# Pairs of commuting nilpotent matrices and Hilbert functions

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## Abstract

We denote by $\mathcal{N}_B$ the nilpotent commutator of an $n \times n$ nilpotent Jordan matrix $B$ of partition $P$, and let $A$ denote a generic elsment of $\mathcal{N}_B$ in a standard form. We denote by $Q(P)$ the partition given by the Jordan form of $A$. Using results of R. Basili, we show that $Q(P) = P$ iff the parts of $P$ differ by at least two. Let $\mathrm{Pow}(P)$ be the $n \times n$ matrix whose $(u, v)$ entry is the smallest non-negative integer $i$ for which $(A^{i+1})_{uv} = 0$. We give an algorithm to determine $\mathrm{Pow}(P)$; and we find the index (largest part) of $Q(P)$.

The Hilbert function $H$ of $K[A, B]$ is a natural invariant of a pair $(A, B)$ of nilpotent commuting matrices. We use standard bases to study the pencil

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[1]Version of March 2, 2007. Section IV has been revised and augmented since the talk.

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$A+\lambda B$, showing that for an open subset of $\lambda \in \mathbb{P}^1$, $P(A+\lambda B$ has the maximum partition $P(H)$ with diagonal lengths $H$. Thus, $Q(P)$ has decreasing parts. NOTE: Prof. B. A. Sethuraman kindly showed us after our talk a preprint of Polona Oblak in which she had determined the index of $Q(P)$ [Oblak 2007].

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**I.** Given the partition $P = P_B$, $B$ nilpotent Jordan, find $Q(P) = P_A$, for a generic $A \in \mathcal{N}_B$, the nilpotent commutator of $B$.

**II.** The integer matrix $\mathrm{Pow}_A$, for a nilpotent matrix $A$.

**III.** The matrix $\mathrm{Pow}(P)$ and the index of $Q(P)$.

**IV.** The Hilbert function of $K[A, B]$: $Q(P)$ has decreasing parts.

# 1 Given a partition $P = P_B$, $B$ nilpotent Jordan, find $Q(P) = P_A$, for a generic $A \in \mathcal{N}_B$.

Let $K =$ algebraically closed field,

$\mathcal{N}(n, K) = \{n \times n$ nilpotent matrices with entries in $K\}$.

Fix $B \in \mathcal{N}(n, K)$ in Jordan form, of partition $P = (u_1, \ldots, u_t)$.

$\mathcal{N}_B = \{A \in \mathcal{N}(n, K) \mid AB = BA\}$.

[Basili 2000] using [Turnb, Aitken 1931] shows:

**Thm 1.1.** $\mathcal{N}_B$ *is irreducible.*

**Problem 1.2.** Determine the Jordan partition $Q(P)$ of a generic element of $\mathcal{N}_B$. Determine all others for $A \in \mathcal{N}_B$.

Note: $Q(P) \geq P$. Open[2]: Is $Q(Q(P)) \neq Q(P)$ in general?

**Def.** $S$ a *string*: (max part of $S$ − min part $S$) $\leq 1$.

Let $r_P = \min\{k \mid P = P_1 \cup \cdots \cup P_k\}$, each $P_i$ a string.

**Thm 1.3.** *[Basili 2003] The # parts of $Q(P) = r_P$.*

*So $\exists$ a dense open of $A \in \mathcal{N}_B \mid \operatorname{rank}(A) = n - r_B$.*

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[2]This has since been answered by P. Oblak and T. Košir: see Theorem 4.8 and [KoOb 07]

Let $s_P = \max\{\#$ parts of $S$ for any string $S \subset P\}$.

**Thm 1.4.** *[Basili 2003] Let $A \in \mathcal{N}(B)$. Then*

$$rank(A^{s_P})^m \leq rank(B^m).$$

Notation: $2P = (u_1, u_1; u_2, u_2; \ldots; u_t, u_t)$.

By $P = (u, 1^s)$ we mean $P$ is

$$\left(u, \underbrace{1, \ldots, 1}_{s}\right)$$

**Cor 1.5.** *[Basili-I1] $P$ is* stable *under $P \to Q(P) \Leftrightarrow$*

*$s_P = 1$ (i.e. the parts of $P$ differ by at least two).*

*Also $P$ stable, $c > 0 \Rightarrow Q(cP) = (cu_1, cu_2, \ldots, cu_t)$.*

**Lem 1.6.** *(Basili) $P = (u, 1^s)$, and $u \geq 3 \Rightarrow$*

*$Q(P) = (\max\{u, s + 2\}, \min\{s, u - 2\})$.*

**Ex 1.7.** $P = (4, 4, 1, 1) \Rightarrow Q(P) = (8, 2)$.

But $P = (4, 1^3) \Rightarrow Q(P) = (5, 2)$. Also, $P = (5, 4, 3) \Rightarrow$

$Q(P) = (9, 3)$, and $P = (7, 2, 1^4) \Rightarrow Q(P) = (8, 5)$.

**Moral**: $P$ having parts of different multiplicities makes $Q(P)$ more complex.

## 2 The integer matrix $\mathrm{Pow}_A$, for $A$ a non-negative nilpotent matrix.

$\mathcal{N}(n, R) = \{\text{nilpotent matrices in } M_n(R), R = \mathbb{R} \text{ or } \mathbb{Z}[X],$ $X = (x_1, \ldots, x_e)\}$. (See Remark 3.18 for a comment on $R$).

Define $\mathrm{Pow}_A$, a matrix of non-negative integers as follows:

$$(\mathrm{Pow}_A)_{uv} = \min\{k \mid A_{uv}^{k+1} = 0\}. \tag{2.1}$$

**Lem 2.1.** *Let $A \in \mathcal{N}(n, R), R = \mathbb{R}$ or $\mathbb{Z}[X]$), have non-negative coefficients, and assume that*

$$A_{uv} = 0 \Rightarrow (A^2)_{uv} = 0.$$

*Then, for $k \geq 0$,*

$$(A^k)_{uv} = 0 \Rightarrow (A^{k+1})_{uv} = 0. \tag{2.2}$$

*Also, $\mathrm{Pow}_A$ depends only on the pattern of zero, non-zero entries of $A$.*

We now justify the last statement. First define

$$S(\text{Pow}_A, u, v) = \{\text{pairs } ((\text{Pow}_A)_{uk}, (\text{Pow}_A)_{kv}), 1 \leq k \leq n\}.$$

$$(2.3)$$

**Lem 2.2.** *Suppose that $A$ satisfies the hypotheses of Lemma 2.1. Then we have*

    I. *Let $(\text{Pow}_A)_{uv} = s > 0$. Then each integer $t, 0 \leq t < s$ appears in the $u$ row and in the $v$ column of $\text{Pow}(A)$.*

    II. *Let $(\text{Pow}_A)_{uv} = s$. Then the set of pairs*

$$\{(t, s - t), \ 1 \leq t \leq s - 1\} \subset S(\text{Pow}_A, u, v). \quad (2.4)$$

    *Conversely, if the set of pairs on the left of (2.4) is the complete set of those elements of $S(\text{Pow}_A, u, v)$ not having a zero component, then $(\text{Pow}_A)_{uv} = s$.*

This and an induction on $s$ shows that $\text{Pow}(A)$ depends only on the pattern of zero, nonzero entries of $A$.

