On the resolution of path ideals of cycles

Ali Alilooee* Sara Faridi†

January 3, 2013

Abstract

We give a formula to compute all the top degree graded Betti numbers of the path ideal of a cycle. Also we will find a criterion to determine when Betti numbers of this ideal are non zero and give a formula to compute its projective dimension and regularity.

1 Introduction

Path ideals of graphs were first introduced by Conca and De Negri [3] in the context of monomial ideals of linear type. Simply put, path ideals are ideals whose monomial generators correspond to vertices in paths of a given length in a graph. Conca and De Negri showed that path ideals of trees have normal and Cohen-Macaulay Rees rings. More recently Bouchat, Hà and O'Keefe [1] and He and Van Tuyl [6] studied invariants related to resolutions of path ideals of certain graphs. In his thesis, Jacques [7] used beautiful techniques to compute Betti numbers of edge ideals of several classes of graphs. Edge ideals can be considered as path ideals of length 2. Our paper extends Jacque's techniques to higher dimensions. Essentially, we consider the path ideal of a graph as a disjoint union of connected components. We then use homological methods to glue these components back together, and using Hochster's formula we compute all graded Betti numbers of degree n, projective dimension and regularity of path ideals of cycles. The paper is organized as follows. In Section 2 we recall some notation and basic algebraic and combinatorial concepts used in other next chapters. In Section 3 we study the connected components of path ideals which will provide us with the key to our homological computations later in Section 4. Section 5 is where we apply the homological results of Section 4 along to give a criterion to determine all non zero Betti numbers and projective dimension of path ideals of cycles. While working on this paper the computer algebra systems CoCoA [9] and Macaulay2 [4] were used to test examples. We acknowledge the immense help that they have provided us in this project.

2 Preliminaries

Throughout we assume that K is a field and $R = K[x_1, \dots, x_n]$ is a polynomial ring in n variables.

Simplicial complexes and monomial ideals

Definition 2.1. An abstract **simplicial complex** on vertex set $\mathcal{X} = \{x_1, \dots, x_n\}$ is a collection Δ of subsets of $\mathcal{X}$ satisfying

*Department of Mathematics and Statistics, Dalhousie University, Halifax, Canada, alilooee@mathstat.dal.ca.
†Department of Mathematics and Statistics, Dalhousie University, Halifax, Canada, faridi@mathstat.dal.ca.

i. $\{x_i\} \in \Delta$ for all i,

ii. $F \in \Delta, G \subset F \Longrightarrow G \in \Delta$.

The elements of Δ are called **faces** of Δ and the maximal faces under inclusion are called **facets** of Δ. We denote the simplicial complex Δ with facets $F_1, \ldots, F_s$ by $\langle F_1, \ldots, F_s \rangle$. We call $\{F_1, \ldots, F_s\}$ the facet set of Δ and is denoted by $F(\Delta)$. The vertex set of Δ is denoted by Vert(Δ).

Definition 2.2. A **subcollection** of a simplicial complex Δ with vertex set $\mathcal{X}$ is a simplicial complex whose facet set is a subset of the facet set of Δ. For $\mathcal{Y} \subseteq \mathcal{X}$, an **induced subcollection** of Δ on $\mathcal{Y}$, denoted by $\Delta_{\mathcal{Y}}$, is the simplicial complex whose vertex set is a subset of $\mathcal{Y}$ and facet set is

$$\{F \in F(\Delta) \mid F \subseteq \mathcal{Y}\}.$$

If F is a face of $\Delta = \langle F_1, \ldots, F_s \rangle$, we define the **complement** of F in Δ to be

$$F^c_{\mathcal{X}} = \mathcal{X} \setminus F \quad \text{and} \quad \Delta^c_{\mathcal{X}} = \langle (F_1)^c_{\mathcal{X}}, \ldots, (F_s)^c_{\mathcal{X}} \rangle.$$

Also if $\mathcal{X} \subsetneqq \text{Vert}(\Delta)$, then $\Delta^c_{\mathcal{X}} = (\Delta_{\mathcal{X}})^c_{\mathcal{X}}$.

Definition 2.3. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, and I an ideal in R minimally generated by square-free monomials $m_1, \ldots, m_s$. One can associate two simplicial complexes to I.

i. The **Stanley-Reisner complex** Δ_I associated to I has vertex set $V = \{x_i \mid x_i \notin I\}$ and is defined as
$$\Delta_I = \{\{x_{i_1}, \ldots, x_{i_k}\} \mid i_1 < i_2 < \cdots < i_k, x_{i_1} \ldots x_{i_k} \notin I\}.$$

ii. The **facet complex** $\Delta(I)$ associated to I has vertex set $\{x_1, \ldots, x_n\}$ and is defined as

$$\Delta(I) = \langle F_1, \ldots, F_s \rangle \text{ where } F_i = \{x_j \mid x_j | m_i,\ 1 \leq j \leq n\},\ 1 \leq i \leq s.$$

Conversely to a simplicial complex Δ one can associate two monomial ideals.

Definition 2.4. Let Δ be a simplicial complex with vertex set $x_1, \ldots, x_n$ and $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K.

i. The **Stanley-Reisner** ideal of Δ is defined as

$$I_\Delta = (\prod_{x \in F} x \mid F \notin \Delta).$$

ii. The **facet ideal** of Δ is defined as

$$I(\Delta) = (\prod_{x \in F} x \mid \ F \text{ is a facet of} \Delta).$$

Note that there is a one-to-one correspondence between monomial ideals and simplicial complexes via each of these methods.

Definition 2.5. Let Δ be a simplicial complex with vertex set $\mathcal{X}$. The **Alexander Dual** Δ^* is defined to be the simplicial complex with faces

$$\Delta^* = \{F \subset \mathcal{X} \mid F^c_{\mathcal{X}} \text{ is not face of } \Delta\}.$$

To prove some of the results in this paper we need the definition of a cone and its properties.

Definition 2.6. Let Δ_1 and Δ_2 be two simplicial complexes on disjoint vertex sets V and W. The **join** $\Delta_1 * \Delta_2$ is the simplicial complex on $V \bigsqcup W$ with faces $F \cup G$ where $F \in \Delta_1, G \in \Delta_2$. The **cone** $Cn(\Delta)$ of Δ is the simplicial complex $\omega * \Delta$, where ω is a new vertex.

Given a simplicial complex Δ, we denote by $\mathcal{C}.(\Delta)$ the reduced chain complex and by $\widetilde{H}_i(\Delta) = Z_i(\Delta)/B_i(\Delta)$ the i - th reduced homology groups of Δ with coefficients in field K. For the proof of the following fact, see for example Villarreal [10], Proposition 5.2.5.

Proposition 2.7. *If Δ is a simplicial complex we have $\widetilde{H}_i(Cn(\Delta)) = 0$ for all i.*

Betti numbers

For any homogeneous ideal I of the polynomial ring $R = K[x_1, \dots, x_n]$ there exists a **graded minimal finite free resolution**

$$0 \to \bigoplus_d R(-d)^{\beta_{p,d}} \to \cdots \to \bigoplus_d R(-d)^{\beta_{1,d}} \to R \to R/I \to 0$$

of R/I in which $R(-d)$ denotes the graded free module obtained by shifting the degrees of elements in R by d. The numbers $\beta_{i,d}$, which we shall refer to as the i-th $\mathbb{N}$-**graded Betti numbers** of degree d of R/I, are independent of the choice of graded minimal finite free resolution.

