# Cohen-Macaulay Properties of Square-Free Monomial Ideals

Sara Faridi*

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**Abstract**

In this paper we study simplicial complexes as higher dimensional graphs in order to produce algebraic statements about their facet ideals. We introduce a large class of square-free monomial ideals with Cohen-Macaulay quotients, and a criterion for the Cohen-Macaulayness of facet ideals of simplicial trees. Along the way, we generalize several concepts from graph theory to simplicial complexes.

## 1 Introduction

From the point of view of commutative algebra, the focus of this paper is on finding square-free monomial ideals that have Cohen-Macaulay quotients. In [Vi1] Villarreal proved a criterion for the Cohen-Macaulayness of edge ideals of graphs that are trees. Edge ideals are square-free monomial ideals where each generator is a product of two distinct variables of a polynomial ring. These ideals have been studied extensively by Villarreal, Vasconcelos and Simis among others. In [Fa] we studied a generalization of this concept; namely the *facet ideal* of a simplicial complex. By generalizing the definition of a "tree" to simplicial complexes, we extended the results of [SVV] from the class of edge ideals to all square-free monomial ideals.

Below we investigate the structure of simplicial complexes in order to show that Villarreal's Cohen-Macaulay criterion for graph-trees extends to simplicial trees (Corollary 8.3). This is of algebraic and computational significance, as it provides an effective criterion for Cohen-Macaulayness that works for a large class of square-free monomial ideals. We introduce a condition on a simplicial complex that ensures the Cohen-Macaulayness of its facet ideal, and a method to build a Cohen-Macaulay ideal from any given square-free monomial ideal. Along the road to the algebraic goal, this study sheds light on the beautiful combinatorial structure of simplicial complexes.

The paper is organized as follows: Sections 2 to 4 review the basic definitions and cover the elementary properties of trees. In Section 5 we draw comparisons between graph theory and simplicial complex theory, and prove a generalized version of König's Theorem in graph theory for simplicial complexes. We then go on to prove a structure theorem for unmixed trees in Section 6. We introduce the notion of a *grafted* simplicial complex in Section 7, and show that for simplicial trees, being grafted and being unmixed are

*University of Ottawa, 585 King Edward Ave, Ottawa, ON K1N 6N5, Canada. email: *faridi@ uottawa.ca.*
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equivalent conditions. The notion of grafting brings us to Section 8, where we prove that grafted simplicial complexes are Cohen-Macaulay, from which it follows that a simplicial tree is unmixed if and only if it is Cohen-Macaulay.

## 2 Definitions and notation

In this section we define the basic notions that we will use later in the paper. Some of the proofs that appeared earlier in [Fa] have been omitted here; we refer the reader to the relevant sections of [Fa] when that is the case.

**Definition 2.1 (simplicial complex, facet and more).** A *simplicial complex* $\Delta$ over a set of vertices $V = \{v_1, \dots, v_n\}$ is a collection of subsets of $V$, with the property that $\{v_i\} \in \Delta$ for all $i$, and if $F \in \Delta$ then all subsets of $F$ are also in $\Delta$ (including the empty set). An element of $\Delta$ is called a *face* of $\Delta$, and the *dimension* of a face $F$ of $\Delta$ is defined as $|F| - 1$, where $|F|$ is the number of vertices of $F$. The faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively, and $\dim \emptyset = -1$.

The maximal faces of $\Delta$ under inclusion are called *facets* of $\Delta$. The dimension of the simplicial complex $\Delta$ is the maximal dimension of its facets; in other words

$$\dim \Delta = \max\{\dim F \mid F \in \Delta\}.$$

We denote the simplicial complex $\Delta$ with facets $F_1, \dots, F_q$ by

$$\Delta = \langle F_1, \dots, F_q \rangle$$

and we call $\{F_1, \dots, F_q\}$ the *facet set* of $\Delta$.

A simplicial complex with only one facet is called a *simplex*.

**Definition 2.2 (subcollection).** By a *subcollection* of a simplicial complex $\Delta$ we mean a simplicial complex whose facet set is a subset of the facet set of $\Delta$.

**Definition 2.3 (connected simplicial complex).** A simplicial complex $\Delta = \langle F_1, \dots, F_q \rangle$ is *connected* if for every pair $i, j$, $1 \leq i < j \leq q$, there exists a sequence of facets

$$F_{t_1}, \dots, F_{t_r}$$

of $\Delta$ such that $F_{t_1} = F_i$, $F_{t_r} = F_j$ and

$$F_{t_s} \cap F_{t_{s+1}} \neq \emptyset$$

for $s = 1, \dots, r-1$.

An equivalent definition is stated on page 222 of [BH]: $\Delta$ as above is *disconnected* if its vertex set $V$ can be partitioned as $V = V_1 \cup V_2$, where $V_1$ and $V_2$ are nonempty subsets of $V$, such that no facet of $\Delta$ has vertices in both $V_1$ and $V_2$. Otherwise $\Delta$ is *connected*.

**Definition 2.4 (facet ideal, non-face ideal).** Let $\Delta$ be a simplicial complex over $n$ vertices labeled $v_1, \dots, v_n$. Let $k$ be a field, $x_1, \dots, x_n$ be indeterminates, and $R$ be the polynomial ring $k[x_1, \dots, x_n]$.
(a) We define $\mathcal{F}(\Delta)$ to be the ideal of $R$ generated by all the square-free monomials $x_{i_1} \dots x_{i_s}$, where $\{v_{i_1}, \dots, v_{i_s}\}$ is a facet of $\Delta$. We call $\mathcal{F}(\Delta)$ the *facet ideal* of $\Delta$.
(b) We define $\mathcal{N}(\Delta)$ to be the ideal of $R$ generated by all the square-free monomials $x_{i_1} \dots x_{i_s}$, where $\{v_{i_1}, \dots, v_{i_s}\}$ is not a face of $\Delta$. We call $\mathcal{N}(\Delta)$ the *non-face ideal* or the *Stanley-Reisner ideal* of $\Delta$.

We refer the reader to [S] and [BH] for an extensive coverage of the theory of Stanley-Reisner ideals.

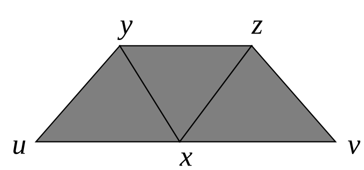
Throughout this paper we often use $x_1, \dots, x_n$ to denote both the vertices of $\Delta$ and the variables appearing in $\mathcal{F}(\Delta)$.

**Definition 2.5 (facet complex, non-face complex).** Let $I = (M_1, \dots, M_q)$ be an ideal in a polynomial ring $k[x_1, \dots, x_n]$, where $k$ is a field and $M_1, \dots, M_q$ are square-free monomials in $x_1, \dots, x_n$ that form a minimal set of generators for $I$.
(a) We define $\delta_{\mathcal{F}}(I)$ to be the simplicial complex over a set of vertices $v_1, \dots, v_n$ with facets $F_1, \dots, F_q$, where for each $i$, $F_i = \{v_j \mid x_j | M_i,\ 1 \leq j \leq n\}$. We call $\delta_{\mathcal{F}}(I)$ the *facet complex* of $I$.
(b) We define $\delta_{\mathcal{N}}(I)$ to be the simplicial complex over a set of vertices $v_1, \dots, v_n$, where $\{v_{i_1}, \dots, v_{i_s}\}$ is a face of $\delta_{\mathcal{N}}(I)$ if and only if $x_{i_1} \dots x_{i_s} \notin I$. We call $\delta_{\mathcal{N}}(I)$ the *non-face complex* or the *Stanley-Reisner complex* of $I$.

Facet ideals give a one-to-one correspondence between simplicial complexes and square-free monomial ideals.

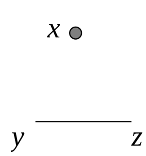
Notice that given a square-free monomial ideal $I$ in a polynomial ring $k[x_1, \dots, x_n]$, the vertices of $\delta_{\mathcal{F}}(I)$ are those variables that divide a monomial in the generating set of $I$; this set may not necessarily include all elements of $\{x_1, \dots, x_n\}$. The fact that some extra variables may appear in the polynomial ring does not affect the algebraic or combinatorial structure of $\delta_{\mathcal{F}}(I)$. On the other hand, if $\Delta$ is a simplicial complex, being able to consider the facet ideals of its subcomplexes as ideals in the same ring simplifies many of our discussions.

**Example 2.6.** Let $\Delta$ be the simplicial complex below.

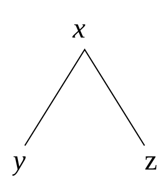



Here $\mathcal{N}(\Delta) = (yv, zu, uv)$, $\mathcal{F}(\Delta) = (xyu, xyz, xzv)$ are ideals in the polynomial ring $k[x, y, z, u, v]$.

**Example 2.7.** If $I = (xy, xz) \subseteq k[x, y, z]$, then $\delta_{\mathcal{N}}(I)$ is the 1-dimensional simplicial complex:




and $\delta_{\mathcal{F}}(I)$ is the simple graph




In this special case $I$ is also called the *edge ideal* of the graph $\delta_{\mathcal{F}}(I)$ (this terminology is due to Villarreal; see [Vi1]).

We now generalize some notions from graph theory to simplicial complexes.

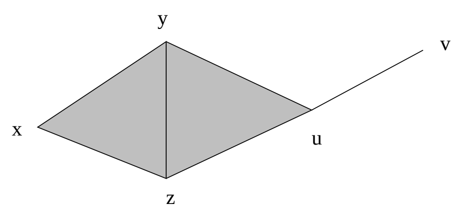
**Definition 2.8 (minimal vertex cover, vertex covering number, unmixed).** Let $\Delta$ be a simplicial complex with vertex set $V$ and facets $F_1, \ldots, F_q$. A *vertex cover* for $\Delta$ is a subset $A$ of $V$, with the property that for every facet $F_i$ there is a vertex $v \in A$ such that $v \in F_i$. A *minimal vertex cover* of $\Delta$ is a subset $A$ of $V$ such that $A$ is a vertex cover, and no proper subset of $A$ is a vertex cover for $\Delta$. The smallest cardinality of a vertex cover of $\Delta$ is called the *vertex covering number* of $\Delta$ and is denoted by $\alpha(\Delta)$.

A simplicial complex $\Delta$ is *unmixed* if all of its minimal vertex covers have the same cardinality.

Note that a simplicial complex may have several minimal vertex covers.

**Definition 2.9 (independent set, independence number).** Let $\Delta$ be a simplicial complex. A set $\{F_1, \ldots, F_u\}$ of facets of $\Delta$ is called an *independent set* if $F_i \cap F_j = \emptyset$ whenever $i \neq j$. The maximum possible cardinality of an independent set of facets in $\Delta$, denoted by $\beta(\Delta)$, is called the *independence number* of $\Delta$. An independent set of facets which is not a proper subset of any other independent set is called a *maximal independent set* of facets.