# 3   The integer matrix $\mathrm{Pow}(P)$

Recall from R. Basili's talk, the form in a good basis $E$, of $A \in \mathcal{N}_B$, where $B$ is a nilpotent Jordan matrix of partition $P$. (See [Basili 2003, Lemma 2.3]). Given $E$ we write $A \in \mathcal{N}_{B,sp}$

**Ex 3.1.** Let $P = (3, 3, 2)$. Then there is basis $E$ for which $A \in \mathcal{N}_{B,sp}$ has the following form:

$$
\left(
\begin{array}{ccc|ccc|cc}
0 & a_{11}^2 & a_{11}^3 & a_{12}^1 & a_{12}^2 & a_{12}^3 & a_{13}^1 & a_{13}^2 \\
0 & 0 & a_{11}^2 & 0 & a_{12}^1 & a_{12}^2 & 0 & a_{13}^1 \\
0 & 0 & 0 & 0 & 0 & a_{12}^1 & 0 & 0 \\
\hline
0 & a_{21}^2 & a_{21}^3 & 0 & a_{22}^2 & a_{22}^3 & a_{23}^1 & a_{23}^2 \\
0 & 0 & a_{21}^2 & 0 & 0 & a_{22}^2 & 0 & a_{23}^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & a_{31}^2 & a_{31}^3 & 0 & a_{32}^2 & a_{32}^3 & 0 & a_{33}^2 \\
0 & 0 & a_{31}^2 & 0 & 0 & a_{32}^2 & 0 & 0
\end{array}
\right)
$$

with entries in the ring $\mathbb{Z}[a_{11}^2, \ldots, a_{33}^2]$ in 18 variables.

**Lem 3.2.** *Let $A$ be generic in $\mathcal{N}_{B,sp}$. Then $A_{uv} = 0 \Rightarrow A^2{}_{uv} = 0$. When* char $K = 0$, $(A^k)_{uv} = 0 \Rightarrow (A^{k+1})_{uv} = 0$.

**Def.** $\text{Pow}(P) = \text{Pow}_A$, for $A$ generic in $\mathcal{N}_B$.

Index $i(Q) = $ largest part of $Q$.

Note: index of $Q(P) = 1+$ maximum entry of $\text{Pow}(P)$

**Ex 3.3.** For $P = (3, 3, 2)$, we have $\text{Pow}(P)$ is

$$
\left(
\begin{array}{ccc|ccc|cc}
0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\
0 & 0 & 3 & 0 & 1 & 4 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 2 & 5 & 0 & 3 & 6 & 1 & 4 \\
0 & 0 & 2 & 0 & 0 & 3 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 4 & 0 & 2 & 5 & 0 & 3 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
\end{array}
\right).
$$

The index $i(Q(P)) = 7 + 1 = 8$ and $Q(P) = (8)$

Since $r_P = 1$ this follows also from Thm. 1.3.

**Notation**: We introduce a compressed notation $\mathrm{CPow}(P)$ listing the top rows of key small blocks of $\mathrm{Pow}(P)$, one for each pair of integers $(q, p)$ occurring as parts of $P$. For $q < p$ the key $(q, p)$ block $B^{qp}$ of $\mathrm{Pow}(P)$ is the one in the lower left corner of the set of $q \times p$ small blocks.

When $q = p$ we take for $B^{q,q}$ any diagonal small block among the $q \times q$ blocks in $\mathrm{Pow}(P)$.

We also include the top row of the small block $B^{pq}$ in the upper right corner of the $p \times q$ small blocks, for $p > q$.

We arrange the top rows of these key blocks according to their relative positions in $\mathrm{Pow}(P)$.

**Ex.** For $P = (3, 3, 2)$ the compressed notation is

$$
\left(
\begin{array}{ccc|cc}
0 & 3 & 6 & 2 & 5 \\
0 & 1 & 4 & 0 & 3
\end{array}
\right).
$$

(See the $\mathrm{Pow}(P)$ matrix on just previous page.)

**Ex 3.4.** For $P = (4, 2, 2, 2)$, we have $\mathrm{Pow}(P)$ is

$$
\left(
\begin{array}{cccc|cc|cc|cc}
0 & 1 & 4 & 7 & 1 & 4 & 2 & 5 & 3 & 6 \\
0 & 0 & 1 & 4 & 0 & 1 & 0 & 2 & 0 & 3 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 3 & 6 & 0 & 3 & 1 & 4 & 2 & 5 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 & 2 \\
\hline
0 & 0 & 2 & 5 & 0 & 2 & 0 & 3 & 1 & 4 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 1 & 4 & 0 & 1 & 0 & 2 & 0 & 3 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\right).
$$

The index $i(Q(P)) = 7 + 1 = 8$; since $r_P = 2$, $Q(P) = (8, 6)$.

The matrix $\text{CPow}(P)$ for $P = (4, 2, 2, 2)$ is

$$\begin{pmatrix} 0 & 1 & 4 & 7 & 3 & 6 \\ 0 & 0 & 1 & 4 & 0 & 3 \end{pmatrix}.$$

**Ex 3.5.** For $P = (5, 3^2, 1^a), a \geq 2$ we have $i(Q(P)) = (6+a)$,

and $\text{CPow}(P)$ is

$$\begin{pmatrix} 0 & 1 & 3 & 5 & 5+a & 2 & 4 & 4+a & 2+a \\ 0 & 0 & 1 & 3 & 3+a & 0 & 2 & 2+a & 1+a \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since $r_P = 3, Q(P)$ has three parts. Is $Q(P) = (6+a, 4, 1)$ or

$(6+a, 3, 2)$?[3]

But for $P = (5, 3^2, 1)$ we have $\text{CPow}(P)$ is

$$\begin{pmatrix} 0 & 1 & 3 & 5 & 7 & 2 & 4 & 6 & 3 \\ 0 & 0 & 1 & 3 & 5 & 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

and $Q(P) = (8, 3, 1)$.

[3]Theorem 4.8 of T. Košir and P. Oblak implies $Q(P) = (6 + a, 4, 1)$.

For $P = (5, 3^2, 1^3)$ we have for $\mathrm{Pow}(P)$

$$\left(\begin{array}{ccccc|ccc|ccc|ccc}
0 & 1 & 3 & 5 & 8 & 1 & 3 & 6 & 2 & 4 & 7 & 3 & 4 & 5 \\
0 & 0 & 1 & 3 & 5 & 0 & 1 & 3 & 0 & 2 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 2 & 4 & 7 & 0 & 2 & 5 & 1 & 3 & 6 & 2 & 3 & 4 \\
0 & 0 & 0 & 2 & 4 & 0 & 0 & 2 & 0 & 1 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 3 & 6 & 0 & 1 & 4 & 0 & 2 & 5 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 5 & 0 & 0 & 3 & 0 & 0 & 4 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0
\end{array}\right)$$

**Thm.** $\mathrm{CPow}(P)$ and $P$ determine both $\mathrm{Pow}(P)$ and the index $i(Q(P))$.  (Drawing of fish – What is fishy about this "theorem"?)