For computing the $\mathbb{N}$-graded Betti numbers of the Stanley-Reisner ring of a simplicial complex we use an equivalent form of Hochster's formula.

Theorem 2.8. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K, and I be a pure square-free monomial ideal in R. Then the $\mathbb{N}$-graded Betti numbers of R/I are given by*

$$\beta_{i,d}(R/I) = \sum_{\Gamma \subset \Delta(I), |\mathit{Vert}(\Gamma)| = d} \dim_K \widetilde{H}_{i-2}(\Gamma^c_{\mathit{Vert}(\Gamma)})$$

where the sum is taken over the induced subcollections Γ of $\Delta(I)$ which have d vertices.

Proof. Hochster's formula (see for example Corollary 5.12 of [8]) says

$$\beta_{i,d}(R/I) = \sum_{W \subset \mathrm{Vert}(\Delta_I), |W| = d} \dim_K \widetilde{H}_{d-i-1}(\{\Delta_I\}_W)$$

where $\{\Delta_I\}_W = \{F \in \Delta_I \mid F \subset W\}$. On the other hand from [2] Lemma 5.5.3 we have

$$\widetilde{H}_{d-i-1}(\{\Delta_I\}_W) \cong \widetilde{H}^{i-2}(\{\Delta_I\}_W{}^*) \cong \widetilde{H}_{i-2}(\{\Delta_I\}_W{}^*). \tag{2.1}$$

Suppose $m_1, m_2, \dots, m_r$ is a minimal monomial generating set for I and correspondingly, $\Delta(I) = \langle F_1, \dots, F_s \rangle$. We now claim $\{\Delta_I\}_W{}^* = (\Delta(I)_W)^c_W$ for $W \subset \mathrm{Vert}(\Delta_I)$.

$$\begin{aligned} F \in \{\Delta_I\}_W{}^* &\iff W \setminus F \notin \{\Delta_I\}_W \\ &\iff W \backslash F \notin \Delta_I \\ &\iff \prod_{x \in W \backslash F} x \in I = (m_1, m_2, \dots, m_r) \\ &\iff m_s | \prod_{x \in W \backslash F} x, \text{ for some } s \in \{1, \dots, r\} \\ &\iff F_s \subset W \backslash F \subset W, \text{ for some } s \in \{1, \dots, r\} \\ &\iff F \subset W \backslash F_s \in (\Delta(I)_W)^c_W, \text{ for some } s \in \{1, \dots, r\}. \end{aligned}$$

Now Hochster's formula and (2.1) imply that

$$\beta_{i,d}(R/I) = \sum_{W \subset \text{Vert}(\Delta), |W|=d} dim_K \widetilde{H}_{i-2}((\Delta(I)_W)^c_W).$$

If we assume Vert$(\Delta(I)_W) \neq W$ then clearly we have $(\Delta(I)_W)^c_W$ is a cone and by Proposition 2.7 it contributes zero to the sum. So we have

$$\beta_{i,d}(R/I) = \sum_{\Gamma \subset \Delta(I), |\text{Vert}(\Gamma)|=d} dim_K \widetilde{H}_{i-2}(\Gamma^c_{\text{Vert}(\Gamma)}).$$

where the sum is taken over the induced subcollections Γ of $\Delta(I)$ which have d vertices. $\square$

Based on Theorem 2.8, from here on all induced subcollections $\Gamma = \Delta_{\mathcal{Y}}$ of a simplicial complex Δ that we consider will have the property that $\mathcal{Y} = \text{Vert}(\Gamma)$.

3 Path ideals and runs

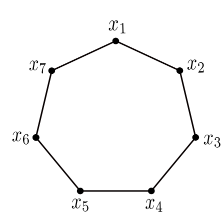
We now focus on path ideals, path complexes, and their structures.

Definition 3.1. Let $G = (\mathcal{X}, E)$ be a finite simple graph and t be an integer such that $t \geq 2$. If x and y are two vertices of G, a **path** of length $(t-1)$ from x to y is a sequence of vertices $x = x_{i_1}, \ldots, x_{i_t} = y$ of G such that $\{x_{i_j}, x_{i_{j+1}}\} \in E$ for all $j = 1, 2, \ldots, t-1$. We define the **path ideal** of G, denoted by $I_t(G)$ to be the ideal of $K[x_1, \ldots, x_n]$ generated by the monomials of the form $x_{i_1}x_{i_2}\ldots x_{i_t}$ where $x_{i_1}, x_{i_2}, \ldots, x_{i_t}$ is a path in G. The facet complex of $I_t(G)$, denoted by $\Delta_t(G)$, is called the **path complex** of the graph G.

Two special cases that we will be considering in this paper are when G is a **cycle** C_n, or a **line graph** L_n on vertices $\{x_1, \ldots, x_n\}$.

$$C_n = \langle x_1x_2, \ldots, x_{n-1}x_n, x_nx_1 \rangle \text{ and } L_n = \langle x_1x_2, \ldots, x_{n-1}x_n \rangle.$$

Example 3.2. Consider the cycle C_7 with vertex set $\mathcal{X} = \{x_1, \ldots, x_7\}$



Then we have

$I_4(C_7) = (x_1x_2x_3x_4, x_2x_3x_4x_5, x_3x_4x_5x_6, x_4x_5x_6x_7, x_1x_5x_6x_7, x_1x_2x_6x_7, x_1x_2x_3x_7)$
$D_4(C_7) = \langle \{x_1, x_2, x_3, x_4\}, \{x_2, x_3, x_4, x_5\}, \{x_3, x_4, x_5, x_6\}, \{x_4, x_5, x_6, x_7\}, \{x_1, x_5, x_6, x_7\},$
$\{x_1, x_2, x_6, x_7\}, \{x_1, x_2, x_3, x_7\} \rangle.$

Notation 3.3. Let i and n be two positive integers. For (a set of) labeled objects we use the notation $\bmod n$ to denote

$$x_i \mod n = \{x_j \mid 1 \leq j \leq n, i \equiv j \mod n\}$$

and

$$\{x_{u_1}, x_{u_2}, \ldots, x_{u_t}\} \mod n = \{x_{u_j} \mod n \mid j = 1, 2, \ldots, n\}.$$

Note 3.4. Let C_n be a cycle on vertex set $\mathcal{X} = \{x_1, \dots, x_n\}$ and $t < n$. The facets of the path complex $\Delta_t(C_n) = \langle F_1, \dots, F_n \rangle$ can be labeled as

$$F_1 = \{x_1, \dots, x_t\}, \dots, F_{n-(t-1)} = \{x_{n-(t-1)}, \dots, x_n\}, \dots, F_n = \{x_1, \dots, x_{t-1}, x_n\}$$

such that $F_i = \{x_i, x_{i+1}, \dots, x_{i+t-1}\} \mod n$ for all $1 \leq i \leq n$. This labeling is called the **standard labeling** of $\Delta_t(C_n)$.

Since for each $1 \leq i \leq n$ we have

$$F_{i+1} \setminus F_i = \{x_{t+i}\} \quad \text{and} \quad F_i \setminus F_{i+1} = \{x_i\} \qquad \mod n,$$

it follows that $|F_i \setminus F_{i+1}| = 1$ and $|F_{i+1} \setminus F_i| = 1 \quad \mod n$ for all $1 \leq i \leq n-1$.