**Example 2.10.** If $\Delta$ is the simplicial complex




then $\beta(\Delta) = 2$. Also, $\Delta$ is unmixed as its minimal vertex covers, listed below, all have cardinality equal to two:

$$\{x, u\}, \{y, u\}, \{y, v\}, \{z, u\}, \{z, v\}$$

This, by the way, is an example of a "grafted" tree (see Definitions 3.5 and 7.1). We show later in the paper that all grafted trees are unmixed.

The graph $\delta_{\mathcal{F}}(I)$ in Example 2.7 however is not unmixed. This is because $\{x\}$ and $\{y, z\}$ are both minimal vertex covers for $\delta_{\mathcal{F}}(I)$ of different cardinalities. In this case $\alpha(\delta_{\mathcal{F}}(I)) = \beta(\delta_{\mathcal{F}}(I)) = 1$. The same argument shows that the simplicial complex in Example 2.6 is not unmixed.

The following is an easy but very useful observation; see Proposition 1 in [Fa] for a proof.

**Proposition 2.11.** *Let $\Delta$ be a simplicial complex over $n$ vertices labeled $x_1, \ldots, x_n$. Consider the ideal $I = \mathcal{F}(\Delta)$ in the polynomial ring $R = k[x_1, \ldots, x_n]$ over a field $k$. Then an ideal $p = (x_{i_1}, \ldots, x_{i_s})$ of $R$ is a minimal prime of $I$ if and only if $\{x_{i_1}, \ldots, x_{i_s}\}$ is a minimal vertex cover for $\Delta$.*

We say that a simplicial complex $\Delta$ over a set of vertices $x_1, \ldots, x_n$ is Cohen-Macaulay if for a given field $k$, the quotient ring

$$k[x_1, \ldots, x_n]/\mathcal{F}(\Delta)$$

is Cohen-Macaulay. It follows directly from Proposition 2.11, or from an elementary duality with Stanley-Reisner theory discussed in Corollary 2 of [Fa], that in order for $\Delta$ to be Cohen-Macaulay, it has to be unmixed.

**Proposition 2.12 (A Cohen-Macaulay simplicial complex is unmixed).** *Suppose that $\Delta$ is a simplicial complex with vertex set $x_1, \ldots, x_n$. If $k[x_1, \ldots, x_n]/\mathcal{F}(\Delta)$ is Cohen-Macaulay, then $\Delta$ is unmixed.*

**Discussion 2.13.** It is worth observing that for a square-free monomial ideal $I$, there is a natural way to construct $\delta_{\mathcal{N}}(I)$ and $\delta_{\mathcal{F}}(I)$ from each other using the structure of the minimal primes of $I$. To do this, consider the vertex set $V$ consisting of all variables that divide a monomial in the generating set of $I$. The following correspondence holds:

$$F = \text{facet of } \delta_{\mathcal{N}}(I) \longleftrightarrow V \setminus F = \text{minimal vertex cover of } \delta_{\mathcal{F}}(I)$$

Also

$$I = \bigcap p$$

where the intersection is taken over all prime ideals $p$ of $k[V]$ that are generated by a minimal vertex cover of $\delta_{\mathcal{F}}(I)$ (or equivalently, primes $p$ that are generated by $V \setminus F$, where $F$ is a facet of $\delta_{\mathcal{N}}(I)$; see [BH] Theorem 5.1.4).

Regarding the dimension and codimension of $I$, note that by Theorem 5.1.4 of [BH] and the discussion above, setting $R = k[V]$ as above, we have

$$\dim R/I = \dim \delta_{\mathcal{N}}(I) + 1 = |V| - \text{ vertex covering number of } \delta_{\mathcal{F}}(I)$$

and

$$\text{height } I \ = \ \text{vertex covering number of } \delta_{\mathcal{F}}(I).$$

We illustrate all this through an example.

**Example 2.14.** For $I = (xy, xz)$, where $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ are drawn in Example 2.7, we have:

| facets of $\delta_{\mathcal{N}}(I)$ | minimal vertex covers of $\delta_{\mathcal{F}}(I)$ |
| --- | --- |
| $\{x\}$ | $\{y, z\}$ |
| $\{y, z\}$ | $\{x\}$ |

Note that $I = (x) \cap (y, z)$, and

$$\dim k[x, y, z]/(xy, xz) = 2$$

as asserted in Discussion 2.13 above.

A notion crucial to the rest of the paper is "removing a facet". We want the removal of a facet from a simplicial complex to correspond to dropping a generator from its facet ideal. We record this definition.

**Definition 2.15 (facet removal).** Suppose $\Delta$ is a simplicial complex with facets $F_1, \ldots, F_q$ and $\mathcal{F}(\Delta) = (M_1, \ldots, M_q)$ its facet ideal in $R = k[x_1, \ldots, x_n]$. The simplicial complex obtained by *removing the facet* $F_i$ from $\Delta$ is the simplicial complex

$$\Delta \setminus \langle F_i \rangle = \langle F_1, \ldots, \hat{F}_i, \ldots, F_q \rangle.$$

Note that $\mathcal{F}(\Delta \setminus \langle F_i \rangle) = (M_1, \ldots, \hat{M}_i, \ldots, M_q)$.

Also note that the vertex set of $\Delta \setminus \langle F_i \rangle$ is a subset of the vertex set of $\Delta$.

**Example 2.16.** let $\Delta$ be the simplicial complex in Example 2.10 with facets $F = \{x, y, z\}$, $G = \{y, z, u\}$ and $H = \{u, v\}$. Then $\Delta \setminus \langle F \rangle = \langle G, H \rangle$ is a simplicial complex with vertex set $\{y, z, u, v\}$.

# 3 Trees

In [Fa] we extended the notion of a "tree" from graphs to simplicial complexes. The construction, at the time, was motivated by two factors: the restriction to graphs should produce the classic graph-theoretical definition of a tree, and the new structure should fit into a machinery that proves that the facet ideal of a tree satisfies Sliding Depth condition (Theorem 1 of [Fa]).

The resulting definition not only satisfies those two properties, but as we prove later in this paper, it also generalizes graph-trees in the sense of Cohen-Macaulayness, which confirms that algebraically this in fact is the optimal way to extend the definition of a tree.

Recall that a connected graph is a tree if it has no cycles; for example, a triangle is not a tree. An equivalent definition states that a connected graph is a tree if every subgraph has a *leaf*, where a leaf is a vertex that belongs to only one edge of the graph. This latter description is the one that we adapt, with a slight change in the definition of a leaf, to the class of simplicial complexes.

**Definition 3.1 (leaf, joint, universal set).** Suppose that $\Delta$ is a simplicial complex. A facet $F$ of $\Delta$ is called a *leaf* if either $F$ is the only facet of $\Delta$, or there exists a facet $G$ in $\Delta \setminus \langle F \rangle$, such that

$$F \cap F' \subseteq F \cap G$$

for every facet $F' \in \Delta \setminus \langle F \rangle$.

In other words, $F$ is a leaf of $\Delta$ if it intersects $\Delta \setminus \langle F \rangle$ in a face of $\Delta \setminus \langle F \rangle$.

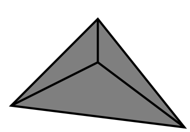
The set of all $G$ as above is denoted by $\mathcal{U}_\Delta(F)$ and called the *universal set* of $F$ in $\Delta$. If $G \in \mathcal{U}_\Delta(F)$ and $F \cap G \neq \emptyset$, then $G$ is called a *joint* of $F$.

Another way to describe a leaf is the following: (with assumptions as above) $F$ is a leaf if either $F$ is the only facet of $\Delta$ or the intersection of $F$ with the simplicial complex $\Delta \setminus \langle F \rangle$ is a face of $\Delta \setminus \langle F \rangle$.

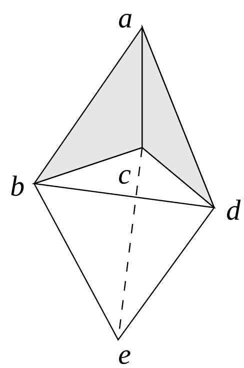
**Definition 3.2 (free vertex).** A vertex of a simplicial complex $\Delta$ is *free* if it belongs to exactly one facet of $\Delta$.

In order to be able to quickly identify a leaf in a simplicial complex, it is important to notice that a leaf must have a free vertex. This follows easily from Definition 3.1: otherwise, a leaf $F$ would be contained in its joints, which would contradict the fact that a leaf is a facet.

**Example 3.3.** The simplicial complex in example 2.6 has two leaves: $\{x, y, u\}$ and $\{x, z, v\}$. The one below has no leaves, because every vertex is shared by at least two facets.



**Example 3.4.** In the simplicial complex below with facets $F_1 = \{a, b, c\}$, $F_2 = \{a, c, d\}$ and $F_3 = \{b, c, d, e\}$, the only candidate for a leaf is the facet $F_3$ (as it is the only facet with a free vertex), but neither one of $F_1 \cap F_3$ or $F_2 \cap F_3$ is contained in the other, so there are no leaves.




**Definition 3.5 (tree).** Suppose that $\Delta$ is a connected simplicial complex. We say that $\Delta$ is a *tree* if every nonempty subcollection of $\Delta$ (including $\Delta$ itself) has a leaf.

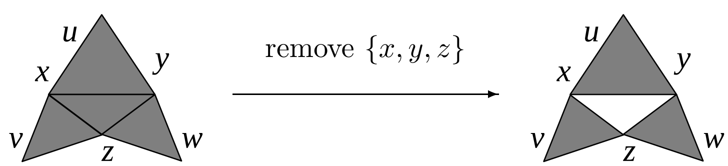
Equivalently, a connected simplicial complex $\Delta$ is a tree if every nonempty *connected* subcollection of $\Delta$ has a leaf.

**Definition 3.6 (forest).** A simplicial complex $\Delta$ with the property that every connected component of $\Delta$ is a tree is called a *forest*. In other words, a forest is a simplicial complex with the property that every nonempty subcollection has a leaf.

The simplicial complex in Example 2.6 above is a tree, whereas the ones in Examples 3.3 and 3.4 are not, as they have no leaves.

Here is a slightly less straightforward example:

**Example 3.7.** The simplicial complex on the left is not a tree, because although all three facets $\{x, y, u\}$, $\{x, v, z\}$ and $\{y, z, w\}$ are leaves, if one removes the facet $\{x, y, z\}$, the remaining simplicial complex (on the right) has no leaf.




Notice that in the case that $\Delta$ is a graph, Definition 3.5 agrees with the definition of a tree in graph theory, with the difference that now the term "leaf" refers to an edge, rather than a vertex.

# 4 Basic properties of trees

**Lemma 4.1 (A tree has at least two leaves).** *Let $\Delta$ be a tree of two or more facets. Then $\Delta$ has at least two leaves.*

*Proof.* Suppose that $\Delta$ has $q$ facets $F_1, \ldots, F_q$ where $q \geq 2$. We prove the lemma by induction on $q$.