*Proof.* Trivial, as $P$ determines both $\text{Pow}(P)$ and $Q(P)$! $\quad\square$

However, we will develop a more precise version.

**Def.** Let $q \leq p$ be integers occurring as parts of $P$. For $q \leq p$ denote by $U(q, p)$ the $q \times p$ matrix that is zero except for an upper triangular rightmost $q \times q$ submatrix, whose nonzero entries, all entries above or on its main diagonal, equal one.

We let $U(q) = U(q, q)$ and $SU(q) = U(q) - I_q$.

**Definition 3.6.** A *translation* of the key $q \times p, q \leq p$ block $B^{qp}$ by $s \geq 0$ is the block

$$T_s(B^{qp}) = B^{qp} + s \cdot U(q, p).$$

A *translation* of a $q \times q$ diagonal block $B^{qq}$ by $s$ is adding $s \cdot U(q)$ for $s > 0$, and subtracting $|s| \cdot SU(q)$ for $s < 0$.

A *reflection* of $B^{qp}, q < p$ (or, respectively, of $T_s(B^{qp})$) is the $p \times q$ block whose leading (top) $q \times q$ subblock is the rightmost $q \times q$ submatrix of $B^{qp}$ (respectively, of $T_s(B^{qp})$), and whose last $p - q$ rows are zero.

**Thm 3.7.** *Let $q \leq p$ be integers occurring as parts of $P$.*

A. *When $q < p$ all the small $q \times p$ blocks of $\mathrm{Pow}(P)$ may be obtained from the key $q \times p$ block $B^{qp}$ by translation, taking $s$ to be the rook distance between the subblocks.*

B. *All $p \times q$ subblocks of $\mathrm{Pow}(P)$ may be obtained from the appropriate $q \times p$ subblock by reflection. All $q \times q$ small blocks of $\mathrm{Pow}(P)$ may be obtained from $B^{qq}$ by appropriate positive or negative translation by $|s|$.*

   *(Here $|s|$ is the distance from the main diagonal.)*

C. *All long diagonals (parallel to the main diagonal) of the set of $q \times p, q \leq p$ blocks of $\mathrm{Pow}(P)$ are constant.*

**Question.** The theorem shows how $\mathrm{CPow}(P)$ and $P$ determine $\mathrm{Pow}(P)$. But how do we determine $\mathrm{CPow}(P)$?. We begin with the special cases $P = m^a$ and $P = (m^a, n^b)$.

**Notation.** Henceforth in this section (only) we consider $N \times N$ matrices, and partitions of length $N$, as we use $n$ as a part of $P$.

**Thm 3.8.** *Let $P = (m^a)$. Then* $\mathrm{CPow}(P)$ *is*

$$(0, a, 2a, \ldots, (m-1)a),$$

*the largest entry of* $\mathrm{Pow}(P)$ *is* $ma - 1$, *and* $Q(P) = (ma)$.

*Note: Add $a - 1$ to each entry of the key row above to obtain the top row of the rightmost $m$-block, $(a - 1, 2a - 1, \ldots, ma - 1)$.*

**Thm 3.9.** *Let $P = (m^a, n^b), m \geq n + 2$. We have $r_P = 2$,*

$$Q(P) = (\max\{ma, nb + 2a\}, \min\{nb, (m-2)a\}).$$

*For* $\mathrm{CPow}(P)$ *we break into cases.*

**A.** *Suppose $a \geq b$, and $m \geq 2n$, $(m, n) \neq (2, 1)$ then* $\mathrm{CPow}(P)$ *is*

$$\begin{pmatrix} (0, a, 2a, \ldots, (m-1)a) & (a+b-1, 2a+b-1, \ldots, na+b-1 \\ (0^{m-n}, 1, 1+a, \ldots, 1+(n-1)a) & (0, b, 2b, \ldots (n-1)b) \end{pmatrix}$$

**B.** *Suppose $a \geq b$ and $n + 2 \leq m < 2n$, then* $\mathrm{CPow}(P)$ *is*

$$\begin{pmatrix} (0, a, 2a, \ldots, (m-1)a) & (a+b-1, 2a+b-1, \ldots, na+b-1 \\ ((0^{m-n}, 1, 1+a, \ldots, 1+(n-1)a) & (0, b, 2b, \ldots (m-n-1)b, t_1, \ldots, t_{2n-m}) \end{pmatrix}$$

*where $t_k = \max\{(m-n-1+k)b, ka + b\}$.*

**C.** *Suppose $a \leq b$, and $m \geq n + 2$, then, letting*

$$k = m - n - 1,$$

$$\alpha_i = \alpha_i(a, b, m, n) = \max\{(i-1)a, a + (i - (m-n))b\},$$

$\mathrm{CPow}(P)$ *is (note the first entry is $(\alpha_1, \ldots, \alpha_m)$)*

$$
\begin{pmatrix}
(0, a, \ldots, ka = \alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_m) & (a + b - 1, a + 2b - 1, \ldots, a + nb - 1) \\
(0^{m-n}, 1, 1 + b, \ldots, 1 + (n-1)b) & (0, b, 2b, \ldots (n-1)b)
\end{pmatrix}
$$

**D.** *Arranging the small blocks of $\mathrm{Pow}(P)$ in quadrants*

$$
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
$$

*we have, the $n \times n$ sub-subblocks $F^{ij}$ topmost of each $m \times n$ subblock of $F$, and the $n \times n$ rightmost sub-subblocks $G^{ij}$ of each $n \times m$ block of $G$ satisfy*

$$F^{ij} = G^{b+1-j, a+1-i} \mid 1 \leq j \leq b, 1 \leq i \leq a. \qquad (3.1)$$

*Proof.* By induction on $| a - b |$. For the equal multiplicity case $b = a$, $P = (m^a, n^a)$, $m \geq n + 2$, $\mathrm{Pow}(P)$ has a simple form, and by Cor 1.5 we have $Q(P) = (m^a, n^a)$.

*Second proof:* see algorithm for the general case, below.[4] □

MORAL: Oldies of age $m$ cease to influence youngies of age $n$ when their ages satisfy $m \geq 2n$, even if there are many more oldies. Youngies always influence oldies whenever there are more youngies.

OR: When $a > b$, and $2n > m \geq n + 2$ the last $2n - m$ entries of the diagonal $n$-blocks are affected by the $a$-multiplicity. When $b > a$ the last $n$ entries of the top row of the diagonal $m$ block are the max of the sequence $((m-n-1)a, \ldots, (m-1)a)$ and the translation $T_s(B^{nn})$ by $s = a + b - 1$.

**Cor 3.10.** *Let $P = (m^a, n^b)$, $m \geq n + 2$; then $Q(Q(P)) = Q(P)$. Equivalently (in this case) $Q(P)$ has two parts, that differ by at least two (See Cor. 1.5).*

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[4] A variation uses an involution $\sigma$ on $\mathcal{N}_{B,sp}$. See Remark 3.18 H.