It is clear that each induced subgraph of a graph-cycle is a disjoint union of paths. Borrowing the terminology from S. Jacques [7], we call the path complex of a line a "run", and show that every induced subcollection of the path complex of a cycle is a disjoint union of runs.

Definition 3.5. Given an integer t, we define a **run** to be the path complex of a line graph. A run which has p facets is called a **run of length** p and corresponds to $\Delta_t(L_{p+t-1})$. Therefore a run of length p has $p+t-1$ vertices.

Proposition 3.6 below shows that every proper induced subcollection of a path complex is a disjoint union of runs.

Proposition 3.6. *Let C_n be a cycle with vertex set $\mathcal{X} = \{x_1, \dots . x_n\}$ and $2 \leq t < n$. Let Γ be a proper induced connected subcollection of $\Delta_t(C_n)$ on $U \subsetneq \mathcal{X}$. Then Γ is of the form $\Delta_t(L_{|\Gamma|})$, where $L_{|\Gamma|}$ is the line graph on $|\Gamma|$ vertices.*

Proof. Suppose $\Delta_t(C_n) = \langle F_1, \dots, F_n \rangle$ has standard labeling and $\Gamma = \langle F_{i_1}, \dots, F_{i_r} \rangle$. It is clear that there exists the facet $F_a \in \Gamma$ for $1 \leq a \leq n$ such that $F_{a+1} \notin \Gamma \mod n$, because otherwise $\Gamma = \Delta_t(C_n)$. Therefore from Note 3.4 we have

$$\{x_a, x_{a+1}, \dots, x_{a+t-1}\} \subset U \quad \text{and} \quad x_{a+t} \notin U. \tag{3.1}$$

Let r be the largest non-negative integer such that $x_{a-i} \in U \mod n$ for $0 \leq i \leq r$ so that

$$\underbrace{x_{a-r-1}}_{\notin U}, \underbrace{x_{a-r}, \dots, x_{a-1}, x_a, x_{a+1}, \dots, x_{a+t-1}}_{\in U}, \underbrace{x_{a+t}}_{\notin U}. \tag{3.2}$$

It follows that since Γ is an induced subcollection of $\Delta_t(C_n)$ on U, $F_{a-r}, F_{a-r+1}, \dots, F_a \in \Gamma$. We now show that

$$F_i \notin \Gamma \text{ for all } i \notin \{a-r, a-r+1, \dots, a\} \quad \mod n.$$

This follows from the fact that Γ is connected: if any F_i (except for $a-r \leq i \leq a \mod n$) intersects some of the facets $F_{a-r}, \dots, F_a \mod n$, then it must contain x_{a-r-1} or x_{a+t} (as otherwise it would be one of $F_{a-r}, \dots, F_a \mod n$), and hence $F_i \notin \Gamma$.

We have therefore shown that

$$\Gamma = \langle F_{a-r}, F_{a-r+1}, \dots, F_a \rangle \quad \mod n.$$

Next we prove $\Gamma = \Delta_t(L_{|U|})$. Without loss of generality we can assume that $a - r = 1$, so that $\Gamma = \langle F_1, \dots, F_{r+1} \rangle$. Since $\Gamma \neq \Delta_t(C_n)$ we can say that $r+1 < n$ and therefore we have

$$\text{Vert}(\Gamma) = \{x_1, x_2, \dots, x_{r+t}\}.$$

Since Γ is induced and proper we have $r + t < n$, and therefore we can conclude that

$$\Gamma = \Delta_t(L_{\{x_1,x_2,\ldots,x_{t+r}\}}).$$

$\square$

Example 3.7. Consider the cycle C_7 on vertex set $\mathcal{X} = \{x_1, \ldots x_7\}$ and the simplicial complex $\Delta_4(C_7)$. The following induced subcollections are two runs in $\Delta_4(C_7)$

$$\begin{array}{rcl} \Delta_1 & = & \langle \{x_1, x_2, x_3, x_4\}, \{x_2, x_3, x_4, x_5\} \rangle \\ \Delta_2 & = & \langle \{x_1, x_2, x_6, x_7\}, \{x_1, x_2, x_3, x_7\}, \{x_1, x_2, x_3, x_4\} \rangle. \end{array}$$

Lemma 3.8. *Let Γ and Λ be two induced subcollections of $\Delta_t(C_n) = \langle F_1, F_2, \ldots, F_n \rangle$ which are both composed of runs of lengths $s_1, \ldots, s_r$. Then Γ and Λ are homeomorphic as simplicial complexes. In particular the two simplicial complexes Γ^c and Λ^c are homeomorphic and have the same reduced homologies.*

Proof. First we suppose $\Delta_t(C_n) = \langle F_1, F_2, \ldots, F_n \rangle$ has standard labeling. If we denote each run of length s_i in Γ and Λ by R_i and R'_i, respectively, we have

$$\Gamma = \langle R_1, R_2, \ldots, R_r \rangle \text{ and } \Lambda = \langle {R_1}', {R_2}', \ldots, {R_r}' \rangle$$

where, using the standard labeling, for $1 \leq j_i, h_i \leq n$ we have

$$R_i = \langle F_{j_i}, F_{j_i+1}, \ldots, F_{j_i+s_i-1} \rangle \text{ and } {R_i}' = \langle F_{h_i}, F_{h_i+1}, \ldots, F_{h_i+s_i-1} \rangle \mod n.$$

Then clearly

$$\text{Vert}(\Gamma) = \bigcup_{i=1}^{r} \text{Vert}(R_i) = \bigcup_{i=1}^{r} \{x_{j_i}, x_{j_i+1}, \ldots, x_{j_i+s_i+t-2}\}$$

and

$$\text{Vert}(\Lambda) = \bigcup_{i=1}^{r} \text{Vert}({R_i}') = \bigcup_{i=1}^{r} \{x_{h_i}, x_{h_i+1}, \ldots, x_{h_i+s_i+t-2}\}.$$

Now we define the function $\varphi : \text{Vert}(\Gamma) \longrightarrow \text{Vert}(\Lambda)$ where

$$\varphi(x_{j_i+u}) = x_{h_i+u} \text{ for } 0 \leq u \leq s_i + t - 2.$$

Since φ is a bijective map between vertex set Γ and Λ which preserves faces, we can conclude Γ and Λ are homeomorphic. Therefore, two simplicial complexes Γ^c and Λ^c are homeomorphic as well and have the same reduced homology. $\square$

Therefore, by applying Theorem 2.8 and Lemma 3.8 all the information we need to compute the homologies of induced subcollections of $\Delta_t(C_n)$ depends on the number and the lengths of the runs.

Definition 3.9. For a fixed integer $t \geq 2$, let the pure $(t-1)$-dimensional simplicial complex $\Gamma = \langle F_1, \ldots, F_s \rangle$ be a disjoint union of runs of length $s_1, \ldots, s_r$. Then the sequence of positive integers $s_1, \ldots, s_r$ is called a **run sequence** on $\mathcal{Y} = \text{Vert}(\Gamma)$, and we use the notation

$$E(s_1, \ldots, s_r) = \Gamma^c_{\mathcal{Y}} = \langle (F_1)^c_{\mathcal{Y}}, \ldots, (F_s)^c_{\mathcal{Y}} \rangle.$$

4 Reduced homologies for Betti numbers

Let $I = I_t(C_n)$ be the path ideal of the the cycle C_n for some $t \geq 2$. By applying Hochster's formula (Theorem 2.8), we see that to compute the Betti numbers of R/I, we need to compute the reduced homologies of complements of induced subcollections of Δ. This section is devoted to complex homological calculations. The results here will allow us to compute all Betti numbers of R/I (and more) in the sections that follow. We begin by recalling the Mayer-Vietoris sequence; for more details see for example Hatcher [5] Chapter 2.

