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Secant Varieties and Inverse Systems

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$\mathbb{X} \subset \mathbb{P}^n$ non-degenerate, reduced, irreducible projective variety.

Definitions:
1) *Secant* $\mathbb{P}^{s-1}$ *to* $\mathbb{X}$: linear subspace, Π, of $\mathbb{P}^n$ generated by s linearly independent points of $\mathbb{X}$.
2) $(s-1)^{st}$ *Secant Variety to* $\mathbb{X}$:

$$Sec_{s-1}(\mathbb{X}) = \mathbb{X}^s :=$$

$$\overline{\cup \{ \ P \in \Pi \ \ | \, \Pi \text{ a secant } \mathbb{P}^{s-1} \text{ to } \mathbb{X} \ \}}$$

Why "*closure*" in the definition?

Example: $\mathbb{X}$ rational normal cubic in $\mathbb{P}^3$ i.e. let $R = \mathbb{C}[x_0, x_1] = \oplus_{i=0}^{\infty} R_i$, then $\mathbb{X}$ is the image of:

$$\mathbb{P}^1 = \mathbb{P}(R_1) \ \ \{= \text{ projective space of linear forms } \}$$

$$\downarrow$$

$$\mathbb{P}^3 = \mathbb{P}(R_3) \ \ \{= \text{ projective space of cubic forms } \}$$

described by

$$[L] \rightarrow [L^3].$$

When is $P \in \mathbb{P}^3$ on a secant line to $\mathbb{X}$? ($P \notin \mathbb{X}$)

Consider the projection of $\mathbb{P}^3$ into $\mathbb{P}^2$ from P and restrict it to $\mathbb{X}$. The image of $\mathbb{X}$ is a plane rational cubic curve, hence a cubic curve with a singularity i.e. with either a node or a cusp.

The singularity of that curve is a node iff the point P lies on a secant line to $\mathbb{X}$; and a cusp iff the point P lies on a tangent line to $\mathbb{X}$ (by Bezout, a point P cannot lie on both a secant line and a tangent line simultaneously).

But, a point P is on a tangent line to $\mathbb{X}$ iff it is on the tangent developable to $\mathbb{X}$, which is a surface $\mathbb{Y}$ in $\mathbb{P}^3$. So, the points which lie on (true!) secants to $\mathbb{X}$ form an open set ($\mathbb{P}^3 \setminus \mathbb{Y}$) (whose closure is all of $\mathbb{P}^3$).

The Main Problem: Given $\mathbb{X}$, what is the dimension of $\mathbb{X}^s = Sec_{s-1}(\mathbb{X})$?

(Subproblem: What's the least s for which $\mathbb{X}^s = \mathbb{P}^n$?)

The Naive Answer! It has the dimension it ought to have: in other words count parameters!

To describe a secant $\mathbb{P}^{s-1}$ to $\mathbb{X}$ we need s general points of $\mathbb{X}$, i.e. a point in $\mathbb{X} \times \mathbb{X} \times \cdots \times \mathbb{X}$ (s-times) (i.e. "$s \dim \mathbb{X}$" parameters) PLUS we have to add the fact that we can choose any point on the secant $\mathbb{P}^{s-1}$ itself, another $s-1$ parameters. So, the dimension of $\mathbb{X}^s$ should be

$$s \dim \mathbb{X} + (s-1).$$

Wait !!! of course $\mathbb{X}^s$ is also in $\mathbb{P}^n$, so, the naive parameter count gives:

$$min\{n, s \dim \mathbb{X} + (s-1)\}.$$

The Naive Answer is not always correct!

If naive answer correct $\Rightarrow$ secant line variety to a surface in $\mathbb{P}^5$ is all $\mathbb{P}^5$; so, find a surface in $\mathbb{P}^5$ whose secant line variety isn't $\mathbb{P}^5$.

Veronese Surface in $\mathbb{P}^5$: $R = \mathbb{C}[x, y, z] = \oplus_{i=0}^{\infty} R_i$; the Veronese surface in $\mathbb{P}^5$ is the image of $\mathbb{P}^2$ via

$$\phi : \mathbb{P}^2 = \mathbb{P}(R_1) \longrightarrow \mathbb{P}^5 = \mathbb{P}(R_2) \text{ by } [L] \longrightarrow [L^2].$$

The secant line variety is the closure of the set of all $[Q] \in \mathbb{P}^5$ (Q a quadratic form in R, i.e. an element of R_2) such that $[Q] = [L_1^2 + L_2^2]$.