**Ex 3.11.** Let $P = (6^7, 4^2)$. Then $Q(P) = (42, 8)$ and $\mathrm{CPow}(P)$ is

$$\begin{pmatrix} (0, 7, 14, 21, 28, 35) & (8, 15, 22, 29) \\ (0, 0, 1, 8, 15, 22) & (0, 2, 9, 16) \end{pmatrix}$$

Note that the topmost right 6-block of $\mathrm{Pow}(P)$ has top row $(6, 13, 20, 27, 34, 41)$, being a translation by $a - 1 = 6$ of the top row $(0, 7, 14, 21, 28, 35)$ of the key 6-block $B^{6,6}$.

Since $2n - m = 2$, the last two entries of $B^{44}$ are affected by $a$. Thus, the entry 9 can be understood as a *product* of the top row $(0, 0, 1, 8, 15, 22)$ of $B^{46}$ times the third column $(22, 15, 8, 0, 0, 0)^T$ of $B^{64}$: that is,

$$9 = \max\{0 \cdot 22, 0 \cdot 15, 1 + 8, 8 \cdot 0, 15 \cdot 0, 22 \cdot 0)\}.$$

Likewise, the last entry 16 in the top row of $B^{44}$ can be understood as the lead entry of the fourth column of the product of $B^{46}$ and $B^{64}$. (See $*$ product defined before Thm 3.15).

But the second entry 2 of the top row of $B^{44}$ agrees with that entry of the one-row $\mathrm{CPow}(4^2) = (0, 2, 4, 6)$.

**Ex 3.12.** Let $P = (6^3, 4^5)$. Then $Q(P) = (26, 12)$, and $\mathrm{CPow}(P)$ is

$$
\begin{pmatrix}
(0, 3, 8, 13, 18, 23) & (7, 12, 17, 22) \\
(0, 0, 1, 6, 11, 16) & (0, 5, 10, 15)
\end{pmatrix}
$$

Here in $(m^a, n^b)$ we have $b = 5 > a = 3$ so there is an effect of $4^5$ on the last four entries of the top left row $r(6)$ of $\mathrm{CPow}(P)$, which come from the block $B^{66}$ in $\mathrm{Pow}(P)$. The third entry is (see Theorem 3.9 C),

$$
8 = \max\{2 \cdot 3, 3 + 5\},
$$

here one more than 7, the first entry of $B^{64}$, which is $a + b - 1$.

**Lem 3.13 ($\mathrm{CPow}(P)$ for a String $P$).** *Let $P = (m^a, n^b), n = m - 1$ and set $c = a + b$. Then $r_P = 1$, $Q(P) = (ma + nb)$, and $\mathrm{CPow}(P)$ is*

$$
\begin{pmatrix}
(0, c, 2c, \ldots, (m - 1)c) & (c - 1, 2c - 1, \ldots, nc - 1) \\
(0, 1, c + 1, \ldots, (n - 1)c + 1) & (0, c, 2c, \ldots, (n - 1)c)
\end{pmatrix}
$$

**Def.** We let $m = u_1$ be the largest part of $P$, and we use the exponent notation, $n_i$ is the number of parts equal to $i$ in $P$:

$$P = (m^{n_p}, (m-1)^{n_{p-1}}, \ldots, 2^{n_2}, 1^{n_1}), n_i \geq 0, \qquad (3.2)$$

a partition of $N = \sum_{i=1}^{m} i \cdot n_i$.

The *jump index* of the occurring part $i$ of $P$ is

$$j_i(P) = \max\{n_i + n_{i-1}, n_i + n_{i+1}\}$$

We let

$$s_i = \sum_{k>i} n_k.$$

**Thm 3.14 (Index of $Q(P)$).** *We have*

$$\text{index}(Q(P)) = \max_{1 \leq i \leq m}\{2s_i + n_i + (i-1)j_i\}. \qquad (3.3)$$

*Comment.* This follows from the next theorem. Note that, since the maximum entry of $\text{Pow}(P)$ occurs in the top right-most $m \times m$ block of $\text{Pow}(P)$, we need only consider the effect of $n_i$ for smaller parts $i$ on the last entry in the top row of $B^{mm}$, then add the multiplicity $n_m$ to the result. $\square$

The principle for the following theorem can be seen in the case $P = (m^a, n^b), m > n$. In particular

- The $a$-th nonzero entry (so $a + 1$-st entry) of the diagonal $q$ row of $\mathrm{CPow}(P)$ is influenced only by the multiplicity $n_k$ in $P$ of integers $k$ satisfying $q - a \le k \le q + a$.

- We give an order of steps to construct $\mathrm{CPow}(P)$. It suffices to specify the non-zero entries.

1. We specify the first nonzero entry of all rows at once;

2. We specify inductively the $a + 1$-st non-zero entry of each row as a maximum of integers arising from "products" involving previously chosen partial rows.

**Def.** For rows $a = (a_1, a_2, \ldots, a_p)$ and $b = (b_1, b_2, \ldots b_q)$ we define the "$*$-product"

$$a *_k b = \max'\{a_1 + b_k, a_2 + b_{k-1}, \ldots\}$$

where $\max'$ excludes pairs with a zero summend.

We label the rows of $\mathrm{CPow}(P)$ by $r(q, p)$ or $r(q)$ for $r(q, q)$.

**Thm 3.15 (Algorithm).** *We construct* $\mathrm{CPow}(P)$:

*A. The first non-zero entries of rows of* $\mathrm{CPow}(P)$ *satisfy*

    *i. For $r(q)$, the jump index $j_q$. (it is 2nd entry)*

    *ii. For $r(q, p)$, $q < p$, $1 + s_q - s_{p-1} = 1 + n_{q+1} + \cdots + n_{p-1}$.*

    *(it is the $(p + 1 - q)$-th entry).*

    *iii. For $r(p, q)$, $p > q$, $(n_p + n_{p-1} \cdots + n_q) - 1$.*

*B. For $a > 1$ the $a$-th non-zero entries of rows satisfy*

    *i. For each $r(q)$ row, the maximum of*

        1. $a \cdot j_q$,

        2. $\max_{2q > p > q+1} \{r(q, p) *_{a+1} r(p, q)\}$,

        3. $\max_{q-a \leq i < q-1} \{r(i)_{i+a+1-q} + n_i + 2s_i - 2s_q - n_q\}$

    *(it is the $(a + 1)$ entry of $r(q)$)*

*ii. For each $r(q, p), q < p$ row, the maximum of*

$$1. \ r(q)_{a+1} + 1$$

$$2. \ r(q, p) *_{p-q+a} r(p).$$

**Note**. The integers to be maximized in B(1)(iii) involve the $a + (i - (q - 1)) \leq a$-th entry of each $r(i), i < q$; these are the $a - 1$ or earlier non-zero entries of $r(i), i < q$, which have already been decided.