Suppose Δ is a simplicial complex and Δ_1 and Δ_2 are two subcollections such that $\Delta = \Delta_1 \cup \Delta_2$. We have the exact sequence of the chain complexes

$$0 \to \mathcal{C}.(\Delta_1 \cap \Delta_2) \to \mathcal{C}.(\Delta_1) \oplus \mathcal{C}.(\Delta_2) \to \mathcal{C}.(\Delta) \to 0$$

which produces the following long exact sequence, called the **Mayer-Vietoris** sequence

$$\cdots \to \widetilde{H}_i(\Delta_1 \cap \Delta_2) \to \widetilde{H}_i(\Delta_1) \oplus \widetilde{H}_i(\Delta_2) \to \widetilde{H}_i(\Delta) \to \widetilde{H}_{i-1}(\Delta_1 \cap \Delta_2) \to \dots .$$

We make a basic observation.

Lemma 4.1. *Let $E_1, \dots, E_s$ be subsets of the finite set V where $s \geq 2$ and suppose that $\mathcal{E} = \langle (E_1)^c_V, (E_2)^c_V, ..., (E_s)^c_V \rangle$. Then for any i we have*

i. Suppose $V \setminus \bigcup_{j=2}^{s} E_j \neq \emptyset$. If $\mathcal{E}_1 = \langle (E_1)^c_V \rangle$ and $\mathcal{E}_2 = \langle (E_2)^c_V, ..., (E_s)^c_V \rangle$ then

$$\begin{aligned}\widetilde{H}_i(\mathcal{E}) = \widetilde{H}_i(\mathcal{E}_1 \cup \mathcal{E}_2) &\cong \widetilde{H}_{i-1}(\mathcal{E}_1 \cap \mathcal{E}_2)\\ &= \widetilde{H}_{i-1}(\langle (E_1 \cup E_2)^c_V, ..., (E_1 \cup E_s)^c_V \rangle)\\ &= \widetilde{H}_{i-1}(\langle (E_2)^c_{(V \setminus E_1)}, ..., (E_s)^c_{(V \setminus E_1)} \rangle).\end{aligned}$$

ii. If $E_a \subset E_b$ for some $a \neq b$, then $\mathcal{E} = \langle (E_1)^c_v, \dots, (\widehat{E_b})^c_V, \dots, (E_s)^c_V \rangle$.

The decomposition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ described above is called **standard decomposition** of $\mathcal{E}$.

Proof. The proof of ii is trivial so we shall only prove i. Since $\mathcal{E}_1$ is a simplex, we have $\widetilde{H}_i(\mathcal{E}_1) = 0$. Also since $V \setminus \bigcup_{i=2}^{s} E_i \neq \emptyset$ we have $\mathcal{E}_2$ is a cone, so from Proposition 2.7 we conclude

$$\widetilde{H}_i(\mathcal{E}_2) = 0 \text{ for all } i.$$

Now $\mathcal{E}_1 \cap \mathcal{E}_2 = \langle (E_1 \cup E_2)^c_V, ..., (E_1 \cup E_s)^c_V \rangle$, so by applying the Mayer-Vietoris sequence we reach the following exact sequence

$$\cdots \longrightarrow \widetilde{H}_i(\mathcal{E}_1) \oplus \widetilde{H}_i(\mathcal{E}_2) \longrightarrow \widetilde{H}_i(\mathcal{E}) \longrightarrow \widetilde{H}_{i-1}(\mathcal{E}_1 \cap \mathcal{E}_2) \longrightarrow \widetilde{H}_{i-1}(\mathcal{E}_1) \oplus \widetilde{H}_{i-1}(\mathcal{E}_2) \longrightarrow \cdots$$

which implies that

$$\begin{aligned}\widetilde{H}_i(\mathcal{E}_1 \cup \mathcal{E}_2) \cong &\widetilde{H}_{i-1}(\mathcal{E}_1 \cap \mathcal{E}_2)\\ = &\widetilde{H}_{i-1}(\langle (E_1 \cup E_2)^c_V, ..., (E_1 \cup E_s)^c_V \rangle).\end{aligned}$$

□

Proposition 4.2. *Let $\Gamma = \langle E_1, \dots, E_s \rangle$ be a pure simplicial complex of dimension $t-1$ over the vertex set $V = \{x_1, \dots, x_n\}$ where $2 \le t \le n$. Suppose the connected components of Γ are runs of lengths $s_1, \dots, s_r$, and $\mathcal{E} = E(s_1, \dots, s_r)$. Let $s_j = (t+1)p_j + d_j$ where $p_j \ge 0$ and $0 \le d_j \le t$ and $1 \le j \le r$. Then for all i, we have*

$i.$ *If $s_j \ge t+2$ then* $\widetilde{H}_i(\mathcal{E})) \cong \widetilde{H}_{i-2}(E(s_1, \dots, s_j - (t+1), \dots, s_r))$;

$ii.$ *If $d_j \neq 1, 2$ then* $\widetilde{H}_i(\mathcal{E}) = 0$;

$iii.$ *If $s_j = 2$ and $r \ge 2$ then* $\widetilde{H}_i(\mathcal{E}) = \widetilde{H}_{i-2}(E(s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_r))$;

$iv.$ *If $s_j = 1$ and $r \ge 2$ then* $\widetilde{H}_i(\mathcal{E}) = \widetilde{H}_{i-1}(E(s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_r))$.

Proof. We assume without loss of generality that $E_1, \dots, E_s$ are ordered such that $E_1, \dots, E_{s_j}$ are the facets of the run of length s_j, and they have standard labeling

$$E_1 = \{x_1, x_2, \dots, x_t\}, E_2 = \{x_2, x_3, \dots, x_{t+1}\}, \dots, E_{s_j} = \{x_{s_j}, x_{s_j+1}, \dots, x_{s_j+t-1}\} \mod n.$$

We have $\mathcal{E} = \langle (E_1)^c_V, (E_2)^c_V, ..., (E_s)^c_V \rangle$. Since $x_1 \in V \setminus \bigcup_{i=2}^s E_i$ there is a standard decomposition

$$\mathcal{E} = \langle (E_1)^c_V \rangle \cup \langle (E_2)^c_V, \dots, (E_s)^c_V \rangle.$$

From Lemma 4.1 (i), setting $V' = V \setminus \{x_1, x_2, \dots, x_t\}$, we have

$$\widetilde{H}_i(\mathcal{E}) \cong \widetilde{H}_{i-1}(\langle (E_2)^c_{V'}, \dots, (E_s)^c_{V'} \rangle). \tag{4.1}$$

If $s_j \ge t+2$ from (4.1) we have

$$\widetilde{H}_i(\mathcal{E})) = \widetilde{H}_{i-1}(\langle \{x_{t+1}\}^c_{V'}, (E_{t+2})^c_{V'}, \dots, (E_s)^c_{V'} \rangle) \tag{4.2}$$

and since the following is a standard decomposition

$$\langle \{x_{t+1}\}^c_{V'} \rangle \cup \langle (E_{t+2})^c_{V'}, \dots, (E_{s_j})^c_{V'}, \dots, (E_s)^c_{V'} \rangle$$

from (4.2), Lemma 4.1 (i) and by setting $V'' = V \setminus \{x_1, x_2, \dots, x_{t+1}\}$, we have

$$\widetilde{H}_i(\mathcal{E}) \cong \widetilde{H}_{i-2}(\langle (E_{t+2})^c_{V''}, \dots, (E_{s_j})^c_{V''}, \dots, (E_s)^c_{V''} \rangle).$$