The case $q = 2$ follows from the definition of a leaf.

To prove the general case suppose that $F_1$ is a leaf of $\Delta$ and $G_1 \in \mathcal{U}_\Delta(F_1)$. Consider the subcomplex $\Delta' = \langle F_2, \dots, F_q \rangle$ of $\Delta$. By induction hypothesis $\Delta'$ has two distinct leaves; say $F_2$ and $F_3$ are those leaves. At least one of $F_2$ and $F_3$ must be different from $G_1$; say $F_2 \neq G_1$. We show that $F_2$ is a leaf for $\Delta$.

Let $G_2 \in \mathcal{U}_{\Delta'}(F_2)$. Given any facet $F_i$ with $i \neq 1, 2$, we already know by the fact that $F_2$ is a leaf of $\Delta'$

$$F_i \cap F_2 \subseteq G_2 \cap F_2.$$

We need to verify this for $i = 1$.

Since $F_1$ is a leaf for $\Delta$ and $F_2 \neq F_1$,

$$F_2 \cap F_1 \subseteq G_1 \cap F_1.$$

Intersecting both sides of this inclusion with $F_2$, we obtain

$$F_2 \cap F_1 \subseteq G_1 \cap F_1 \cap F_2 \subseteq G_1 \cap F_2 \subseteq G_2 \cap F_2$$

where the last inclusion holds because $G_1 \neq F_2$ and $F_2$ is a leaf of $\Delta'$.

It follows that $F_2$, as well as $F_1$, is a leaf for $\Delta$.

□

A promising property of trees from an algebraic point of view is that they behave well under localization, i.e. the localization of a tree is a forest. This property is in particular useful when making inductive arguments on trees, as localization usually corresponds to reducing the number of vertices of a simplicial complex. Before proving this, we first determine what the localization of a simplicial complex precisely looks like.

**Discussion 4.2 (On the localization of a simplicial complex).** Suppose that

$$\Delta = \langle F_1, \dots, F_q \rangle$$

is a simplicial complex over the vertex set $V = \{x_1, \dots, x_n\}$. Let $p$ be a prime ideal of $k[x_1, \dots, x_n]$ generated by a subset of $\{x_1, \dots, x_n\}$ that contains $I = \mathcal{F}(\Delta)$ (We show later in the proof of Lemma 4.5 that this is the main case that we need to study). We would like to see what the simplicial complex associated to $I_p$ looks like.

So

$$p = (x_{i_1}, \dots, x_{i_r}).$$

Now suppose

$$I = (M_1, \dots, M_q)$$

where each $M_i$ is the monomial corresponding to the facet $F_i$. It follows that

$$I_p = (M'_1, \dots, M'_q)$$

where each $M'_i$ is obtained by dividing $M_i$ by the product of all the variables in $V \setminus \{x_{i_1}, \dots, x_{i_r}\}$ that appear in $M_i$. Some of the monomials in the generating set of $I_p$ are redundant after this elimination, so without loss of generality we can write:

$$I_p = (M'_1, \dots, M'_t) \tag{1}$$

where $M'_{t+1}, \dots, M'_q$ are the redundant monomials.

We use the notation $\delta_{\mathcal{F}}(I_p)$ to indicate the simplicial complex associated to the monomial ideal with the same generating set as the one described for $I_p$ in (1), in the polynomial ring $k[x_{i_1}, \dots, x_{i_r}]$. It follows that:

$$\delta_{\mathcal{F}}(I_p) = \langle F'_1, \dots, F'_t \rangle$$

where for each $i$,

$$F'_i = F_i \cap \{x_{i_1}, \dots, x_{i_r}\}$$

and $F'_{t+1}, \dots, F'_q$ each contain at least one of $F'_1, \dots, F'_t$. This simplicial complex is called the *localization* of $\Delta$ at the prime ideal $p$.

Note that every minimal vertex cover $A$ of $\Delta$ that is contained in $\{x_{i_1}, \dots, x_{i_r}\}$ remains a minimal vertex cover of $\delta_{\mathcal{F}}(I_p)$, as the minimal prime over $I$ generated by the elements of $A$ remains a minimal prime of $I_p$.

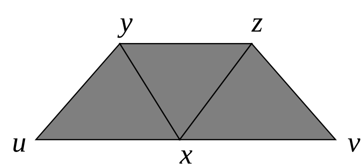
Moreover if $\Delta$ is unmixed then $\delta_{\mathcal{F}}(I_p)$ is also unmixed. Algebraically, this is easy to see, as the height of the minimal primes of $I_p$ remain the same. One can also see it from a combinatorial argument: If $B \subseteq \{x_{i_1}, \dots, x_{i_r}\}$ is a minimal vertex cover for $\delta_{\mathcal{F}}(I_p)$, then $B$ covers all facets $F'_1, \dots, F'_t$, and therefore $F'_{t+1}, \dots, F'_q$, as well. Therefore $B$ covers all of $F_1, \dots, F_q$, and so has a subset $B'$ of cardinality $\alpha(\Delta)$ that is a minimal vertex cover for $\Delta$, and so $B'$ must cover $\delta_{\mathcal{F}}(I_p)$ as well. Therefore $B' = B$.

We have thus shown that:

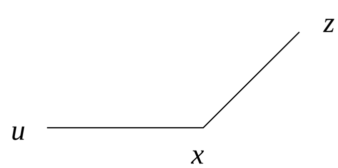
**Lemma 4.3 (Localization of an unmixed simplicial complex is unmixed).** *Let $\Delta$ be an unmixed simplicial complex with vertices $x_1, \dots, x_n$, and let $I = \mathcal{F}(\Delta)$ be the facet ideal of $\Delta$ in the polynomial ring $R = k[x_1, \dots, x_n]$ where $k$ is a field. Then for any prime ideal $p$ of $R$, $\delta_{\mathcal{F}}(I_p)$ is unmixed with $\alpha(\delta_{\mathcal{F}}(I_p)) = \alpha(\Delta)$.*

We examine a specific case:

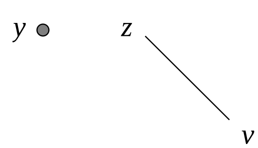
**Example 4.4.** Let $\Delta$ be the simplicial complex below with $I = (xyu, xyz, xzv)$ its facet ideal in the polynomial ring $R = k[x, y, z, u, v]$.




Let $p = (u, x, z)$ be a prime ideal of $R$. Then $I_p = (xu, xz, xz) = (xu, xz)$. The tree $\delta_{\mathcal{F}}(I_p)$, shown below, has minimal vertex covers $\{x\}$ and $\{u, z\}$, which are the generating sets for the minimal primes of $I_p$.




If $q = (y, z, v)$ then $I_q = (y, yz, zv) = (y, zv)$ which corresponds to the forest $\delta_{\mathcal{F}}(I_q)$ drawn below with minimal vertex covers $\{y, z\}$ and $\{y, v\}$.



Example 4.4 above also demonstrates the following lemma.

**Lemma 4.5 (Localization of a tree is a forest).** *Let $\Delta$ be a tree with vertices $x_1, \ldots, x_n$, and let $I = \mathcal{F}(\Delta)$ be the facet ideal of $\Delta$ in the polynomial ring $R = k[x_1, \ldots, x_n]$ where $k$ is a field. Then for any prime ideal $p$ of $R$, $\delta_{\mathcal{F}}(I_p)$ is a forest.*

*Proof.* The first step is to show that it is enough to prove this for prime ideals of $R$ generated by a subset of $\{x_1, \ldots, x_n\}$. To see this, assume that $p$ is a prime ideal of $R$ and that $p'$ is another prime of $R$ generated by all $x_i \in \{x_1, \ldots, x_n\}$ such that $x_i \in p$ (recall that the minimal primes of $I$ are generated by subsets of $\{x_1, \ldots, x_n\}$). So $p' \subseteq p$. If $I = (M_1, \ldots, M_q)$, then

$$I_{p'} = ({M_1}', \ldots, {M_q}')$$

where for each $i$, ${M_i}'$ is the image of $M_i$ in $I_{p'}$. In other words, ${M_i}'$ is obtained by dividing $M_i$ by the product of all the $x_j$ such that $x_j | M_i$ and $x_j \notin p'$. But $x_j \notin p'$ implies that $x_j \notin p$, and so it follows that ${M_i}' \in I_p$. Therefore $I_{p'} \subseteq I_p$. On the other hand since $p' \subseteq p$, $I_p \subseteq I_{p'}$, which implies that $I_{p'} = I_p$ (the equality and inclusions of the ideals here mean equality and inclusion of their generating sets).

We now prove the theorem for $p = (x_{i_1}, \ldots, x_{i_r})$. Following the setup in Discussion 4.2, we let

$$\Delta = \langle F_1, \ldots, F_q \rangle$$

$$I_p = (M_1', \ldots, M_t')$$

$$\Delta' = \delta_{\mathcal{F}}(I_p) = \langle F_1', \ldots, F_t' \rangle.$$

for some $t \leq q$.

To show that $\Delta'$ is a forest, we need to show that every nonempty subcollection of $\Delta'$ has a leaf.

Let

$$\Delta_1' = \langle F_{j_1}', \ldots, F_{j_s}' \rangle$$

be a subcollection of $\Delta'$ where $F_{j_1}', \ldots, F_{j_s}'$ are distinct facets. If $s = 1$, $F_{j_1}'$ is obviously a leaf and so we are done; so suppose $s > 1$. Consider the corresponding subcollection

$$\Delta_1 = \langle F_{j_1}, \ldots, F_{j_s} \rangle$$

of $\Delta$, which has a leaf, say $F_{j_1}$. So there exists $G \in \Delta_1 \setminus \langle F_{j_1} \rangle$, such that

$$F_{j_1} \cap F \subseteq F_{j_1} \cap G$$

for every facet $F \in \langle F_{j_2}, \ldots, F_{j_s} \rangle$. Now since each of the $F_{j_u}'$ is a nonempty facet of $\Delta_1'$ and $G' \neq F_{j_1}'$, the same statement holds in $\Delta_1'$; so

$$F_{j_1}' \cap F' \subseteq F_{j_1}' \cap G'$$

for every facet $F' \in \Delta_1' \setminus \langle F_{j_1}' \rangle$. This implies that $F_{j_1}'$ is a leaf for $\Delta_1'$.

□

## 5 Simplicial complexes as higher dimensional graphs

In this section we study simplicial complexes as graphs with higher dimension, drawing results that will help us later in inductive arguments on unmixed trees.