**Ex 3.16.** Let $P = (5^2, 4^3, 2^4, 1^7)$. Then $Q(P) = (25, 12)$ and $\mathrm{CPow}(P)$ is

$$
\begin{pmatrix}
(0, 5, 10, 15, 23) & (4, 9, 14, 22) & (8, 19) & 15 \\
(0, 1, 6, 11, 19) & (0, 5, 10, 18) & (6, 17) & 13 \\
(0, 0, 0, 4, 15) & (0, 0, 1, 12) & (0, 11) & 10 \\
(0, 0, 0, 0, 8) & (0, 0, 0, 5) & (0, 1) & 0
\end{pmatrix}
$$

**Ex 3.17.** $P = (10^7, 7^4), Q(P) = (70, 28)$. Then $\mathrm{CPow}(P)$ is

$$
\begin{pmatrix}
(0, 7, 14, 21, 28, 35, 42, 49, 56, 63) & (10, 17, 24, 31, 38, 45, 52) \\
(0, 0, 0, 1, 8, 15, 22, 29, 36, 43) & (\underbrace{0, 4, 8,}_{m-n} \underbrace{12, \underline{18}, 25, 32}_{t_1, t_2, t_3, t_4})
\end{pmatrix}
$$

This is the case $a > b$, and $n < m < 2n$ when the $(n, n)$ entry

of $\mathrm{CPow}(P)$ is affected by $m^a$. By formula, we have

$$t_2 = \max(4b, b + 2a) = \max(16, 18) = 18.$$

Using the $*$ product underlying this formula we have, $t_2$ is the

maximum of two such products,

$$(0, 4, 8, 12, ?\ldots) *_5 (0, 4, 8, 12, ?)$$

$$= \max\{0 \cdot ?, 4 + 12, 8 + 8, 12 + 4, ? \cdot 0\} = 16$$

and

$$= (0, 0, 0, 1, 8, ?\ldots) *_5 (10, 17, 24, 31, 38, ?\ldots)$$

$$= \max\{0 \cdot 38, 0 \cdot 31, 0 \cdot 24, 1 + 17, 8 + 10\} = 18.$$

**Note:** The star product is just a convenient way of noting the

Hankel property of $\mathrm{Pow}(P)$ that each small block has constant

diagonals.

**Remark 3.18.** [5]

A. Typically, algebraic closure of $K$ is needed to assure that $A$ has a Jordan form; however, since $A$ is nilpotent, it is not needed for this purpose. The results of R. Basili that we use concerning the form of $A \in \mathcal{N}_B$ and the invariants $r_P, s_P$ are valid for all fields $K$.

B. The argument used in Lemma 2.1 concerning $\mathrm{Pow}_A$ requires a sum of "positive" terms to have no cancellation, so it does not work in characteristic $p$. However, for $\mathrm{Pow}(P)$ it appears that one can show there is no cancellation, by keeping some track of the sums of monomoials comprising the entries $(A^k)_{uv}$ for $A$ generic in $\mathcal{N}_B$: such an argument would extend Lemma 3.2 to all characteristics. We plan to do this, but have not yet; there is no issue for $k = 1$.

C. R. Basili's Theorem 1.3 gives $Q(P)$ regular when $r_P = 1$. When $r_P = 2$, knowing the index determines $Q(P)$: these cases are readily enumerated and involve $P$ having no more than four nonzero $n_i$.

D. The statements in Corollary 1.5 readily generalize to give $Q(P) = (|S_1|, \ldots, |S_t|)$ for $P$ a union of strings separated from each other by at least two.

---

[5]Added January 31, 2007, revised March 2, 2007.

In general, the map $P \to Q(P)$ remains mysterious to us.

E. The scheme $\mathcal{N}_B$ is irreducible, but it is not in general closed (see Example 4.2). Thus, even if we know $Q(P)$ the set $\mathcal{P}\mathcal{N}(P) = \{P_A, A \in \mathcal{N}_B\}$ is not understood. Let $\mathcal{N}_{P,Q} = \{A \in \mathcal{N}_B, \mid P_A = Q\}$. What can we say about the schemes $\mathcal{N}_{P,Q}$ and their closures $\overline{\mathcal{N}_{P,Q}}$? Can we determine their dimensions? Are the closures Cohen-Macaulay?

F. A. Zelevinksy has pointed out to us the article of S. Poljak [Pol 1989], obtaining the maximum possible rank of the power $A^p$ of a matrix $A$ with a given pattern of zero, non-zero entries, in terms of the number of independent $p$-walks on a digraph giving the pattern. See also [KnZe, p. 278 ff].

G. Since the talk, P. Oblak and T. Kosir have shown that $Q(P)$ is stable $Q(Q(P)) = Q(P)$, by showing that for a generic $A \in \mathcal{N}_B$ the ring $K[A, B]$ is Gorenstein. See Theorem 4.8 and [KoOb 07].

H. There is an involution on $\mathcal{N}_{B,sp}$ that underlies several of the symmetries in $\mathrm{Pow}(A)$, in particular the "reflections", and diagonal "translations". We will treat this in [Basili-I1].

I. An auxiliary matrix

$$\text{Powx}(P) : \text{Powx}(P)_{uv} = A^s_{uv} \mid \text{Pow}(P)_{uv} = s$$

has a quite simple structure that underlies the symmetries of $\text{Pow}(P)$.

See [Basili-I2].

J. T. Harima and J. Watanabe have also studied the structure of $\mathcal{N}_B$

[HW1 2005, HW2 2006].

# 4 The Hilbert function of $K[A, B]$; $Q(P)$ has decreasing parts.

In this section[6] we assume that $K$ is an algebraically closed field, and denote by $R = K\{x, y\}$ the power series ring, i.e. the completed local ring at $(0, 0)$) of the polynomial ring $K[x, y]$. We denote by $M = (x, y)$ the maximal ideal of $R$.

**Def..** Given a pair of commuting $n \times n$ nilpotent matrices $(A, B)$, consider the Artin algebra

$$\mathcal{A} = \mathcal{A}_{A,B} = R/I, I = \ker(\theta), \theta : R \to k[A, B].$$

Let $\mathcal{H}(n, K) = \{(A, B) \in \mathcal{N}(n, K) \mid AB - BA = 0\}$ and $\mathcal{U}(n, K)$ be the open subset such that $\dim_K(\mathcal{A}_{A,B}) = n$. We denote by $H = H(\mathcal{A})$ the *Hilbert function* of $\mathcal{A}$.

---

[6]Augmented and substantially rewritten after the Workshop

## 4.1  Note re inverse systems and punctual Hilbert scheme

The inverse system $I^\perp$ of $\mathcal{A}$ is generated by elements of the divided power ring $K_{DP}[X, Y]$ in dual variables. In the complete intersection (CI) case, where $I$ has two generators, there is a unique polynomial generator of $I^\perp$. Then there is a further structure on $\mathcal{A}^* = Gr_M(\mathcal{A})$, the associated graded algebra (see [I2]). The CI case was studied in the early 1900's by F.H.S. Macaulay, and C.A. Scott [Scott 1902, Mac 1904].

However, the Hilbert Burch structure theorem for height two ideals, and as well the parameters for $\text{Hilb}^H(R)$ of J. Briançon á la Hironaka, [Br 77, I1] have been the main "elementary" tools in studying the Hilbert scheme in two variables, not so far the inverse system. Recently, H. Nakajima, M. Haiman and others have used deeper tools to greatly develop the study of this punctual Hilbert scheme.