Now the connected components of $\langle E_{t+2}, \dots, E_s \rangle$ are runs of lengths $s_1, \dots, s_j - (t+1), \dots, s_r$, and and therefore we can conclude that for all i

$$\widetilde{H}_i(\mathcal{E}) = \widetilde{H}_{i-2}(E(s_1, \dots, s_j - (t+1), \dots, s_r)).$$

This settles Case (i) of the proposition. Now suppose $1 \le s_j < t+2$. In this case by (4.1) and Lemma 4.1 (i) and (ii) we see that

$$\widetilde{H}_i(\mathcal{E}) \cong \widetilde{H}_{i-1}(\langle \{x_{t+1}\}^c_{V'}, (E_{s_j+1})^c_{V'}, \dots, (E_s)^c_{V'} \rangle) \quad \text{for all } i. \tag{4.3}$$

1. If $s_j \ge 3$ since $x_{s_j+t-1} \in V' \setminus (\bigcup_{i=s_j+1}^s E_i \cup \{x_{t+1}\})$ the simplicial complex

$$\langle \{x_{t+1}\}^c_{V'}, (E_{s_j+1})^c_{V'}, \dots, (E_s)^c_{V'} \rangle$$

is a cone and by Proposition 2.7 and (4.3) we have $\widetilde{H}_i(\mathcal{E}) = 0$ for all i.

2. If $s_j = 2$ and $r \geq 2$, since $x_{t+1} \in V' \setminus (\bigcup_{i=s_j+1}^{s} E_i)$ we have

$$\langle \{x_{t+1}\}_{V'}^c \rangle \cup \langle (E_{s_j+1})_{V'}^c, \ldots, (E_s)_{V'}^c \rangle$$

   is a standard decomposition and then by Lemma 4.1 and (4.3) we have

$$\begin{aligned} \widetilde{H}_i(\mathcal{E}) &\cong \widetilde{H}_{i-2}(\langle (E_{s_j+1})_{V''}^c, \ldots, (E_s)_{V''}^c \rangle) \\ &= \widetilde{H}_{i-2}(E(s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_r)) \quad \text{for all } i. \end{aligned}$$

   This settles Case (iii).

3. If $s_j = 1$ and $r \geq 2$ since $E_1 \cap E_h = \emptyset$ for $1 < h \leq s$, and from (4.1) we have

$$\widetilde{H}_i(\mathcal{E}) \cong \widetilde{H}_{i-1}(E(s_1, \ldots s_{j-1}, s_{j+1}, \ldots, s_r)) \text{ for all } i.$$

   This settles Case (iv).

To prove (ii), we use induction on p_j. If $p_j = 0$, then $d_j = s_j \geq 1$. From above we know that $\widetilde{H}_i(\mathcal{E}) = 0$ if $3 \leq s_j \leq t$, and we are done. Now suppose $p_j \geq 1$ and the statement holds for all values less than p_j. We have two cases:

1. If $s_j < t+2$, then since $p_j \geq 1$, we must have $p_j = 1$, $d_j = 0$, and $s_j = t+1$. It was proved above (under the case $3 \leq s_j < t+2$) that $\widetilde{H}_i(\mathcal{E}) = 0$.

2. If $s_j \geq t+2$, by (i) we have

$$\widetilde{H}_i(\mathcal{E}) \cong \widetilde{H}_{i-2}(E(s_1, \ldots, (t+1)(p_j - 1) + d_j, \ldots, s_r)) = 0 \text{ when } d_j \neq 1, 2.$$

This proves (ii) and we are done. □

We conclude that for computing the homology of the induced subcollections of path complexes of cycles or lines, the only cases which have to be considered are those of runs of length one or two. We now set about computing these.

Proposition 4.3. *Let t and n be integers, such that $2 \leq t \leq n$. Let $\alpha, \beta \geq 0$ and consider*

$$\mathcal{E} = E((t+1)p_1 + 1, \ldots, (t+1)p_\alpha + 1, (t+1)q_1 + 2, \ldots, (t+1)q_\beta + 2)$$

for nonnegative integers $p_1 \ldots, p_\alpha, q_1, \ldots, q_\beta$. Then

$$\widetilde{H}_i(\mathcal{E}) = \begin{cases} K & i = 2(P+Q) + 2\beta + \alpha - 2 \\ 0 & \textit{otherwise} \end{cases}$$

where $P = \sum_{i=1}^{\alpha} p_i$ and $Q = \sum_{i=1}^{\beta} q_i$.

From here on, we use the notation $E(1^\alpha, 2^\beta)$ to denote the complex $\mathcal{E}$ described in the statement of Proposition 4.3 in the case where all the p's and q's are zero; i.e. the case of α runs of length one and β runs of length two.

Proof. First we prove the two cases $\alpha = 0$, $\beta = 1$ and $\alpha = 1$, $\beta = 0$.

1. If $\alpha = 1$, $\beta = 0$, then $\mathcal{E} \cong \langle V \setminus \{x_1, x_2, \dots, x_t\}\rangle = \{\emptyset\}$ where $V = \{x_1, \dots, x_t\}$, and therefore
$$\widetilde{H}_i(\mathcal{E}) = \begin{cases} K & i = -1 \\ 0 & \text{otherwise.} \end{cases}$$
2. If $\alpha = 0$, $\beta = 1$, then $\mathcal{E} \cong \langle (\{x_1, x_2, \dots, x_t\})^c_V, (\{x_2, \dots, x_{t+1}\})^c_V\rangle = \langle \{x_{t+1}\}, \{x_1\}\rangle$ where $V = \{x_1, \dots, x_{t+1}\}$. Since $\mathcal{E}$ is disconnected, we have
$$\widetilde{H}_i(\mathcal{E}) = \begin{cases} K & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

To prove the statement of the proposition, we use repeated applications of Proposition 4.2 (i), p_1 times to the first run, p_2 times to the second run, and so on till q_β times to the last run as follows.

$$\begin{aligned}
\widetilde{H}_i(\mathcal{E}) &= \widetilde{H}_i(E((t+1)p_1+1, \dots, (t+1)p_\alpha+1, (t+1)q_1+2, \dots, (t+1)q_\beta+2)) \\
&\cong \widetilde{H}_{i-2}(E((t+1)(p_1-1)+1, \dots, (t+1)p_\alpha+1, (t+1)q_1+2, \dots, (t+1)q_\beta+2)) \\
&\ \ \vdots \\
&\cong \widetilde{H}_{i-2(P+Q)}(E(1^\alpha, 2^\beta)) && \text{apply Proposition 4.2 (iv)} \\
&\cong \widetilde{H}_{i-2(P+Q)-\alpha}(E(2^\beta)) && \text{apply Proposition 4.2 (iii)} \\
&\cong \widetilde{H}_{i-2(P+Q)-\alpha-2\beta+2}(E(2)) && \text{apply Case 2 above} \\
&= \begin{cases} K & i = 2(P+Q)+\alpha+2\beta-2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

□

An immediate consequence of the above calculations is the homology of the complement of a run, or equivalently, the path complex of any line graph.