**Lemma 5.1.** *If $\Delta$ is a simplicial complex that has a leaf $F$ with joint $G$, then $\alpha(\Delta \setminus \langle G\rangle) = \alpha(\Delta)$.*

*Proof.* Suppose $\alpha(\Delta) = r$. Let $\Delta' = \Delta \setminus \langle G\rangle$ and let $A$ be a vertex cover of minimal cardinality for $\Delta'$, which implies that $|A| \leq r$, as any vertex cover of $\Delta$ has a subset that is a vertex cover of $\Delta'$. Since $F$ is a facet of $\Delta'$, there exists a vertex $x \in A$ that belongs to $F$. If $x$ is a free vertex of $F$, we may replace it by a non-free vertex of $F$ to get a vertex cover $A''$ of $\Delta'$, with a subset $A'$ that is a minimal vertex cover of $\Delta'$, and so $|A'| \leq |A|$. But now $A'$ is a minimal vertex cover for all of $\Delta$, and so $|A'| = |A| = r$ which implies that $\alpha(\Delta') = \alpha(\Delta) = r$. $\square$

**Corollary 5.2.** *If the simplicial complex $\Delta$ is a tree and $G \in \Delta$ is a joint, then $\alpha(\Delta\setminus\langle G\rangle) = \alpha(\Delta)$.*

This means that in a tree with more than one facet, it is always possible to remove a facet without reducing the vertex covering number. Moreover we show in Proposition 6.6 that if $\Delta$ is an unmixed tree with a joint $G$, then $\Delta \setminus \langle G\rangle$ is also unmixed. As a result, one can use induction on the number of facets of an unmixed tree. Note that all these arguments remain valid for a forest.

We are now ready to extend König's theorem from graph theory.

**Theorem 5.3 (A generalization of König's theorem).** *If $\Delta$ is a simplicial complex that is a tree (forest) and $\alpha(\Delta) = r$, then $\Delta$ has $r$ independent facets, and therefore $\alpha(\Delta) = \beta(\Delta) = r$.*

*Proof.* We use induction on the number of facets $q$ of $\Delta$. If $q = 1$, then there is nothing to prove since $\alpha(\Delta) = \beta(\Delta) = 1$.

Suppose that the theorem holds for forests with less than $q$ facets and let $\Delta$ be a forest with $q$ facets. If every connected component of $\Delta$ has only one facet, our claim follows immediately. Otherwise, by Corollary 5.2 one can remove a joint of $\Delta$ to get a forest $\Delta'$ with $\alpha(\Delta') = r$, and so by induction hypothesis $\Delta'$ has $r$ independent facets, which are also independent facets of $\Delta$; so $\alpha(\Delta) \leq \beta(\Delta)$. On the other hand it is clear that $\alpha(\Delta) \geq \beta(\Delta)$, and so the assertion follows. $\square$

## 6 The structure of an unmixed tree

This section is the combinatorial core of the paper. Here we give a precise description of the structure of an unmixed tree. It turns out that a tree is unmixed if and only if it is "grafted" (see Definition 7.1). The notion of grafting is what eventually builds a bridge between unmixed and Cohen-Macaulay trees.

Below $V(\Delta)$ stands for the vertex set of $\Delta$.

**Lemma 6.1.** *Let $\Delta$ be an unmixed simplicial complex. Suppose that $\alpha(\Delta) = \beta(\Delta) = r$, and $\{F_1, \ldots, F_r\}$ is a maximal independent set of facets of $\Delta$. Then every vertex of $\Delta$ belongs to one of the $F_i$. In other words, the vertex set of $\Delta$ is the disjoint union of the vertex sets of the $F_i$:*

$$V(\Delta) = V(F_1) \cup \ldots \cup V(F_r)$$

*Proof.* Let $x$ be an vertex of $\Delta$ that does not belong to any of the $F_i$. Then one can find a minimal vertex cover $A$ of $\Delta$ containing $x$ (this is always possible). But then $A$ must contain one vertex of each of the $F_i$ as well, which implies that $|A| \geq r+1$. Since $\Delta$ is unmixed, this is not possible. $\square$

**Remark 6.2.** Lemma 6.1 does not hold in general for any unmixed simplicial complex. Take, for example, the case of a complete graph $G$ over 5 vertices labeled $x, y, z, u, v$ (every pair of vertices of $G$ are connected by an edge). This graph is unmixed with $\alpha(G) = 4$ and $\beta(G) = 2$. However, $\{xy, uv\}$ is a maximal independent set of facets and the fifth vertex $z$ of $G$ is missing from the vertex set of the graph $\langle xy, uv\rangle$, which contradicts the claim of Lemma 6.1.

Lemma 6.1 along with Theorem 5.3 provides us with the following property for unmixed trees.

**Corollary 6.3.** *If $\Delta$ is an unmixed tree with $\alpha(\Delta) = r$, and $\{F_1, \ldots, F_r\}$ is a maximal independent set of facets of $\Delta$, then $V(\Delta) = V(F_1) \cup \ldots \cup V(F_r)$.*

**Corollary 6.4.** *If $\Delta$ is an unmixed tree, then any maximal independent set of facets of cardinality $\alpha(\Delta)$ of $\Delta$ contains all the leaves. In particular, the leaves of an unmixed tree are independent.*

*Proof.* Every leaf has a free vertex, and so it follows from above that a independent set of facets of cardinality $\alpha(\Delta)$ must contain all the leaves. The claim then follows. $\square$

**Corollary 6.5.** *If $\Delta$ is an unmixed tree, then a maximal independent set of facets of cardinality $\alpha(\Delta)$ of $\Delta$ cannot contain a joint. In particular, a joint of an unmixed tree cannot be a leaf.*

*Proof.* If $G$ is a joint, it has to intersect a leaf $F$ by definition, and as $F$ is in every maximal independent set of facets of cardinality $\alpha(\Delta)$, $G$ cannot be in any. $\square$

But even more is true. For an unmixed tree $\Delta$, there is only one maximal independent set of facets with $\alpha(\Delta)$ elements, and that is the set consisting of all the leaves. We prove this in Theorem 6.8.

The proposition below allows us to use induction on the number of facets of an unmixed tree.

**Proposition 6.6.** *Let $\Delta$ be an unmixed tree with a leaf $F$, and let $G$ be a joint of $F$. Then $\Delta' = \Delta \setminus \langle G\rangle$ is also unmixed.*

*Proof.* We use induction on the number of vertices of $\Delta$. Let

$$\Delta = \langle F_1, \ldots, F_q\rangle$$

and

$$V = \{x_1, \ldots, x_n\}$$

be the vertex set for $\Delta$.

The case $n = 1$ is clear.

Suppose that $\alpha(\Delta) = r$ and $A$ is a minimal vertex cover for $\Delta'$. By Corollary 5.2 $\alpha(\Delta') = r$ as well. If $A$ contains any vertex of $G$, then it is also a minimal vertex cover for $\Delta$ and hence of cardinality $r$. So suppose that

$$A \cap G = \emptyset \text{ and } |A| > r.$$

**Claim:** *There is a vertex $x \in V \setminus (A \cup G)$.*

*Proof of Claim:* If not, then

$$V = A \cup G. \tag{2}$$

We show that this is not possible.

Notice that for any $y \in A$ there is a facet $H \in \Delta'$ such that $H \cap A = \{y\}$ (If no such $H$ existed, then $A \setminus \{y\}$ would also be a vertex cover).

From (2) it follows that

$$H = (G \cap H) \cup \{y\}. \tag{3}$$

On the other hand using Theorem 5.3 we can assume $\{F_1, \dots, F_r\}$ is a maximal independent set of facets in $\Delta$. By Corollary 6.5

$$G \notin \{F_1, \dots, F_r\}.$$

As $|A| > r$, one of the $F_i$, say $F_r$, has to contain more than one element of $A$, so suppose

$$A \cap F_r = \{y_1, \dots, y_s\}$$

where $s > 1$ and $y_1, \dots, y_s$ are distinct elements of $A$. It follows from (2) that

$$F_r = (F_r \cap G) \cup \{y_1, \dots, y_s\}. \tag{4}$$

From the discussion preceding (3) above, one can pick $H_1, \dots, H_s$ to be facets of $\Delta'$ such that

$$H_i = (G \cap H_i) \cup \{y_i\} \tag{5}$$

for $i = 1, \dots, s$, and consider the tree

$$\langle G, F_r, H_1, \dots, H_s \rangle$$

which by Lemma 4.1 is supposed to have two leaves. But based on the descriptions of $F_r, H_1, \dots, H_s$ in (4) and (5), only one facet of this tree, namely $G$, could possibly have a free vertex, which is a contradiction. This proves the claim.

We now proceed to showing that $|A| > r$ is not possible.

Let $x \in V \setminus (A \cup G)$. We localize at the prime ideal $p$ generated by $V \setminus \{x\}$, and use the induction hypothesis.

Let

$$I = \mathcal{F}(\Delta) \text{ and } I' = \mathcal{F}(\Delta')$$

and let

$$\Delta_p = \delta_{\mathcal{F}}(I_p) \ \text{ and } \ \Delta'_p = \delta_{\mathcal{F}}(I'_p)$$

and

From Discussion 4.2 we know that, without loss of generality, for some $t \leq q$

$$\Delta_p = \langle \tilde{F}_1, \dots, \tilde{F}_t \rangle$$

where $\tilde{F}_i = F_i \setminus \{x\}$, and each of $\tilde{F}_{t+1}, \dots, \tilde{F}_q$ contains at least one of $\tilde{F}_1, \dots, \tilde{F}_t$.

We also know by Lemma 4.5 that $\Delta_p$ is a forest whose vertex set is a proper subset of $V$.

By Lemma 4.3 $\Delta_p$ is unmixed with $\alpha(\Delta_p) = r$

We now focus on $\Delta'_p$. Besides possibly $\tilde{G}$, all other facets of $\Delta_p$ and $\Delta'_p$ are the same. We show why this is true.

Let $\tilde{F}_i \in \Delta'_p$. Then clearly

$$\tilde{F}_j \not\subseteq \tilde{F}_i \text{ for all } F_j \in \Delta', \ j \neq i.$$

On the other hand, as $\tilde{G} = G$ and $G \not\subseteq F_i$, we have

$$\tilde{G} \not\subseteq \tilde{F}_i$$

and so $\tilde{F}_i \in \Delta_p$.

Conversely, if $\tilde{F}_i \in \Delta_p$, then

$$\tilde{F}_j \not\subseteq \tilde{F}_i \text{ for all } F_j \in \Delta, \ j \neq i$$

which implies the same for all $F_j \in \Delta'$, and therefore $\tilde{F}_i \in \Delta'_p$.

So there are two possible scenarios:

*Case 1.* If $\tilde{G} \notin \Delta_p$, then

$$\Delta_p = \Delta'_p$$

which implies that $A$ is also a minimal vertex cover of $\Delta_p$, which is unmixed, and hence $|A| = r$; a contradiction.

*Case 2.* If $\tilde{G} \in \Delta_p$ then

$$\tilde{F} \in \Delta_p.$$

If not, then for some facet $H$ of $\Delta$, we have $\tilde{H} \subseteq \tilde{F}$, so $H \cap F \neq \emptyset$ and therefore, since $G$ is a joint of the leaf $F$,

$$H \cap F \subseteq G \cap F$$

which immediately results in

$$\tilde{H} \subseteq \tilde{G}$$

which is not possible.