## 4.2 Irreducibility of $\mathrm{Hilb}^n R$ and $\mathcal{H}(n, K)$

. J. Briançon and subsequently M. Granger of the Nice school, showed that the family $\mathrm{Hilb}^n R$ of length-$n$ Artinian quotients of $R$ is irreducible, in characteristic zero [Br 77, Gr 83]. It was a slight extension to show their proofs applied to char $K > n$ [I1].

V. Baranovsky, R. Basili, and A. Premet connected this problem to that of the irreducibility of $\mathcal{H}(n, K)$ [Bar2001, Basili 2003, Premet 2003]. Recall that we set

$$\mathcal{U}(n, K) = \{(A, B \in \mathcal{H}(n, K) \mid \dim_K(\mathcal{A}_{A,B}) = n\}.$$

By the universal property of the Hilbert scheme, there is a morphism,

$$\pi : \mathcal{U}(n, K) \to \mathrm{Hilb}^n(R) : \pi(A, B) = \mathcal{A}_{A,B}.$$

It follows that the irreducibility of $\mathcal{H}(n, K)$ was equivalent to that of $\mathrm{Hilb}^n(R)$. V. Baranovsky used this to prove the irreducibility of $\mathcal{H}(n, K)$, for char $K = 0$ and char $K > n$.

R. Basili gave an "elementary" proof of the irreducibility of $\mathcal{H}(n, K)$, that worked also for char $K \geq n/2$. A. Premet later gave a Lie algebra proof of the irreducibility of $\mathcal{H}(n, K)$ that is valid in all characteristics. Incidently, that $K$ be algebraically closed is necessary as the $\mathrm{Hilb}^n(R)$ is reducible over the reals (see [I1]).

It would be of interest to see how the R. Basili or A. Premet proofs would look, directly applied to Artinian quotients of $R$.

### 4.3   Hilbert function strata:

Let $H$ be a fixed Hilbert function sequence. There has been little study of the connection between the Hilbert function strata $Z_H = \mathrm{Hilb}^H(R)$, and the analogous subscheme

$$\mathcal{H}^H(n, K) = \pi^{-1}(Z_H) = \{\text{pairs } (A, B) \mid H(\mathcal{A}_{A,B}) = H\}.$$

$Z_H$ is irreducible, and has a cover by affine spaces [Br 77]. We have the projection

$$\pi : Z_H \to G_H, \mathcal{A} \to Gr_M(\mathcal{A})$$

to the irreducilbe projective variety $G_H$ parametrizing graded quotients of $R$ having Hilbert function $H$. $Z_H, G_H$ and the fibres of $\pi$ each have covers by opens in affine spaces of known dimension [I1]. The homology groups of $G_H$ are known but the homology ring structure is understood only in a few special cases ([IY]). The Nice school studied specializations of $Z_H$, see work of M. Granger [Gr 83] and J. Yaméogo [Yaméogo 1994a] but the problem of understanding the intersection $\overline{Z_H} \cap Z_{H'}$ is in general difficult and unsolved. Let $Z_{\nu,n}$ parametrize order $\nu$ colength $n$ ideals in $R = K\{x, y\}$ (completed local ring at the origin of $K[x, y]$): that is

$$Z_{\nu,n} = \{I \mid M^\nu \supset I, M^{\nu+1} \not\supseteq I\}.$$

J. Briançon's irreducibility result can be stated,

$$\mathrm{Hilb}^n R = \overline{Z_{1,n}}.$$

M. Granger showed, more generally,

$$\overline{Z_{\nu,n}} \supset Z_{\nu+1,n}, \nu \geq 1.$$

The Hilbert function of an Artinian quotient $\mathcal{A} = R/I$ of $R = K\{x, y\}$ satisfies,

$$H = (1, 2, \ldots, \nu, h_\nu, \ldots, h_c), \nu \geq h_\nu \geq \ldots \geq h_c > 0, \quad (4.1)$$

or, when $\nu(I) = 1$ (i.e. $I \not\subseteq (x, y)^2$), $H = (1, 1, \ldots, 1)$.

We denote by $\Delta H$ the difference sequence $\Delta H_i = H_{i-1} - H_i$.

**Thm 4.1.** *[I1, IY] The dimension of $G_H$ satisfies*

$$\dim G_H = \sum_{i \geq d}(\Delta H_i + 1)(\Delta H_{i+1}).$$

**Def.** The *diagonal lengths $H_P$* of a partition $P$ are the lengths of the lower left to upper right diagonals of a Ferrer's graph of $P$ having largest part at the top.

We denote by $P(H)$ the maximum partition of diagonal lengths $H$: it satisfies $u_i(P(H)) = $ length of the $i$-th row of the bar graph of $H$.

**Ex.** $P = (3, 3, 3)$ has $H_P = (1, 2, 3, 2, 1)$. $H = (1, 2, 3, 2, 1), P(H) = (5, 3, 1)$.

**Note.** The length $n$ Hilbert functions satisfying (4.1) correspond 1-1 via $H \to P(H)$ to the partitions of $n$ having decreasing parts.

Let $I$ be an ideal of colength $n$ in $R = K[x, y]$ and let $H = H(A), A = R/I$. Consider the deg lex order, and the monomial initial form ideal $E_\lambda$ for $I$ using the standard basis for $I$ in the direction $y + \lambda x, \lambda \in \mathbb{P}^1$. The monomial cobasis $E_\lambda^c = N^2 - E_D$ may be seen as the Ferrer's graph of a partition $P(E_\lambda^c)$ of diagonal lengths $H$.

We now give an example illustrating the connection between Hilbert function strata $Z_H$ of Artinian algebras and those of commuting nilpotent matrices. Here are some features. As-

sume $k[A, B] \in \mathcal{H}^H(n, K)$. Then

i. The ideals that occur in writing $k[A, B] \cong R/I$ are in general non-graded.

ii. The partition $P$ need not have diagonal lengths $P(H)$.

iii. The partition $P_\lambda$ arising from the action of $B + \lambda A, \lambda$ satisfies $P_\lambda = P(H)$ for a generic $\lambda$ (all but a finite number).

iv. The closure of the orbit of $P$ includes a partition of diagonal lengths $P(H)$.

**Ex 4.2 (Pencil and specialization).** Recall that for $P = (3, 1, 1)$ we have $Q(P) = (4, 1)$ by Lemma 1.6. Take for $B$ the Jordan matrix of partition $(3, 1, 1)$. By [Basili 2003, Lemma 2.3], a good basis may be chosen so that $A \in \mathcal{N}_B$ satisfies

$$
B = \left(\begin{array}{ccc|cc}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right), \quad
A = \left(\begin{array}{ccc|cc}
0 & a & b & f & g \\
0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & e & 0 & c \\
0 & 0 & d & 0 & 0
\end{array}\right).
$$

We send $A \to x$, $B \to y$. Let $\beta = 1/(cdf)$, and let

$$g_2 = y^2 - \beta x^3, g_1 = y - a\beta x^2, g_0 = 1.$$

Considering the standard basis for $I$ in the $x$ direction (see equation (4.2) in the proof of Lemma 4.4 below) we have

$$\mathcal{A}_{A,B} = K[A, B] \cong R/I, I = (g_2, xg_1, x^4 g_0).$$

and $H(K[A, B]) = (1, 2, 1, 1)$. The product action of the generic $A = m_x$ on the classes $\langle 1, x, x^2, x^3; g_1 \rangle$ in $\mathcal{A}$ illustrates the $(4, 1)$ Jordan form.