Recall that quadratic forms (over $\mathbb{C}$) are (after a change of basis) either L^2 or $L_1^2+L_2^2$ or a sum of three squares of linear forms (depending on the rank of the associated symmetric matrix). So, for a quadratic form to lie on a "true" secant, we must have that (under some change of variables) we can write $[Q] = [L_1^2 + L_2^2]$. But, that would mean that the rank of the associated matrix was 2, i.e. the determinant of that matrix is 0. So, the secant line variety is a cubic hypersurface of $\mathbb{P}^5$ and has dimension 4, not 5.

A key tool in trying to figure out the dimensions of Secant Varieties is the Lemma of Terracini:

Lemma: (Terracini) Let $\mathbb{X} \subset \mathbb{P}^n$ be as above. The dimension of $\mathbb{X}^{s+1}$ is the same as the dimension of the linear span of the tangent spaces to $s+1$ general points of $\mathbb{X}$.

(...because that linear space **IS** the tangent space to $Sec_s(\mathbb{X})$ at a general point of the secant $\mathbb{P}^s$ generated by those $s+1$ general points.)

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I want to show today how Inverse Systems enter in considering the Main Problem for the collection of all the Veronese varieties. I.e. let $R = \mathbb{C}[x_0, \ldots, x_n]$, define $\nu_d(\mathbb{P}^n)$ as the image of

$$\nu_d : \mathbb{P}^n = \mathbb{P}(R_1) \to \mathbb{P}(R_d) = \mathbb{P}^{N_d}, N_d = \binom{d+n}{n} - 1$$

where

$$\nu_d(L) = L^d.$$

In this case, the variety we are interested in is:

$$Sec_s(\nu_d(\mathbb{P}^n)) = \overline{\{\ [F]\ \mid\ F = L_0^d + \cdots + L_s^d\}}\ .$$

(Waring Problem for Forms.)

By Terracini's Lemma we need a good way to describe the tangent space at a general point of $\nu_d(\mathbb{P}^n)$.

Let's consider ν_d affinely, i.e. we have

$$\nu_d : R_1 \simeq \mathbb{A}^{n+1}(\mathbb{C}) \longrightarrow R_d \simeq \mathbb{A}^{N_d+1}$$

and we want to understand the tangent space at a point of the image of this map.

I.e. if $P \in \mathbb{A}^{n+1}$ we are interested in the image of the linear transformation $(d\nu_d)_P$, where

$$(d\nu_d)_P : T_P(\mathbb{A}^{n+1}) \to T_{\nu_d(P)}(\mathbb{A}^{N_d+1}).$$

How does the linear transformation $(d\nu_d)_P$ act on a vector $\mathbf{v}$ in $T_P(\mathbb{A}^{n+1})$?
Take a curve $\mathcal{C}$ in $\mathbb{A}^{n+1}$ through P which has tangent vector $\mathbf{v}$ at P and apply ν_d to $\mathcal{C}$ and obtain a curve $\nu_d(\mathcal{C})$ through $\nu_d(P)$. The tangent vector to $\nu_d(\mathcal{C})$ at $\nu_d(P)$, call it $\mathbf{v}'$, is $(d\nu_d)_P(\mathbf{v})$.

So, let $P = L \in \mathbb{A}^{n+1}$ and choose a vector $\mathbf{v} = M$ in $T_P(\mathbb{A}^{n+1}) \simeq \mathbb{A}^{n+1}$ and consider the line in $\mathbb{A}^{n+1}$ through P in the direction $\mathbf{v}$ (the simplest

curve one can think of which passes through P and has tangent vector $\mathbf{v}$ at P).
The points on this line are parameterized by:

$$t \longrightarrow L + tM.$$

The image of this curve under ν_d is:

$$\nu_d(L + tM) = (L + tM)^d.$$

Now

$$\frac{d}{dt}\left((L + tM)^d\right) = d(L + tM)^{d-1}M$$

and if we evaluate this when $t = 0$ (i.e. at the point $\nu_d(P)$), we get that the tangent space at the image of the point L is

$$\langle dL^{d-1}M \rangle = \langle L^{d-1}M \rangle$$

where M varies over R_1.

Thus, by Terracini's Lemma the dimension of the (affine cone over the) variety $Sec_s(\nu_d(\mathbb{P}^n))$ is the same as the dimension of the vector space

$$V = \langle L_0^{d-1}R_1 + L_1^{d-1}R_1 + \ldots + L_s^{d-1}R_1 \rangle.$$

where $L_0, \ldots, L_s$ correspond to a general set of $s + 1$ points in $\mathbb{P}^n$.

We find this dimension via Inverse Systems (but first we change some notation).