In fact, $\tilde{F}$ remains a leaf in $\Delta_p$, since if $\tilde{H}$ is a facet of $\Delta_p$ such that $\tilde{H} \cap \tilde{F} \neq \emptyset$, then

$$\emptyset \neq H \cap F \subseteq G \cap F \Longrightarrow \tilde{H} \cap \tilde{F} \subseteq \tilde{G} \cap \tilde{F}$$

and so $\tilde{G}$ is a joint of $\Delta_p$.

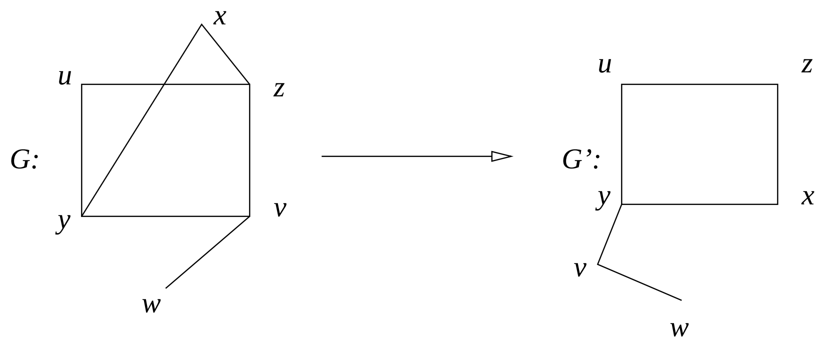
Now by the induction hypothesis,

$$\Delta'_p = \Delta_p \setminus \langle \tilde{G} \rangle$$

is an unmixed forest. This again implies that $|A| = r$; a contradiction.

$\square$

**Example 6.7.** Although not obvious at a first glance, Proposition 6.6 does not necessarily hold if $G$ is not a tree. The following example of an unmixed graph $G$ with a leaf demonstrates this point.




The graph $G$ above was taken from the table of unmixed graphs in [Vi2]. The minimal vertex covers of $G$, all of cardinality 3, are $\{w, z, y\}, \{v, x, u\}$, and $\{v, z, y\}$. But once one removes the joint $\{v, z\}$, $G'$ has minimal vertex covers $\{w, y, z\}$ and $\{w, y, x, u\}$ of different cardinalities, and is therefore not unmixed.

**Theorem 6.8 (Structure theorem for unmixed trees).** *Suppose that $\Delta$ is an unmixed tree with more than one facet such that $\alpha(\Delta) = r$. Then $\Delta$ can be written as*

$$\Delta = \langle F_1, \dots, F_r \rangle \cup \langle G_1, \dots, G_s \rangle$$

*with the following properties:*

*(i)* $F_1, \dots, F_r$ *are all the leaves of* $\Delta$*;*

*(ii)* $\{G_1, \dots, G_s\} \cap \{F_1, \dots, F_r\} = \emptyset$*;*

*(iii) For* $i \neq j$*,* $F_i \cap F_j = \emptyset$*;*

*(iv) If a facet* $H \in \Delta$ *is not a leaf, then it does not contain a free vertex.*

*Proof.* If we prove $(i)$, then parts $(ii)$, $(iii)$ and $(iv)$ will follow from $(i)$, Corollary 6.4 and Corollary 6.3.

We prove part $(i)$ by induction on the number of facets $q$ of $\Delta$. If $q > 1$, then $q \geq 3$ (if $\Delta$ is a tree of two facets, both facets must be leaves by Lemma 4.1, and since $\Delta$ is connected, we can get minimal vertex covers of cardinalities one and two, which means that $\Delta$ is not unmixed).

So the base case for induction is when $q = 3$. In this case, let $F_1$ and $F_2$ be the two disjoint leaves of $\Delta$, and let $G_1$ be the third facet. Since $\Delta$ is connected and unmixed, $G_1$ cannot be a leaf (because the leaves are pairwise disjoint). So $G_1$ is a joint for both $F_1$ and $F_2$ and this settles the case $q = 3$.

For the general case, suppose that $G$ is a joint of $\Delta$. By Corollary 6.5, $G$ is not a leaf. By Corollary 5.2 and Proposition 6.6, if we remove $G$, the forest $\Delta' = \Delta \setminus \langle G \rangle$ is unmixed and $\alpha(\Delta') = r$. By the induction hypothesis,

$$\Delta' = \langle F_1, \dots, F_r \rangle \cup \langle G_1, \dots, G_s \rangle \tag{6}$$

where conditions $(i)$ to $(iv)$ are satisfied. It is easy to see from condition $(iv)$ that if $F$ is a leaf of $\Delta$, then it will still be a leaf of $\Delta'$ (because it has a free vertex).

Our goal is to show that the converse is true, that is, to show that $F_1, \dots, F_r$ are all the leaves of $\Delta$.

We have the following presentation for $\Delta$:

$$\Delta = \langle F_1, \dots, F_r \rangle \cup \langle G_1, \dots, G_s \rangle \cup \langle G \rangle. \tag{7}$$

There are two cases to consider.

*Case 1.* $G$ is the only joint of $\Delta$.

Suppose, without loss of generality, that for some $e$, $F_1, \dots, F_{e-1}$ are leaves of $\Delta$ and $F_e, \dots, F_r$ are not leaves of $\Delta$. Remove $F_1, \dots, F_{e-1}$ from $\Delta$ to obtain the forest

$$\Delta'' = \langle F_e, \dots, F_r \rangle \cup \langle G_1, \dots, G_s \rangle \cup \langle G \rangle.$$

By Lemma 4.1, $\Delta''$ has at least two leaves. Neither one of $G_1, \dots, G_s$ could be a leaf, because neither one of them has a free vertex. To see this, note that by the induction hypothesis on $\Delta'$ and part $(iv)$ of the theorem, $G_1, \dots, G_s$ do not have free vertices in $\Delta'$, and hence they cannot have free vertices in $\Delta$. As facets of $\Delta''$, they still do not have free vertices, because as $G$ is the only joint of $\Delta$,

$$G_i \cap F_j \subseteq G \cap F_j \subseteq G \quad \text{for} \quad 1 \leq i \leq s \quad \text{and} \quad 1 \leq j \leq e-1.$$

Since $G$ is a facet of $\Delta''$ the removal of $F_1, \dots, F_{e-1}$ does not free any vertices of $G_1, \dots, G_s$.

This implies that at least one of $F_e, \dots, F_r$ is a leaf of $\Delta''$. Suppose that $F_e$ is a leaf. Then there exists a facet $G' \in \Delta''$ such that

$$H \cap F_e \subseteq G' \cap F_e \quad \text{for all } H \in \Delta'' \setminus \langle F_e \rangle.$$

Since $F_i \cap F_e = \emptyset$ for $i = 1, \dots e-1$, it follows that

$$H \cap F_e \subseteq G' \cap F_e \quad \text{for all } H \in \Delta \setminus \langle F_e \rangle$$

and so $F_e$ is a leaf of $\Delta$, which is a contradiction.

*Case 2.* $\Delta$ has another joint $G'$ distinct from $G$.

Consider the presentation of $\Delta$ as in (7). As $\{F_1, \dots, F_r\}$ is a maximal independent set of facets in $\Delta$, it cannot contain $G'$ (Corollary 6.5). Therefore

$$G' \in \{G_1, \dots, G_s\}.$$

We show that, say, $F_1$ is a leaf of $\Delta$.

Consider the two unmixed forests

$$\Delta' = \Delta \setminus \langle G \rangle \quad \text{and} \quad \Delta'' = \Delta \setminus \langle G' \rangle.$$

We already know from before that $F_1$ is a leaf of $\Delta'$. From the fact that $\{F_1, \dots, F_r\}$ is a maximal independent set of facets in $\Delta''$ and Corollary 6.4 and the induction hypothesis, it follows that $F_1$ is also a leaf of $\Delta''$.

So, by the definition of a leaf, there is a facet, say $G_1$, in $\Delta'$, such that

$$H \cap F_1 \subseteq G_1 \cap F_1 \quad \text{for all} \quad H \neq G, F_1. \tag{8}$$

and a facet $G_2 \in \Delta''$ such that

$$H \cap F_1 \subseteq G_2 \cap F_1 \quad \text{for all} \quad H \neq G', F_1. \tag{9}$$

The possible scenarios are the following.

(a) $G_1 \neq G'$ or $G_2 \neq G$.
Suppose $G_1 \neq G'$. In this case $G_1 \in \Delta''$, and so because of (9)

$$G_1 \cap F_1 \subseteq G_2 \cap F_1$$

which with (8) and (9) implies that

$$H \cap F_1 \subseteq G_2 \cap F_1 \quad \text{for all} \quad H \neq F_1$$

hence $F_1$ is a leaf of $\Delta$. The case $G_2 \neq G$ is identical.

(b) $G_1 = G'$ and $G_2 = G$.
In this case, Statements (8) and (9), respectively, translate into

$$\begin{cases} H \cap F_1 \subseteq G' \cap F_1 & \text{for all} \quad H \neq G, F_1 \\ H \cap F_1 \subseteq G \cap F_1 & \text{for all} \quad H \neq G', F_1 \end{cases} \tag{10}$$

If $F_1$ is not a leaf of $\Delta$, it follows from (10) that

$$\begin{cases} G \cap F_1 \not\subseteq G' \cap F_1 \\ G' \cap F_1 \not\subseteq G \cap F_1 \\ H \cap F_1 \subseteq (G \cap G') \cap F_1 \quad \text{for all} \quad H \neq G, G', F_1 \end{cases} \tag{11}$$

By (11) there exist

$$x \in (G \cap F_1) \setminus G' \quad \text{and} \quad y \in (G' \cap F_1) \setminus G. \tag{12}$$

**Claim:** *There is a minimal vertex cover for $\Delta \setminus \langle G, G', F_1 \rangle$ that avoids all the vertices in $G$, $G'$ and $F_1$.*