The action of $B = m_y$ on the classes of $\langle 1, y, \beta x^3; x - ay, y^2 \rangle$ in $\mathcal{A}$ (note that $xy - ay^2, y^3 \in I$) illustrates that $P_B = (3, 1, 1)$ of diagonal lengths $(1, 2, 2)$, which is *not* $H(\mathcal{A})$.

Now consider the associated graded algebra $\mathcal{A}^* = R/I^*$: here $I^* = (y^2, xy, x^4)$. The action of $m_y$ on $\langle 1, y, x, x^2, x^3 \rangle$ has Jordan partition $P' = (2, 1, 1, 1)$ of diagonal lengths $H(A) = (1, 2, 1, 1)$. Also, holding $a$ constant, we have

$$I^* = \lim_{\beta \to 0}{}' I,$$

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so $P' = (2, 1, 1, 1)$ is in the closure of the orbit of $B$.

Here $\dim G_H = 1$: a graded ideal of Hilbert function $H$ must satisfy

$$\exists L \in R_1 \mid I = (xL, yL, M^4),$$

so $G_H \cong \mathbb{P}^1$, and $I \in G_H$ is determined by the choice of the linear form $L$, here $L = y$. The fibre of $Z_H$ over a point of $G_H$ is determined here by the choice of $a, \beta$, so has dimension two.

### 4.4 $Q(P)$ has decreasing parts; stability after P. Oblak and T. Košir.

We thank T. Košir and B. A. Sethuraman for pointing out to us the following result from combining [Neub, Salt 1994, Theorem 1.1]. and [Bar2001]. It also follows from considering a suitable monomial ideal in $K\{x, y\}$, determined by $B$ (see (4.2) below). A sharper result is given by P. Oblak and T. Košir below in Theorem 4.8.

**Lem 4.3.** *[Bar, NS] Let $B$ be an $n \times n$ nilpotent Jordan matrix, and let $A$ be generic in $\mathcal{N}_B$. Then*

$$\dim_K K[A, B] = n.$$

*Proof.* V. Baranovsky shows that for $A$ generic in $\mathcal{N}_B$, the ring $k[A, B]$ has a cyclic element; and M. Neubaurer and D. Saltman had shown that this implies $\dim_K K[A, B] = n$. $\qquad \square$

In view of example 4.2 we need the following result.

**Lem 4.4.** *Assume $A, B$ are commuting $n \times n$ nilpotent matrices with $B$ in Jordan form and let $K$ be an algebraically closed field of characteristic zero. Assume further that $\dim_K K[A, B] = n$. Then for a generic $\lambda \in K$, the action of $A + \lambda B$ on $K[A, B] \cong R/I$ has the same Jordan form as its action on the associated graded algebra $Gr_M K[A, B] \cong Gr_M(R/I)$, and has partition $P(H)$.*

*Proof.* For an open dense set of $\lambda' \in \mathbb{P}^1$, the ideal has standard bases in the direction $x' = x + \lambda y$. A standard basis

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$(f_\nu, \ldots, f_0)$ for $I$ in the direction $x$ can be written as follows.

Recall $\nu = \operatorname{order} I : M^\nu \supset I, M^{\nu+1} \not\supseteq I, M = (x, y)$.

$$I = (f_\nu = g_\nu, f_{\nu-1} = x^{k_{\nu-1}} g_{\nu-1}, \ldots, f_0 = x^{k_0} g_0), \text{ where}$$

$$g_i = y^i + h_i, \quad h_i \in M^\nu \cap k[x]\langle y^{i-1}, \ldots, y, 1 \rangle \qquad (4.2)$$

and $k_0 > k_1 > \ldots k_{\nu-1}$ [Br 77, I1]. Considering the action of

$m_x$ on the cyclic subspaces of $R/I$ generated by $1, g_1, \ldots g_{\nu-1}$,

we see that the Jordan partition of $m_x$ is just $Q = (k_0, \ldots, k_{\nu-1})$.

The standard basis for the associated graded ideal is given

by the initial ideal $InI^*$, satisfying

$$InI = (In(f_\nu), \ldots, In(f_1), f_0),$$

where here $Inf$ denotes the lowest degree graded summend of

$f$. So the Jordan partition for the action of $m_x$ on $R/I^*$ is

also $Q$. $\qquad\square$

Recall that $P(H)$ is the maximum partition of diagonal

lengths $H$. Let $H = H(K[A, B])$. Using the connection be-

tween $Z_H$ and $\mathcal{H}^H(n, K)$ we have

**Thm 4.5.** *Assume that $B$ is the Jordan matrix of partition $P$, and that $A \in \mathcal{N}_B$ satisfies $\dim K[A, B] = n$. Then for $\lambda \in \mathbb{P}^1$ generic, $A + \lambda B$ has Jordan blocks $P(H)$. The closure of the orbit of $B$ contains a nilpotent matrix of partition $P'$ having diagonal lengths $H$. These conclusions apply to the pair $(A, B)$ when $A$ is generic in $\mathcal{N}_B$.*

*Proof.* It follows from the assumptions and Lemma 4.4 that $C_\lambda = A + \lambda B$ for $\lambda$ generic satisfies, $P(C_\lambda) = P(H)$. Since the algebra $\mathcal{A} = \mathcal{A}_{A,B} = k[A, B]$ is a deformation of the associated graded algebra, $\mathcal{A}*$ the multiplication $m_y$ on $\mathcal{A}$ is a deformation of the action $m_y$ on $\mathcal{A}*$, so the orbit $P'$ of the latter is in the closure of the orbit of $P$. By Theorem 4.3 $A$ generic in $\mathcal{N}_B$ implies that $\dim K[A, B] = n$. $\square$

**Thm 4.6.** *Let $B$ be nilpotent of partition $P$, and let $Q(P)$ be the partition giving the Jordan block decomposition for the generic element $A \in \mathcal{N}_B$. Then $Q(P)$ has decreasing*

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*parts and satisfies*

$$Q(P) = \sup\{P(H) \mid \exists A \in \mathcal{N}(B), \dim K[A,B] = n, H = H(K[A,B])\}.$$

*Proof.* This follows from Theorem 4.5 and the irreducibility of $\mathcal{N}_B$. $\qquad\square$

There is a natural order on the set $\mathcal{H}(n)$ of Hilbert functions of length $n$ of codimension two $(4.1)$ or one $(H = (1,1,\ldots,1))$, defined by

$$H \leq H' \Leftrightarrow \forall u, 0 \leq u < n, \sum_{k \leq u} H_k \leq \sum_{k \leq u} H'_k.$$

For example, $(1,1,1,1,1) < (1,2,1,1) < (1,2,2)$.

