$, $(R/I_{\triangle^k})_d = (\overline{R/I_{\triangle^k}})_d$.

Theorem. For all $k \geq 2$, $\mathrm{reg}(\triangle^k) = 2$.

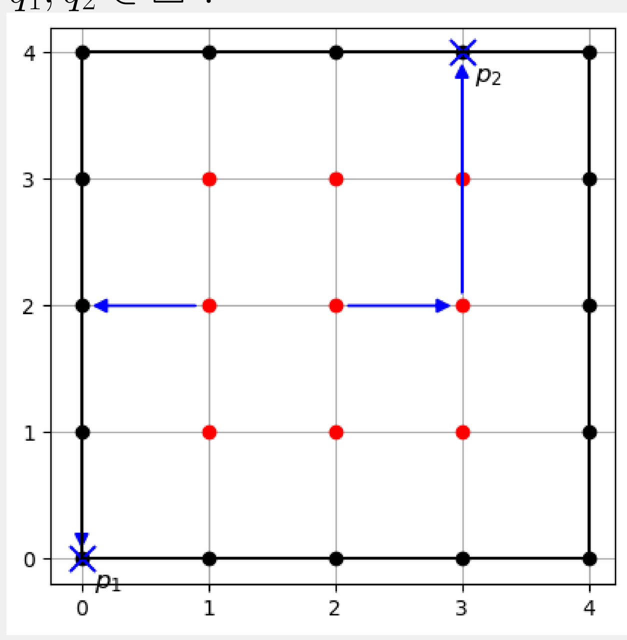
The same results hold for $\square^k$.

Proof Sketch of Theorem

- Use the short exact sequence of sheaves
$$0 \to \mathscr{I}_{\square^k}(d) \to \mathscr{O}_{\mathbb{P}^{4k-1}}(d) \to \mathscr{O}_{\square^k}(d) \to 0$$
to eventually get a short exact sequence
$$0 \to R/I_{\square^k} \to \overline{R/I_{\square^k}} \to N \to 0.$$
- By the lemma, N is generated is only generated by degree 1 monomials.
- $\mathrm{reg}(R/I_{\square^k}) \leq \max(\mathrm{reg}(\overline{R/I_{\square^k}}), N) = \mathrm{reg}(\overline{R/I_{\square^k}}) = 2.$

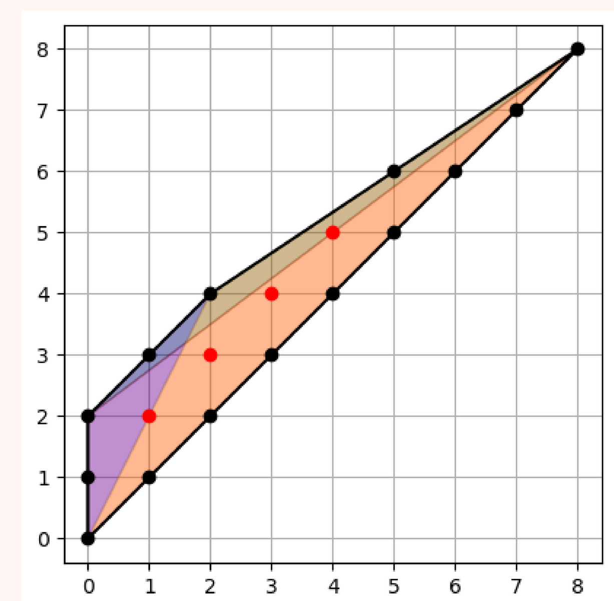
Proof Sketch of Lemma

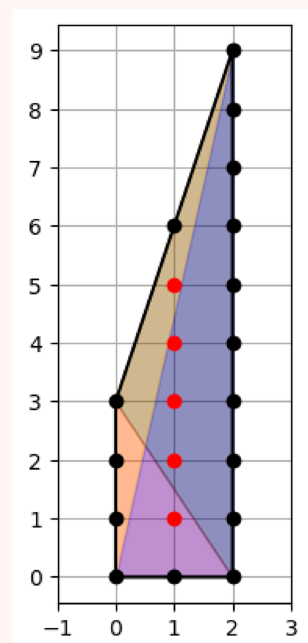
- Showing $(R/I_{\square^k})_d = (\overline{R/I_{\square^k}})_d$ for $d \geq 2$ amounts to a computation with the lattice points of $\square^k$.
- We are done if for any $p_1, p_2 \in \overline{\square^k}$, we can write $p_1 + p_2 = q_1 + q_2$ with $q_1, q_2 \in \square^k$.



$$p_1 + p_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 9 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

Smooth is Not Enough





	0	1	2	3	4	5	...
total	1	54	385	1462	3608	6456	...
0	1	.	.	.	.	.	...
1	.	52	280	730	1128	1050	...
2	.	.	81	600	2040	4416	...
3	.	2	24	132	440	990	...

	0	1	2	3	4	5	...
total	1	74	633	2883	8593	18953	...
0	1	.	.	.	.	.	...
1	.	73	486	1627	3388	4620	...
2	.	1	144	1218	4983	13541	...
3	.	.	3	38	222	792	...