*Proof of Claim:* We first show that there is no facet of $\Delta \setminus \langle G, G', F_1 \rangle$ that has all its vertices in $G \cup G'$. Suppose that $H$ is such a facet:

$$H = (H \cap G) \cup (H \cap G') \tag{13}$$

and consider the tree

$$\Delta_1 = \langle G, G', F_1, H \rangle.$$

By Lemma 4.1, $\Delta_1$ must have two leaves. Note that $H$ cannot be a leaf, since because of (13) it has no free vertices. If $F_1$ is a leaf, then it cannot have $G$ or $G'$ as its joint, since that violates the first two conditions in (11), and so $H$ must be its joint. But then it follows that

$$G \cap F_1 \subseteq H \cap F_1.$$

This implies that $x \in H$ (where $x$ is defined in (12)), which along with the third part of (11), results in $x \in G'$, which is a contradiction.
So $G$ and $G'$ are the two leaves of $\Delta_1$. Consider $G$ first. If $G'$ is a joint for $G$, it follows that

$$F_1 \cap G \subseteq G' \cap G \subseteq G'$$

which contradicts (11).
If $H$ is a joint of $G$, then

$$F_1 \cap G \subseteq H \cap G$$

which implies that $x \in H$, but this again means $x \in G'$ (because of (11)), which is a contradiction. So $F_1$ is the only possible joint for $G$.
With an identical argument for $G'$, it follows that $F_1$ is a joint for both $G$ and $G'$ in $\Delta_1$, and therefore

$$H \cap G \subseteq F_1 \cap G \text{ and } H \cap G' \subseteq F_1 \cap G'$$

which along with (13) implies that

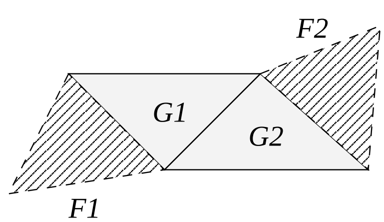
$$H \subseteq F_1$$

which is impossible since $H$ and $F_1$ are both facets of $\Delta$.
So we have shown that every facet of $\Delta$ other than $G$, $G'$ and $F_1$, has at least one vertex outside $G$ and $G'$ (and therefore by the third condition in (11), outside $F_1$).
For each facet $H$ of $\Delta \setminus \langle G, G', F_1 \rangle$, pick a vertex $z \in H$ that avoids all three facets $G$, $G'$ and $F_1$. The set of these vertices is a vertex cover for $\Delta \setminus \langle G, G', F_1 \rangle$, and so it has a subset that is a minimal vertex cover. This proves the claim.

Now let $A$ be a minimal vertex cover for $\Delta \setminus \langle G, G', F_1 \rangle$ that avoids all the vertices in $G$, $G'$ and $F_1$. Since $\Delta \setminus \langle G, G', F_1 \rangle$ has $r-1$ independent facets, $|A| \geq r-1$. Now $A \cup \{x, y\}$ is a minimal vertex cover for $\Delta$ with more than $r$ vertices, which contradicts the fact that $\Delta$ is unmixed with vertex covering number equal to $r$ (Note that $x$ and $y$ do not belong to any facet of $\Delta \setminus \langle G, G', F_1 \rangle$, as this would contradict the third condition in (11)).

So both cases 1 and 2 lead to contradictions, therefore all of $F_1, \ldots, F_r$ must be leaves of $\Delta$, which proves the theorem. $\square$

**Example 6.9.** The simplicial complex $\Delta$ shown below is an unmixed tree, satisfying properties (i) to (iv) of Theorem 6.8.

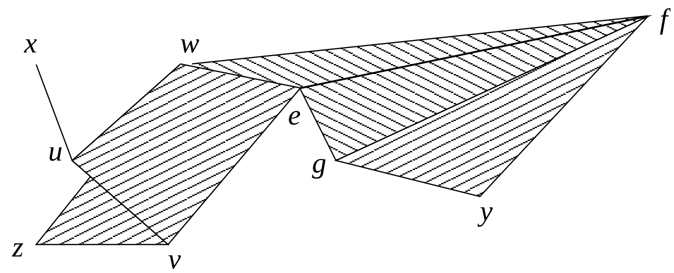


It is important to notice that Theorem 6.8 does not suggest that every facet in an unmixed tree is either a leaf or a joint (See Example 6.10 below). On the other hand two different leaves in an unmixed tree may share a joint, as is the case with the unmixed graph $\langle xy, yz, zu \rangle$. For these reasons the two numbers $r$ and $s$ in the statement of Theorem 6.8 that count the number of leaves and non-leaves, respectively, do not seem to have any particular relationship to one another.

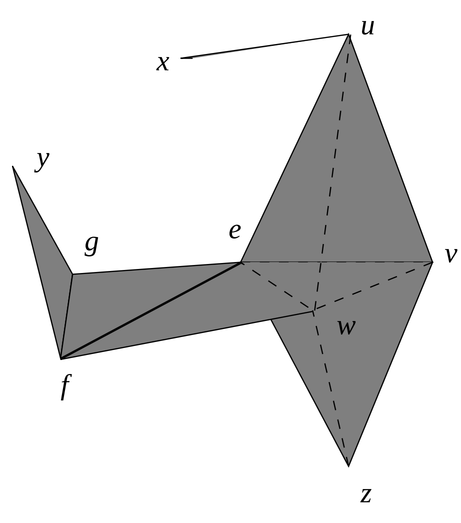
**Example 6.10.** The following simplicial complex, which is the facet complex of the ideal

$$(xu, uvew, zvew, efw, efg, fgy)$$

is an unmixed tree with a facet $\{e, f, w\}$ that is neither a leaf nor a joint. In fact, the two leaves $\{x, u\}$ and $\{z, v, e, w\}$ share a joint $\{u, v, e, w\}$.




Above, for simplicity, an $n$-dimensional facet (simplex) is drawn as a shaded polygon with $n + 1$ vertices. The picture in 3D is as follows:




# 7 Grafting simplicial complexes

All that we proved in the previous section about unmixed trees can be put into one definition– namely that of a grafted tree. In fact, the method of grafting works as an effective way to build an unmixed simplicial complex from any given simplicial complex by adding new leaves (Theorem 7.6). It turns out that a grafted simplicial complex is Cohen-Macaulay (Theorem 8.2).

**Definition 7.1 (grafting).** A simplicial complex $\Delta$ is a *grafting* of the simplicial complex $\Delta' = \langle G_1, \dots, G_s \rangle$ with the simplices $F_1, \dots, F_r$ (or we say that $\Delta$ is *grafted*) if

$$\Delta = \langle F_1, \dots, F_r \rangle \cup \langle G_1, \dots, G_s \rangle$$

with the following properties:

(i) $V(\Delta') \subseteq V(F_1) \cup \ldots \cup V(F_r)$;

(ii) $F_1, \dots, F_r$ are all the leaves of $\Delta$;

(iii) $\{G_1, \dots, G_s\} \cap \{F_1, \dots, F_r\} = \emptyset$;

(iv) For $i \neq j$, $F_i \cap F_j = \emptyset$;

(v) If $G_i$ is a joint of $\Delta$, then $\Delta \setminus \langle G_i \rangle$ is also grafted.

Note that a simplicial complex that consists of only one facet or several pairwise disjoint facets is indeed grafted, as it could be considered as a grafting of the empty simplicial complex. It is easy to check that conditions (i) to (v) above are satisfied in this case.

It is also clear that the union of two or more grafted simplicial complexes is itself grafted.

**Remark 7.2.** Condition (v) above implies that if $F$ is a leaf of a grafted $\Delta$, then all the facets $H$ that intersect $F$ have embedded intersections; in other words if $H \cap F$ and $H' \cap F$ are both nonempty, then

$$H \cap F \subseteq H' \cap F \quad \text{or} \quad H' \cap F \subseteq H \cap F.$$

This implies that there is a chain of intersections

$$H_1 \cap F \supseteq \ldots \supseteq H_t \cap F$$

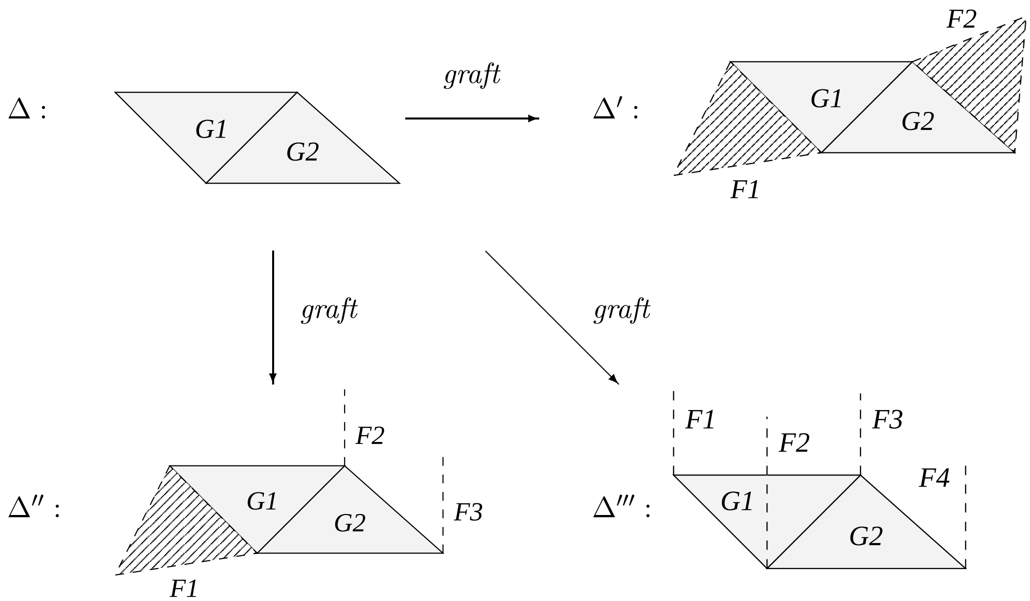
where $H_1, \dots, H_t$ are all the facets of $\Delta$ that intersect $F$.

**Remark 7.3.** Condition (v) in Definition 7.1 can be replaced by "$\Delta \setminus \langle G_i \rangle$ is grafted for all $i = 1, \dots, s$". This is because even if $G_i$ is not a joint of $\Delta$, $\Delta \setminus \langle G_i \rangle$ satisfies properties (i), (iii) and (iv), and it satisfies (ii) and (v) because of Remark 7.2, and so $\Delta \setminus \langle G_i \rangle$ is grafted.

**Remark 7.4 (A grafting of a tree is also a tree).** If $\Delta'$ in Definition 7.1 is a tree, then $\Delta$ is also a tree. To see this, consider any subcollection $\Delta''$ of $\Delta$. If $\Delta''$ contains $F_i$ for some $i$, then by remarks 7.2 and 7.3 $F_i$ is a leaf of $\Delta''$. If $\Delta''$ contains neither of the $F_i$, then it is a subcollection of the tree $\Delta'$, which implies that $\Delta''$ has a leaf.

The "suspension" of a graph, as defined in [Vi1], is also a grafting of that graph.

**Example 7.5.** The tree $\langle F_1, F_2, G_1, G_2 \rangle$ that appeared in Example 6.9 above is a grafting of the tree $\langle G_1, G_2 \rangle$ with the leaves $F_1$ and $F_2$. In fact, there may be more than one way to graft a given simplicial complex. For example, some possible ways of grafting $\langle G_1, G_2 \rangle$ are shown below:




**Theorem 7.6 (A grafted simplicial complex is unmixed).** *Let*

$$\Delta = \langle F_1, \dots, F_r \rangle \cup \langle G_1, \dots, G_s \rangle$$

*be a grafting of the simplicial complex $\langle G_1, \dots, G_s \rangle$ with the simplices $F_1, \dots, F_r$. Then $\Delta$ is unmixed, and $\alpha(\Delta) = r$.*

*Proof.* If $\langle G_1, \dots, G_s \rangle$ is the empty simplicial complex, the claim is immediate, so we assume that it is nonempty.