Corollary 4.4. *Let t, p and d be integers such that $t \geq 2$, $p \geq 0$,and $0 \leq d \leq t$. Then*
$$\widetilde{H}_i(E((t+1)p+d)) = \begin{cases} K & d = 1,\ i = 2p-1 \\ K & d = 2,\ i = 2p \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. By Proposition 4.2 (ii), if $d \neq 1, 2$ the homology is zero. In the cases where $d = 1, 2$, the result follows directly from Proposition 4.3. □

We end this section with the calculation of the homology of the complement of the whole path complex of a cycle; this will give us the top degree Betti numbers of the path ideal of a cycle. We will first need a technical lemma.

Lemma 4.5. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K, and suppose $\Delta_t(C_n) = \langle F_1, F_2, \dots, F_n\rangle$ is the path complex of a cycle C_n with standard labeling. Let $a, k, s, t \in \{1, \dots, n\}$ be such that $k < t$, and $a+s+t-1 < n$. Suppose $s = (t+1)p+d$ where $p \geq 0$ and $0 \leq d < t+1$. Set $V = \{x_a, x_{a+1}, \dots, x_{a+s+t-1}\}$ and*
$$\mathcal{E} = \langle (F_a)^c_V, \dots, (F_{a+s-1})^c_V, \{x_{a+s+t-k}, x_{a+s+t-k+1}, \dots, x_{a+s+t-1}\}^c_V\rangle.$$
Then for all i we have
$$\widetilde{H}_i(\mathcal{E}) = \begin{cases} K & d = 1, i = 2p \\ K & d = k+1, i = 2p+1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Without loss of generality we can assume $a = 1$ so that $V = \{x_1, \dots, x_{s+t}\}$ and

$$\mathcal{E} = \langle (F_1)^c_V, \dots, (F_s)^c_V, \{x_{s+t-k+1}, \dots, x_{s+t}\}^c_V \rangle.$$

Since $x_{s+t} \notin F_h$ for $1 \le h \le s$, $\mathcal{E}$ has standard decomposition

$$\mathcal{E} = \langle (F_1)^c_V, (F_2)^c_V, \dots, (F_s)^c_V \rangle \cup \langle (\{x_{s+t-k+1}, x_{s+t-k+2}, \dots, x_{s+t}\})^c_V \rangle$$

and then from Lemma 4.1 (i) and (ii), setting $V_1 = V \setminus \{x_{s+t-k+1}, x_{s+t-k+2}, \dots, x_{s+t}\}$, we have

$$\begin{aligned}
\widetilde{H}_i(\mathcal{E}) \cong & \widetilde{H}_{i-1} \langle (F_1)^c_{V_1}, (F_2)^c_{V_1}, \dots, (F_{s-k})^c_{V_1}, \{x_{s-k+1}, \dots, x_{s+t-k}\}^c_{V_1}, \{x_{s-k+2}, \dots, x_{s+t-k}\}^c_{V_1}, \\
& \quad , \dots, \{x_{s-1}, \dots, x_{s+t-k}\}^c_{V_1}, \{x_s, \dots, x_{s+t-k}\}^c_{V_1} \rangle) \\
= & \widetilde{H}_{i-1}(\langle (F_1)^c_{V_1}, (F_2)^c_{V_1}, \dots, (F_{s-k})^c_{V_1}, \{x_s, \dots, x_{s+t-k}\}^c_{V_1} \rangle) \qquad (4.4)
\end{aligned}$$

We prove our statement by induction on $|V| = s+t = (t+1)p+d+t$. The base case is $|V| = d+t$, in which case $p = 0$ and $d = s \ge 1$. There are two cases to consider.

1. If $1 \le d \le k$, then $s \le k$, and so by (4.4)

   $$\widetilde{H}_i(\mathcal{E}) = \widetilde{H}_{i-1}(\langle \{x_s, \dots, x_{s+t-k}\}^c_{V_1} \rangle).$$

   The simplex $\{x_s, \dots, x_{s+t-k}\}^c_{V_1}$ is not empty unless $s = d = 1$, and hence we have

   $$\widetilde{H}_i(\mathcal{E}) = \begin{cases} K & d = 1, i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2. If $d > k$, we use (4.4) to note that since $x_{s-k+t} \notin F_1 \cup \ldots \cup F_{s-k}$, the following is a standard decomposition

   $$\langle (F_1)^c_{V_1}, (F_2)^c_{V_1}, \dots, (F_{s-k})^c_{V_1}, \{x_s, \dots, x_{s+t-k}\}^c_{V_1} \rangle.$$

   Using Lemma 4.1 and (4.4) along with the fact that $s = d \le t$, we find that if $V_2 = V \setminus \{x_s, \dots, x_{s+t}\}$, then

   $$\begin{aligned}
   \widetilde{H}_i(\mathcal{E}) & \cong \widetilde{H}_{i-2}(\langle \{x_1, \dots, x_{s-1}\}^c_{V_2}, \{x_2, \dots, x_{s-1}\}^c_{V_2}, \dots, \{x_{s-k}, \dots, x_{s-1}\}^c_{V_2} \rangle) \\
   & \cong \widetilde{H}_{i-2}(\langle \{x_{s-k}, \dots, x_{s-1}\}^c_{V_2} \rangle).
   \end{aligned}$$

   Now the simplex $\{x_{s-k}, \dots, x_{s-1}\}^c_{V_2}$ is nonempty unless $s - k = 1$, or in other words, $d = s = k+1$. Therefore

   $$\widetilde{H}_i(\mathcal{E}) = \begin{cases} K & d = k+1, i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This settles the base case of the induction. Now suppose $|V| = s + t > d + t$ and the theorem holds for all the cases where $|V| < s + t$. Since $|V_1| = (s - k) + t < |V|$ we shall apply (4.4) and use the induction hypothesis on V_1, now with the following parameters: $k_1 = t - k + 1$, $s_1 = s - k = (t + 1)p + d - k$ and

$$d_1 = \begin{cases} d - k & d \ge k \\ d - k + t + 1 & d < k \end{cases} \quad \text{and} \quad p_1 = \begin{cases} p & d \ge k \\ p - 1 & d < k. \end{cases}$$

Applying the induction hypothesis on V_1 we see that $\widetilde{H}_i(\mathcal{E}) = 0$ unless one of the following scenarios happen, in which case $\widetilde{H}_i(\mathcal{E}) = K$.

1. $d_1 = 1$ and $i - 1 = 2p_1$.

   (a) When $d \geq k$, this means that $d = k + 1$ and $i = 2p + 1$.

   (b) When $d < k$, this means $d - k + t + 1 = 1$ which implies that $0 \leq d = k - t \leq 0$, and hence $d = 0$ and $t = k$, which is not possible as we have assumed $k < t$.

2. $d_1 = k_1 + 1$ and $i - 1 = 2p_1 + 1$.

   (a) When $d \geq k$, this means that $d - k = t - k + 1 + 1$ and so $d = t + 2$ which is not possible, as we have assumed $d < t + 1$.

   (b) When $d < k$, this means that $d - k + t + 1 = t - k + 1 + 1$ and so $d = 1$ and $i = 2p_1 + 2 = 2p$.