The openness on $\mathrm{Hilb}^n(R)$ of the condition

$$\dim_K I \cap M^{u+1} > s$$

shows that

$$\overline{Z_H} \cap Z_{H'} \neq \emptyset \Rightarrow H \leq H'. \tag{4.3}$$

**Cor 4.7.** *Let $B$ be Jordan of partition $P$. Then*

$$Q(P) = P(H_{min}(P)), \; H_{min}(P) = \min\{H \mid \exists A, H(K[A,B] = H\}.$$

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*Proof.* This follows from (4.3), Theorem 4.5, and the irreducibility of $\mathcal{N}_B$. $\square$

The following result was proven since the Montreal conference by T. Košir and P. Oblak, who have resolved in a very nice way the question we asked about stability of $Q(P)$ (p. 3).

**Thm 4.8.** *[KoOb 07] Let $A$ be generic in $\mathcal{N}_B$. Then $K[A, B]$ is Gorenstein of length $n$ and $Q(P)$ is stable.*

*Proof idea.* The key step is to extend Baranovsky's result that $K[A, B]$ is cyclic to show it is also cocyclic. It is well known [Mac 1904] that the Hilbert function of a codimension two complete intersection Artinian algebra has first differences at most one, which implies that $P(H)$ has parts differing by at least two. By Corollary 1.5 and Theorem 4.6, $Q(P)$ is stable. $\square$

The following theorem concerns graded ideals. Given a partition $Q$, a difference-one hook of $Q$ is a hook where the (length of arm - length of foot)= 1. We define the *deviation* of a par-

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tition $Q$ of diagonal lengths $H$, from the partition $P(H)$:

#{ difference-one hooks of $P(H)$}−#{difference one hooks of $Q$}.

**Thm 4.9.** *[IY] Fix $\mathcal{A} = R/I, I$ graded. For each $\lambda \in \mathbb{P}^1$ the partition $P(E_\lambda^c)$ has diagonal lengths $H$. For an open dense set of $\lambda \in \mathbb{P}^1$ we have $P(E_\lambda^c) = P(H)$. The total deviation of the partitions arising from the action of $x + \lambda y, \lambda \in \mathbb{P}^1$, from the generically occurring partition $P(H)$, is a function $f(H) = \sum_{i \geq \nu}(H_i)(i + 1 - H_i)$.*

*Proof.* See [IY, Proposition 4.7, and Remark p. 3910]. ☐

**Remark**. Theorem 4.9 together with Theorem 4.5 seem to impose a subtle restriction on the partitions that may occur for $A \in \mathcal{N}_B$ when $H = H(K[A, B])$ is fixed.

# References

[Bar2001]    V. Baranovsky: *The variety of pairs of commuting nilpotent matrices is irreducible*, Transform. Groups

6 (2001), no. 1, 3–8.

[Basili 2000]     R. Basili: *On the irreducibility of varieties of commuting matrices*, J. Pure Appl. Algebra 149(2) (2000), 107–120.

[Basili 2003]     ———: *On the irreducibility of commuting varieties of nilpotent matrices.* J. Algebra 268 (2003), no. 1, 58–80.

[Basili-I1]     R. Basili and A. Iarrobino: *Some commuting k-tuples of matrices*, in preparation, 2007.

[Basili-I2]     R. Basili and A. Iarrobino :*An involution on $\mathcal{N}_B$, the nilpotent commutator of a nilpotent Jordan matrix B*, note, 2007.

[BaH 90]     J. Barria and P. Halmos: *Vector bases for two commuting matrices*, Linear and Multilinear Algebra **27** (1990), 147–157.

[Br 77]     J. Briançon: *Description de* Hilb$^n\mathbb{C}\{x, y\}$, Invent. Math. 41 (1977), no. 1, 45–89.

[Gr 83]      M. Granger: *Géométrie des schémas de Hilbert ponctuels* Mém. Soc. Math. France (N.S.) No. 8, (1983), 84 pp.

[HW1 2005]   T. Harima and J. Watanabe: *The commutator algebra of a nilpotent matrix and an application to the theory of commutative Artinian algebras*, preprint (2005).

[HW2 2006]   T. Harima and J. Watanabe: *The strong Lefschetz propery for Artinian algebras with non-standard grading*, preprint, 2006, to appear, J. Algebra.

[I1]         A. Iarrobino: *Punctual Hilbert Scheme* Mem. Amer. Math. Soc. 10 (1977), no. 188, viii+112 pp.

[I2]         ———: *Associated Graded Algebra of a Gorenstein Artin Algebra*, Memoirs Amer. Math. Society, Vol 107 #514, (1994), Amer. Math. Soc., Providence.

[IY]         A. Iarrobino and J. Yaméogo: *The family $G_T$ of graded Artinian quotients $G_T$ of $k[x,y]$ of given Hilbert function*, Communications in Algebra, Vol. 31 # 8, 3863–3916 (2003).   (ArXiv alg-geom/9709021)

[KoOb 07]    T. Košir and P. Oblak: *A note on commuting pairs of nilpotent matrices*, preprint, 2007.

[KnZe]    H. Knight and A. Zelevinsky: *Representations of Quivers of Type A and the Multisegment Duality*, Advances in Math. 117 #2 (1996), 273–293.

[Mac 1904]    F.H.S. Macaulay: *On a method of dealing with the intersections of plane curves.* Trans. Amer. Math. Soc. 5 (1904), no. 4, 385–410.

[Neub, Salt 1994] M. Neubauer and D. Saltman: *Two-generated commutative subalgebras of $M_nF$*, J. Algebra **164** (1994), 545–562.

[Neub, Seth 1999] M. Neubauer and B.A. Sethuraman: *Commuting pairs in the centralizers of 2-regular matrices*, J. Algebra 214 (1999), 174–181.

[Oblak 2007]    P. Oblak: *The upper bound for the index of nilpotency for a matrix commuting with a given nilpotent matrix*, ArXiv: AC/0701561, January, 2007.

[Pol 1989]        S. Poljak: *Maximum Rank of Powers of a Matrix of Given Pattern*, Proc. A.M.S., 106 #4 (1989), 1137–1144.

[Premet 2003]     A. Premet: *Nilpotent commuting varieties of reductive Lie algebras*, Invent. Math. 154 (2003), no. 3, 653–683.

[Scott 1902]      C.A. Scott: *on a recent method for dealing with the intersections of plane curves*, Trans. AMS 3 (1902), 216-283.

[Turnb, Aitken 1931] H.W. Turnbull, A.C. Aitken: *An introduction to the theory of canonical matrices* Dover, New York, 1961.

[Yaméogo 1994a] J. Yaméogo: *Décomposition cellulaire des variétés paramétrant des idéaux homogènes de $\mathbb{C}[[x,y]]$: Incidence des cellules I*, Compositio Math. 90 (1994): 81–98.

[Yaméogo 1994b] _____: *Décomposition cellulaire des variétés paramétrant des idéaux homogènes de $\mathbb{C}[[x,y]]$: In-*

*cidence des cellules II*, J. Reine Angew. Math. 450 (1994), 123–137.