We argue by induction on the number of facets $q$ of $\Delta$. The first case to consider is $q = 3$. In this case, $\Delta$ must have at least two leaves, as if there were only one leaf $F_1$, i.e. if $\Delta = \langle F_1 \rangle \cup \langle G_1, G_2 \rangle$, then by Condition (i) of Definition 7.1 we would have $G_1 \subseteq F_1$ and $G_2 \subseteq F_1$, which is impossible. So $\Delta = \langle F_1, F_2 \rangle \cup \langle G_1 \rangle$, where $G_1 \subseteq F_1 \cup F_2$ and $F_1 \cap F_2 = \emptyset$. It is now easy to see that $\Delta$ is unmixed with $\alpha(\Delta) = 2$.

Suppose $\Delta$ has $q > 3$ facets, and let $G_1$ be a joint of the leaf $F_1$. By Part (v) of Definition 7.1 $\Delta' = \Delta \setminus \langle G_1 \rangle$ is also grafted, and therefore by the induction hypothesis unmixed with $\alpha(\Delta') = r$.

Let $A$ be a minimal vertex cover of $\Delta$. We already know that $|A| \geq r$ as $F_1, \dots, F_r$ are $r$ independent facets of $\Delta$. Now suppose that $|A| > r$ . Since $A$ is also a vertex cover for $\Delta'$, it has a subset $A'$ that is a minimal vertex cover of $\Delta'$ with $|A'| = r$. Since $A'$ is a proper subset of $A$, it is not a vertex cover for $\Delta$, and therefore $A'$ cannot contain a vertex of $G_1$. So $A'$ contains a free vertex $x$ of $F_1$ (all non-free vertices of $F_1$ are shared with $G_1$). Now $A$ must contain a vertex $y$ of $G_1$; say $y \in G_1 \cap F_2$ ($y \notin F_1$, since in that case $x$ would be redundant). So

$$A = A' \cup \{y\}.$$

On the other hand $A'$ must also contain a vertex of $F_2$, say $z$. So $F_2$ contributes two vertices $y$ and $z$ to $A$; note that neither one of $y$ or $z$ could be a free vertex, as in that case the free one would be redundant.

Now suppose that $G_2$ is a joint of $F_2$. Remove $G_2$ from $\Delta$ to get

$$\Delta'' = \Delta \setminus \langle G_2 \rangle.$$

So $A$ has a subset $A''$, $|A''| = r$, that is a minimal vertex cover for $\Delta''$. But as $A$ already has exactly one vertex in each of $F_1, F_3, \ldots, F_r$, the only way to get $A''$ from $A$ is to remove one of $y$ or $z$, this means that:

$$A'' = A \setminus \{y\} \qquad \text{or} \qquad A'' = A \setminus \{z\}.$$

In either case $A''$ contains a vertex of $G_2$, which implies that $A''$ is a minimal vertex cover for $\Delta$; a contradiction. $\square$

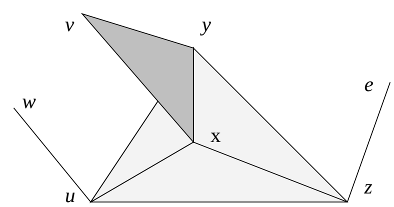
Example 7.5 demonstrates Theorem 7.6: $\Delta = \langle G_1, G_2 \rangle$ is a non-unmixed tree, which gets grafted with some leaves to make the unmixed trees $\Delta'$, $\Delta''$ and $\Delta'''$.

One could graft any simplicial complex, even a badly non-unmixed non-tree.

**Example 7.7.** Let $\Delta'$ be the non-unmixed non-tree in Example 3.3. We could graft $\Delta'$ with three new leaves

$$\{x, y, v\}, \{u, w\}, \{z, e\}$$

The resulting picture below is unmixed, and moreover, as we prove later, Cohen-Macaulay.




In the case of a tree theorems 6.8 and 7.6 put together with Corollary 6.3 produce a much stronger statement:

**Corollary 7.8 (A tree is unmixed if and only if grafted).** *Suppose the simplicial complex $\Delta$ is a tree. Then $\Delta$ is unmixed if and only if $\Delta$ is grafted.*

Grafted simplicial complexes behave well under localization; in other words, the localization of a grafted simplicial complex is also grafted. In the case of trees this follows directly from Corollary 7.8, Lemma 4.3 and Lemma 4.5. But the statement holds more generally.

**Proposition 7.9 (Localization of a grafted simplicial complex is grafted).** *Let $I = \mathcal{F}(\Delta)$ where $\Delta$ is a grafted simplicial complex with vertices labeled $x_1, \ldots, x_n$. Suppose that $k$ is a field and $p$ is a prime ideal of the polynomial ring $k[x_1, \ldots, x_n]$. Then $\delta_{\mathcal{F}}(I_p)$ is a grafted simplicial complex.*

*Proof.* With notation as in Definition 7.1, let

$$\Delta = \langle F_1, \ldots, F_r \rangle \cup \langle G_1, \ldots, G_s \rangle.$$

If $\Delta$ has only one facet, the statement of the theorem follows immediately, so assume that $\Delta$ has two or more facets.

As in the proof of Lemma 4.5, it is enough to assume that $p$ is generated by a subset of $\{x_1, \ldots, x_n\}$, so

$$p = (x_{i_1}, \ldots, x_{i_h}).$$

Following Discussion 4.2, let

$$\Delta_p = \delta_{\mathcal{F}}(I_p) = \langle F'_1, \ldots, F'_t \rangle \cup \langle G'_1, \ldots, G'_u \rangle$$

where for $i = 1, \ldots, r$ and $j = 1, \ldots, s$

$$F_i' = F_i \cap \{x_{i_1}, \ldots, x_{i_h}\} \quad \text{and} \quad G_j' = G_j \cap \{x_{i_1}, \ldots, x_{i_h}\}$$

and $F_{t+1}', \ldots, F_r', G_{u+1}', \ldots, G_s'$ each contain at least one of

$$F_1', \ldots, F_t', G_1', \ldots, G_u'. \tag{14}$$

We now rename the facets of $\Delta_p$ as follows. For $i = 1, \ldots, t$, let

$$H_i = F_i'.$$

For each $i = t+1, \ldots, r$, $F_i'$ contains one of the facets appearing in (14). But as by definition $F_i \cap F_j = \emptyset$ for all $j \neq i$, there must be some $j \leq u$ for which $G_j' \subseteq F_i'$. For this particular $j$, set

$$H_i = G_j'.$$

This choice of $j$ is well-defined: if there were some $f \leq u$ distinct from $j$ such that $G_f' \subseteq F_i'$, then it would follow from Remark 7.2 that either $G_j' \subseteq G_f'$ or $G_f' \subseteq G_j'$, which contradicts the fact that both $G_j'$ and $G_f'$ are facets of $\Delta_p$.

We now represent $\Delta_p$ as

$$\Delta_p = \langle H_1, \ldots, H_r \rangle \cup \langle E_1, \ldots, E_v \rangle$$

where $E_1, \ldots, E_v$ represent all the other facets of $\Delta_p$ that were not labeled by some $H_i$.

Our goal is to show that $\Delta_p$ is a grafting of the simplicial complex $\langle E_1, \ldots, E_v \rangle$ with the simplices $H_1, \ldots, H_r$.

It is clear by our construction that the facets $H_1, \ldots, H_r$ are pairwise disjoint. To see this, notice that for each pair of distinct numbers $i_1, i_2 \leq r$, there is a pair of distinct numbers $j_1, j_2 \leq r$ such that

$$H_{i_1} \subseteq F_{j_1}' \subseteq F_{j_1} \quad \text{and} \quad H_{i_2} \subseteq F_{j_2}' \subseteq F_{j_2}$$

and as $F_{j_1} \cap F_{j_2} = \emptyset$,

$$H_{i_1} \cap H_{i_2} = \emptyset.$$

So Condition (iv) of Definition 7.1 is satisfied.

On the other hand, by Theorem 7.6 $\Delta$ is unmixed, so by Lemma 4.3 $\Delta_p$ is unmixed with $\alpha(\Delta_p) = \alpha(\Delta) = r$. We now apply Lemma 6.1 to $\Delta_p$ to deduce that

$$V(\Delta_p) = V(H_1) \cup \ldots \cup V(H_r),$$

which implies Condition (i) in Definition 7.1. This also implies that $E_1, \ldots, E_v$ cannot have free vertices, and hence cannot be leaves of $\Delta_p$.

Condition (iii) is satisfied by the construction of $\Delta_p$.

We need to show that $H_1, \ldots, H_r$ are all leaves of $\Delta_p$. If $\Delta_p = \langle H_1, \ldots, H_r \rangle$ then $\Delta_p$ is grafted by definition. So suppose that $\Delta_p$ has a connected component $\Delta'$ with two or more facets. As $\Delta'$ is connected, it must contain some of the $E_i$, and as $V(\Delta_p) = V(H_1) \cup \ldots \cup V(H_r)$, $\Delta'$ must also contain some of the $H_j$. So we can without loss of generality assume that

$$\Delta' = \langle H_1, \ldots, H_e \rangle \cup \langle E_1, \ldots, E_f \rangle$$

for some $1 \leq e \leq r$ and $1 \leq f \leq v$.

We now show that, for example, $H_1$ is a leaf for $\Delta'$. There are two cases to consider:

*Case 1.* $H_1 = F'_i$ for some $i$ such that $1 \le i \le t$.

Since $\Delta'$ is connected, it has some facets that intersect $H_i$; suppose that $E_{j_1}, \ldots, E_{j_l}$ are all the facets of $\Delta' \setminus \langle H_1 \rangle$ such that

$$H_1 \cap E_{j_z} \neq \emptyset$$

for $z = 1, \ldots, l$.

For each $z = 1, \ldots, l$ suppose that

$$E_{j_z} = G'_{m_z}.$$

The above paragraph translates into

$$F'_i \cap G'_{m_z} \neq \emptyset$$

and hence

$$F_i \cap G_{m_z} \neq \emptyset$$

for $z = 1, \ldots, l$.

From Remark 7.2 it follows that there is some total order of inclusion on the nonempty sets $F_i \cap G_{m_z}$; we assume that

$$F_i \cap G_{m_1} \supseteq F_i \cap G_{m_2} \supseteq \ldots \supseteq F_i \cap G_{m_l}$$

which after intersecting each set with $\{x_{i_1}, \ldots, x_{i_h}\}$ turns into

$$F'_i \cap G'_{m_1} \supseteq F'_i \cap G'_{m_2} \supseteq \ldots \supseteq F'_i \cap G'_{m_l}$$

which is equivalent to

$$H_1 \cap E_{j_1} \supseteq H_1 \cap E_{j_2} \supseteq \ldots \supseteq H_1 \cap E_{j_l}$$

It follows that $H_1$ is a leaf of $\Delta'$, and in addition, Condition (v) of Definition 7.1 is satisfied.