We conclude that $\widetilde{H}_i(\mathcal{E}) = K$ only when $d = 1$ and $i = 2p$, or $d = k + 1$ and $i = 2p + 1$, and $\widetilde{H}_i(\mathcal{E}) = 0$ otherwise. □

Theorem 4.6. *Let $2 \leq t \leq n$ and $\Delta = \Delta_t(C_n)$ be the path complex of a cycle C_n with vertex set $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$. Suppose $n = (t+1)p + d$ where $p \geq 0$, $0 \leq d \leq t$. Then for all i*

$$\widetilde{H}_i(\Delta^c_{\mathcal{X}}) = \begin{cases} K^t & d = 0,\ i = 2p-2,\ p > 0 \\ K & d \neq 0,\ i = 2p-1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $p = 0$, then $n = t$ and our claim is obvious, so we assume that $p \geq 1$ and therefore $n \geq t + 1$. We define the following simplicial complexes

$$\begin{aligned} E_0 &= \langle (F_1)^c_{\mathcal{X}}, (F_2)^c_{\mathcal{X}}, \ldots, (F_{n-t+1})^c_{\mathcal{X}} \rangle = E(n-t+1) \\ E_k &= E_{k-1} \cup \langle (F_{n-k+1})^c_{\mathcal{X}} \rangle \text{ for } k = 1, 2, \ldots, t-1. \end{aligned} \tag{4.5}$$

Note that $\Delta^c_{\mathcal{X}} = E_{t-1}$. We start with E_0 and apply the Mayer-Vietoris sequence repeatedly to calculate the homologies of the E_k. Since $E_0 = E(n-t+1)$, we find

$$n - t + 1 = (t+1)p + d - t + 1 = \begin{cases} (t+1)p + 1 & d = t \\ (t+1)p & d = t-1 \\ (t+1)(p-1) + d + 2 & d < t-1 \end{cases}$$

which by Corollary 4.4 implies that

$$\widetilde{H}_i(E_0) = \begin{cases} K & d = 0,\ i = 2p-2 \\ K & d = t,\ i = 2p-1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

In order to to find the homologies of E_{t-1} we shall recursively apply the Mayer-Vietoris sequence as follows. For a fixed $1 \leq k \leq t-1$ we have the following exact sequence,

$$\widetilde{H}_i(E_{k-1} \cap \langle (F_{n-k+1})^c_{\mathcal{X}} \rangle) \to \widetilde{H}_i(E_{k-1}) \to \widetilde{H}_i(E_k) \to \widetilde{H}_{i-1}(E_{k-1} \cap \langle (F_{n-k+1})^c_{\mathcal{X}} \rangle). \tag{4.7}$$

We claim that for $0 \leq k \leq t-2$,

$$\widetilde{H}_i(E_k \cap \langle (F_{n-k})^c_{\mathcal{X}} \rangle) = \begin{cases} K & d = 0,\ i = 2p-3 \\ K & d = t-k-1,\ i = 2p-2 \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Setting $\mathcal{X}' = \mathcal{X} \setminus F_{n-k} = \{x_{t-k}, \ldots, x_{n-k-1}\}$ we can write

$$ {}_k \cap \langle (F_{n-k})^c_{\mathcal{X}} \rangle = \langle (F_1)^c_{\mathcal{X}'}, \ldots, (F_{n-t+1})^c_{\mathcal{X}'}, (F_{n-k+1})^c_{\mathcal{X}'}, \ldots, (F_n)^c_{\mathcal{X}'} \rangle. \tag{4.9} $$

We now compute the $(F_h)^c_{\mathcal{X}'} = \{x_h, \ldots, x_{h+(t-1)}\}^c_{\mathcal{X}'}$ appearing in 4.9).

- When $1 \leq h \leq t-k$ it is clear that
$$ (F_h)^c_{\mathcal{X}'} = \{x_{t-k}, x_{t-k+1}, \ldots, x_{t+h-1}\}^c_{\mathcal{X}'}. $$
- When $t-k+1 \leq h \leq n-t-k-1$ then $2t-k \leq h+t-1 \leq n-k-2$, and so
$$ (F_h)^c_{\mathcal{X}'} = \{x_{t-k}, \ldots, x_{h-1}, x_{h+t}, \ldots, x_{n-k-1}\}. $$
- When $n-k-t \leq h \leq n-t+1$, then $n-k-1 \leq h+t-1 \leq n$, and therefore
$$ (F_h)^c_{\mathcal{X}'} = \{x_h, \ldots, x_{n-k-1}\}^c_{\mathcal{X}'}. $$
- When $0 \leq j \leq k-1$, then $t-k \leq -j+(t-1) \leq t-1$ and so we have
$$ F_{n-j} = \{x_{n-j}, \ldots, x_{n-j+(t-1)}\} = \{x_{n-j}, \ldots, x_n, x_1, \ldots, x_{t-j-1}\} \mod n $$
which implies that
$$ (F_{n-j})^c_{\mathcal{X}'} = \{x_{t-k}, x_{t-k+1}, \ldots, x_{t-j-1}\}^c_{\mathcal{X}'}. $$

From the observations above, (4.9) and Lemma 4.1 (ii) we see that

$$ E_k \cap \langle (F_{n-k})^c_{\mathcal{X}} \rangle = \langle \{x_{t-k}\}, F_{t-k+1}, \ldots, F_{n-t-k-1}, \{x_{n-t+1}, \ldots, x_{n-k-1}\} \rangle^c_{\mathcal{X}'}. \tag{4.10} $$

We now consider the following scenarios.

1. Suppose $p = 1$. In this situation, $n = t+d+1 \leq 2t+1$ which implies that $n-t-k-1 \leq t-k$. Therefore, (4.10) becomes
$$ E_k \cap \langle (F_{n-k})^c_{\mathcal{X}} \rangle = \langle \{x_{t-k}\}, \{x_{n-t+1}, \ldots, x_{n-k-1}\} \rangle^c_{\mathcal{X}'}. \tag{4.11} $$
   (a) If $d \leq t-k-2$, then $n-t+1 = t+d+1-t+1 = d+2 \leq t-k$. As well, since $n \geq t+1$, we have $n-k-1 \geq t-k$. It follows that in this situation, $x_{t-k} \in \{x_{n-t+1}, \ldots, x_{n-k-1}\}$ which means that (4.11) becomes $E_k \cap \langle (F_{n-k})^c_{\mathcal{X}} \rangle = \langle \{x_{t-k}\}^c_{\mathcal{X}'} \rangle$. Also note that $\mathcal{X}' = \{x_{t-k}\}$ only when $d = 0$. It follows that
$$ \widetilde{H}_i(E_k \cap \langle (F_{n-k})^c \rangle) \cong \begin{cases} K & d = 0,\ i = -1 \\ 0 & \text{otherwise.} \end{cases} $$
   (b) If $d > t-k-2$. In this situation, $x_{t-k} \notin \{x_{n-t+1}, \ldots, x_{n-k-1}\}$ which means that we can apply Lemma 4.1 (i), with $\mathcal{X}'' = \mathcal{X}' \setminus \{x_{t-k}\}$ to find that for all i
$$ \widetilde{H}_i(E_k \cap \langle (F_{n-k})^c \rangle) = \widetilde{H}_{i-1}(\langle \{x_{n-t+1}, \ldots, x_{n-k-1}\} \rangle^c_{\mathcal{X}''}). $$
   Moreover, $\{x_{n-t+1}, \ldots, x_{n-k-1}\} = \mathcal{X}''$ only when $d = t-k-1$, and so we have
$$ \widetilde{H}_i(E_k \cap \langle (F_{n-k})^c \rangle) \cong \begin{cases} K & d = t-k-1,\ i = 0 \\ 0 & \text{otherwise.} \end{cases} $$