Let

$$R = \mathbb{C}[x_0, \ldots, x_n],\ S = \mathbb{C}[y_0, \ldots, y_n]$$

and let

$$V = \langle L_0^{d-1}S_1 + L_1^{d-1}S_1 + \ldots + L_s^{d-1}S_1 \rangle.$$

where the L_i are general linear forms in S_1. We want to find $\dim V$.

Recall that we can let R act on S via partial differentiation, i.e.

$$R_i \times S_j \to S_{j-i}$$

where , when $i = 1$, we have

$$x_i \times F = \frac{\partial}{\partial y_i} F$$

It is easy to see that

$$R_i \times S_i \to \mathbb{C} = S_0$$

is a *perfect pairing* and so if $V \subset S_i$ then

$$V^{\perp} = \{ \ w \in R_i \ \mid w \circ v = 0 \text{ for all } v \in V \ \}.$$

From standard results in linear algebra we have:

$$\dim V = \dim R_i - \dim V^{\perp}$$

and

$$(V_1 + V_2)^{\perp} = (V_1)^{\perp} \cap (V_2)^{\perp}$$

so, by induction

$$\dim(V_0 + \ldots + V_s) = \dim R_i - \dim((V_0)^{\perp} \cap \ldots \cap (V_s)^{\perp}).$$

This action makes S into a graded R-module (note the direction of the action) so that if M is any graded R-submodule of S then $I = ann_R(M)$ is a homogeneous ideal of R. In the other direction, if I is a homogeneous ideal of R we define the *inverse system of* I, denoted I^{-1} by

$$I^{-1} := \{ \ m \in S \ \mid i \circ m = 0, \ \text{for all } i \in I \ \}.$$

The noted theorem of Macaulay tells us how to calculate I^{-1} very easily:

Theorem: $(I^{-1})_d = I_d^{\perp}$.

The result that brings everything together in this circle of ideas is the calculation of the inverse system for a specific ideal!

Lemma: Let $P = [a_0 : \ldots : a_n] \in \mathbb{P}^n$ and let $\wp \subset R$ be the associated prime ideal. Let $L = a_0 y_0 + \ldots + a_n y_n$ be an element of S_1. Then, if $I = \wp^{t+1}$ then

$$I^{-1} = S_0 \oplus \ldots \oplus S_t \oplus LS_t \oplus L^2 S_t \oplus \ldots$$

i.e.

$$I^{-1} = \begin{cases} S_j & \text{for } j \leq t, \\ \\ L^{j-t} S_t & \text{for } j \geq t+1. \end{cases}$$

(some remarks on the proof of the Lemma).
It is easy to prove the Lemma in case $P = [1 : 0 : \ldots : 0]$, i.e. for $L = y_0$ for then $I = (x_1, \ldots, x_n)$ and it is easy to see the result. One proceeds from this by a change of variable.

Now, let's consider the vector space we were considering above, i.e.

$$V = \langle L_0^{d-1} S_1 + \ldots + L_s^{d-1} S_1 \rangle;$$

by applying the Lemma to this vector space we see that $V = (I^{\perp})_d$ where

$$I = \wp_0^2 \cap \cdots \cap \wp_s^2$$

where the $\wp_i$ are the prime ideals corresponding to the points which correspond to the linear forms L_i, i.e. are a generic set of $s+1$ points in $\mathbb{P}^n$.

We thus obtain the following theorem:

Theorem:

Consider the Veronese Variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^N$, $N = \binom{d+n}{n} - 1$. Then

$$\dim(Sec_s(\nu_d(\mathbb{P}^n))) = \dim_{\mathbb{C}}(R/\wp_0^2 \cap \cdots \cap \wp_s^2)_d - 1.$$

Consequently, the effort in solving the Main Problem for the Veronese Varieties revolves around finding the dimension of this quotient. But, that quotient is nothing more than the Hilbert function of the ideal

$$I = \wp_0^2 \cap \cdots \cap \wp_s^2$$

in a single degree d.

The achievement of J. Alexander and A. Hirschowitz was to find the entire Hilbert function of this "*fat point*" ideal.

I won't go into details here, but with my collaborators, M.V. Catalisano, and A. Gimigliano, we have done a similar thing for the Segre Varieties, i.e. using Inverse systems, we have shown that the dimensions of the secant varieties of the various Segre Embeddings of $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_t}$ are related to calculating the Hilbert functions of certain "*fat linear schemes*" in $\mathbb{P}^{n_1+\ldots+n_t}$. We have, in certain cases, been able to calculate those Hilbert functions and so resolve the problem. But, the results here for the Segre Varieties are very, very far from complete and would take us far from the discussion of Inverse Systems.