*Case 2.* $H_1 = G'_j$ for some $j$ such that $1 \le j \le u$.

In this case for some $i$, $t < i \le r$,

$$H_1 = G'_j \subseteq F'_i.$$

Exactly as above, let $E_{j_1}, \ldots, E_{j_l}$ be all the facets of $\Delta' \setminus \langle H_1 \rangle$ such that $H_1 \cap E_{j_z} \neq \emptyset$, and let $E_{j_z} = G'_{m_z}$ for $z = 1, \ldots, l$.

As all the sets $F_i \cap G_{m_z}$ are nonempty, we follow the exact argument as above to obtain the chain

$$F'_i \cap G'_{m_1} \supseteq F'_i \cap G'_{m_2} \supseteq \ldots \supseteq F'_i \cap G'_{m_l}$$

As $G'_j \subseteq F'_i$, we can intersect all these sets with $G'_j$ to obtain

$$G'_j \cap G'_{m_1} \supseteq G'_j \cap G'_{m_2} \supseteq \ldots \supseteq G'_j \cap G'_{m_l}$$

which is equivalent to

$$H_1 \cap E_{j_1} \supseteq H_1 \cap E_{j_2} \supseteq \ldots \supseteq H_1 \cap E_{j_l}$$

It follows that $H_1$ is a leaf of $\Delta'$, and also Condition (v) of Definition 7.1 is satisfied.

$\square$

# 8 Grafted simplicial complexes are Cohen-Macaulay

We are now ready to show that the facet ideal of a grafted simplicial complex has a Cohen-Macaulay quotient. Besides revealing a wealth of square-free monomial ideals with Cohen-Macaulay quotients, this result implies that all unmixed trees are Cohen-Macaulay.

Let $\Delta$ be a grafted simplicial complex over a vertex set $V = \{x_1, \dots, x_n\}$. By Definition 7.1, $\Delta$ will have the form

$$\Delta = \langle F_1, \dots, F_r \rangle \cup \langle G_1, \dots, G_s \rangle$$

where $\alpha(\Delta) = r$ and $F_1, \dots, F_r$ are the leaves of $\Delta$.

Let

$$\mathcal{R}(\Delta) = k[x_1, \dots, x_n]/\mathcal{F}(\Delta),$$

where $k$ is a field and let

$$\mathbf{m} = (x_1, \dots, x_n)$$

be the irrelevant maximal ideal.

From Discussion 2.13 we know that

$$\dim \mathcal{R}(\Delta) = n - r.$$

In order to show that $\mathcal{R}(\Delta)$ is Cohen-Macaulay, it is enough to show that there is a homogeneous regular sequence in $\mathbf{m}$ of length $n - r$.

It is interesting to observe how the previous sentence follows also from Proposition 7.9: if $m$ is any other maximal ideal of $\mathcal{R}(\Delta)$, from the proof of Lemma 4.5 and Proposition 7.9 we see that if $p = (x_1, \dots, x_e)$ is the ideal generated by all of $x_i$ that belong to $m$, then $I_m = I_p$ is the facet ideal of a grafted simplicial complex over the vertex set $\{x_1, \dots, x_e\}$. So one can write $m = p + q$ where $q$ is a prime ideal of $k[x_{e+1}, \dots, x_n]$. It follows that

$$\mathcal{R}(\Delta)_m = k[x_1, \dots, x_e]_p/I_p \otimes_k k[x_{e+1}, \dots, x_n]_q.$$

As $k[x_{e+1}, \dots, x_n]_q$ is clearly Cohen-Macaulay, by Theorem 5.5.5 of [V], it is enough to show that $k[x_1, \dots, x_e]_p/I_p$ is Cohen-Macaulay in order to conclude that $\mathcal{R}(\Delta)_m$ is Cohen-Macaulay. But this is again the case of localizing at the irrelevant ideal.

Now suppose that for each $i \leq r$,

$$F_i = y_i x_1^i \dots x_{u_i}^i$$

where $y_i$ is a free vertex of the leaf $F_i$, and $y_i, x_1^i, \dots, x_{u_i}^i \in V$. We wish to show that

$$y_1 - x_1^1, \dots, y_1 - x_{u_1}^1, \dots, y_r - x_1^r, \dots, y_r - x_{u_r}^r \tag{15}$$

is a regular sequence in $\mathcal{R}(\Delta)$. This follows from the process of "polarization" that we describe below.

**Proposition 8.1 ([Fr]).** *Let $R$ be the ring $k[x_1, \dots, x_n]/(M_1, \dots, M_q)$, where $M_1, \dots, M_q$ are monomials in the variables $x_1, \dots, x_n$, and $k$ is a field. Then there is an $N \geq n$, and a set of square-free monomials $N_1, \dots, N_q$ in the polynomial ring $k[x_1, \dots, x_N]$, such that*

$$R = R'/(f_1, \dots, f_{N-n})$$

*where $R' = k[x_1, \dots, x_N]/(N_1, \dots, N_q)$ and $f_1, \dots, f_{N-n}$ is a regular sequence of forms of degree one in $R'$.*

For the purpose of our argument, it is instructive to see an outline of the proof of this proposition.

*Sketch of proof.* Suppose, without loss of generality, that $x_1 | M_i$ for $1 \leq i \leq s$, and $x_1 \not| M_j$ for $s < j \leq q$.

For $i = 1, \ldots, s$ we set
$$M_i' = \frac{M_i}{x_1}$$
so that we can write
$$I = (M_1, \ldots, M_q) = (x_1 M_1', \ldots, x_1 M_s', M_{s+1}, \ldots, M_q).$$

Define
$$I_1 = (x_{n+1} M_1', \ldots, x_{n+1} M_s', M_{s+1}, \ldots, M_q) \subseteq k[x_1, \ldots, x_n, x_{n+1}].$$

Then $R = R_1/(x_{n+1} - x_1)$ where
$$R_1 = k[x_1, \ldots, x_n, x_{n+1}]/I_1.$$

It is then shown that $x_{n+1} - x_1$ is a non-zerodivisor in $R_1$. If $I_1$ is square-free, we are done. Otherwise one applies the same procedure to $I_1$ continually until the ideal becomes square-free. $\square$

What we would like to show is that Sequence (15) polarizes the ring
$$S = k[y_1, \ldots, y_r]/(y_1^{u_1+1}, \ldots, y_r^{u_r+1}, E_1, \ldots, E_s)$$
into the ring $\mathcal{R}(\Delta)$, where $E_1, \ldots, E_s$ are monomials corresponding to the facets $G_1, \ldots, G_s$, where each vertex belonging to $F_i$ has been replaced by the free vertex $y_i$. In other words if
$$J = (y_1 - x_1^1, \ldots, y_1 - x_{u_1}^1, \ldots, y_r - x_1^r, \ldots, y_r - x_{u_r}^r),$$
we wish to show that
$$S = \mathcal{R}(\Delta)/J.$$
It will then follow from the proof of Proposition 8.1 (as detailed in [Fr] as well as in [Vi2]) that Sequence (15) is a regular sequence in $\mathcal{R}(\Delta)$.

Intuitively our claim is straightforward to see. The only problem that may arise is if after applying Sequence (15) to $S$, we end up with a permutation of the vertices of $\Delta$. To prevent this from happening, we use the subtle structure of a grafted simplicial complex (Remark 7.2) that the facets intersecting a leaf do so in an embedded (and therefore ordered) manner. In other words, suppose for the leaf $F_i$, the facets $H_1^i, \ldots, H_{e_i}^i$ are all the facets of $\Delta \setminus \langle F_i \rangle$ that intersect $F_i$, with the ordering
$$H_1^i \cap F_i \subseteq \ldots \subseteq H_{e_i}^i \cap F_i. \tag{16}$$

So in Sequence (15), we order
$$y_i - x_1^i, \ldots, y_i - x_{u_i}^i \tag{17}$$
such that if for any $e$ and $f$, $x_e^i \in H_f^i$ then $x_e^i \in H_{f+1}^i$.

We now use induction on the number of facets of $\Delta$. If we remove a joint, say $G_1 \in \mathcal{U}_\Delta(F_1)$, we obtain a grafted simplicial complex

$$\Delta' = \Delta \setminus \langle G_1 \rangle$$

over the same set of vertices $x_1, \ldots, x_n$, with $\alpha(\Delta') = \alpha(\Delta)$ (Lemma 5.1). Therefore if

$$\mathcal{R}(\Delta') = k[x_1, \ldots, x_n]/\mathcal{F}(\Delta')$$

then

$$\dim\, \mathcal{R}(\Delta) = \dim\, \mathcal{R}(\Delta').$$

Moreover, $\Delta'$ has $F_1, \ldots, F_r$ as leaves. So by the induction hypothesis, Sequence (15) polarizes the ring

$$S' = k[y_1, \ldots, y_r]/(y_1^{u_1+1}, \ldots, y_r^{u_r+1}, E_2, \ldots, E_s)$$

into $\mathcal{R}(\Delta')$, or in other words,

$$S' = \mathcal{R}(\Delta')/J.$$

The induction hypothesis has ensured that after applying Sequence (15) to $S'$, all facets of $\Delta'$ are restored to their original positions and labeling. Now it all reduces to showing that during this polarization process, $E_1$ turns into $G_1$.

This is clear, as for every $i$, $G_1 \cap F_i$ has its place in the ordered sequence (16), and so if $|G_1 \cap F_i| = h_i$, then the first $h_i$ applications of Sequence (17) restore $G_1 \cap F_i$ before moving on to facets that have larger intersections with $F_i$. As $G_1$ has disjoint intersections with $F_1, \ldots, F_r$, once Sequence (17) has been applied for all $i$, $G_1$ is restored to its proper position.

We have shown that:

**Theorem 8.2 (Grafted simplicial complexes are Cohen-Macaulay).** *Let $\Delta$ be a grafted simplicial complex over a set of vertices labeled $x_1, \ldots, x_n$, and let $k$ be a field. Then $\mathcal{R}(\Delta) = k[x_1, \ldots, x_n]/\mathcal{F}(\Delta)$ is Cohen-Macaulay.*

Theorem 8.2 along with Proposition 2.12 and Corollary 7.8 imply that for a tree being unmixed and being Cohen-Macaulay are equivalent conditions.

**Corollary 8.3 (A tree is Cohen-Macaulay if and only if unmixed).** *Let $\Delta$ be a tree over a set of vertices $x_1, \ldots, x_n$, and let $k$ be a field. Then the quotient ring $k[x_1, \ldots, x_n]/\mathcal{F}(\Delta)$ is Cohen-Macaulay if and only if $\Delta$ is unmixed.*

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