2. Suppose $p \geq 2$. In this case it is easy to see that $n-t-k-1 > t-k$ and $n-t+1 > t-k$. Therefore, we can apply Lemma 4.1 (i) with $\mathcal{X}'' = \mathcal{X}' \setminus \{x_{t-k}\}$ to (4.10) to conclude that for all i

$$\widetilde{H}_i(E_k \cap \langle (F_{n-k})^c_{\mathcal{X}} \rangle) = \widetilde{H}_{i-1}(\langle F_{t-k+1}, \ldots, F_{n-t-k-1}, \{x_{n-t+1}, \ldots, x_{n-k-1}\} \rangle^c_{\mathcal{X}''}).$$

Now we use Lemma 4.5 with values $a = t-k+1$ and $s = n-2t-1 = (p-2)(t+1)+d+1$ to conclude that

$$\widetilde{H}_i(E_k \cap \langle (F_{n-k})^c_{\mathcal{X}} \rangle) = \begin{cases} K & d = 0,\ i = 2p-3 \\ K & d = t-k-1,\ i = 2p-2 \\ 0 & \text{otherwise} \end{cases}$$

This settles (4.8). We now return to finding $\widetilde{H}_i(E_{t-1})$ by recursively using the Mayer-Vietoris sequence to find $\widetilde{H}_i(E_k)$.

1. If $0 < d < t$ then by (4.8) we know that $\widetilde{H}_i(E_{k-1} \cap \langle (F_{n-k+1})^c_{\mathcal{X}} \rangle)$ is nonzero only when $i = 2p-2$ and $k = t-d$. We apply this observation and (4.6) to the exact sequence (4.7) to see that

$$\widetilde{H}_i(E_k) = \widetilde{H}_i(E_{k-1}) = \widetilde{H}_i(E_0) = 0 \text{ for } 1 \leq k \leq t-d-1.$$

Once again we use (4.7) to observe that

$$\begin{aligned} \widetilde{H}_i(E_k) &= \begin{cases} 0 & 1 \leq k \leq t-d-1 \\ \widetilde{H}_{i-1}(E_{k-1} \cap \langle (F_{n-k+1})^c \rangle) & k = t-d \\ \widetilde{H}_i(E_{t-d}) & t-d < k \leq t-1. \end{cases} \\ &= \begin{cases} K & k \geq t-d,\ i = 2p-1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We can conclude that in this case

$$\widetilde{H}_i(E_{t-1}) = \begin{cases} K & i = 2p-1 \\ 0 & \text{otherwise.} \end{cases}$$

2. If $d = t$ then by (4.8) we know that $\widetilde{H}_i(E_{k-1} \cap \langle (F_{n-k+1})^c_{\mathcal{X}} \rangle)$ is always zero. We apply this fact along with (4.6) to the sequence in (4.7) to observe that

$$\widetilde{H}_i(E_k) \cong \widetilde{H}_i(E_0) = \begin{cases} K & i = 2p-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } k \in \{1, 2, \ldots, t-1\}.$$

3. If $d = 0$ then by (4.8) we know that $\widetilde{H}_i(E_{k-1} \cap \langle (F_{n-k+1})^c_{\mathcal{X}} \rangle)$ is zero unless $i = 2p-3$, and by (4.6) we know $\widetilde{H}_i(E_0)$ is zero unless $i = 2p-2$. Applying these facts to (4.7) we see that

$$\widetilde{H}_i(E_k) = \widetilde{H}_i(E_0) = 0 \quad \text{for } i \neq 2p-2.$$

When $i = 2p-2$, the sequence (4.7) produces an exact sequence

$$0 \longrightarrow \overbrace{\widetilde{H}_{2p-2}(E_0)}^{K} \longrightarrow \widetilde{H}_{2p-2}(E_1) \longrightarrow \overbrace{\widetilde{H}_{2p-3}(E_0 \cap \langle (F_n)^c \rangle)}^{K} \longrightarrow 0.$$

Therefore

$$\widetilde{H}_i(E_1) = \begin{cases} K^2 & i = 2p-2 \\ 0 & \text{otherwise.} \end{cases}$$

We repeat the above method, recursively , for values $k = 2, 3, \ldots, t-1$

$$0 \longrightarrow \overbrace{\widetilde{H}_{2p-2}(E_{k-1})}^{K^k} \longrightarrow \widetilde{H}_{2p-2}(E_k) \longrightarrow \overbrace{\widetilde{H}_{2p-3}(E_{k-1} \cap \langle (F_{n-k+1})^c \rangle)}^{K} \longrightarrow 0$$

and conclude that for $1 \leq k \leq t-1$

$$\widetilde{H}_i(E_k) = \begin{cases} K^{k+1} & i = 2p-2 \\ 0 & \text{otherwise.} \end{cases}$$

We put this all together

$$\widetilde{H}_i(E_{t-1}) = \begin{cases} K^t & d = 0,\ i = 2p-2,\ p > 0 \\ K & d \neq 0,\ i = 2p-1 \\ 0 & \text{otherwise} \end{cases}$$

and this proves the statement of the theorem. □

5 The Betti numbers

We are now ready to apply the homological calculations from the previous section, and compute Betti numbers of path ideals. If I is the degree t path ideal of a cycle, then

$$\beta_{i,j}(R/I) = 0 \text{ for all } i \geq 1 \text{ and } j > ti; \tag{5.1}$$

see for example [7] 3.3.4. By Theorem 2.8, to compute the Betti numbers of I of degree less than n, we should consider the complements of proper induced subcollection of $\Delta = \Delta_t(C_n)$ or $\Delta_t(L_n)$ and, for degree n we should consider Δ^c. We first compute the $\mathbb{N}$-graded Betti numbers of degree n of path ideals.

Theorem 5.1 (**Betti numbers of degree** n)**.** *Let p, t, n, d be integers such that $n = (t+1)p + d$, where $p \geq 0$, $0 \leq d \leq t$, and $2 \leq t \leq n$. If C_n is a cycle over n vertices, then*

$$\beta_{i,n}(R/I_t(C_n)) = \begin{cases} & d = 0,\ i = 2\left(\dfrac{n}{t+1}\right) \\ 1 & d \neq 0,\ i = 2\left(\dfrac{n-d}{t+1}\right) + 1 \\ & \textit{otherwise}. \end{cases}$$

Proof. Suppose $\Delta = \Delta_t(C_n)$. By Theorem 2.8 $\beta_{i,n}(R/I_t(C_n)) = \dim_K \widetilde{H}_{i-2}(\Delta^c_{\mathcal{X}})$ and the result now follows directly from Theorem 4.6. □

We now focus on Betti numbers of degree less than n. From Hochster's formula we can conclude that this comes down to counting induced subcollections of certain kinds. The next theorem, which is essential to our further discussions, is similar to statement proved for the edge ideal of a cycle in [7].

Definition 5.2. Let i and j be positive integers. We call an induced subcollection Γ of $\Delta_t(C_n)$ an (i,j)**-eligible subcollection** of $\Delta_t(C_n)$ if Γ is composed of disjoint runs of lengths

$$(t+1)p_1+1,\dots,(t+1)p_\alpha+1,(t+1)q_1+2,\dots,(t+1)q_\beta+2 \tag{5.2}$$

for nonnegative integers $\alpha,\beta,p_1,p_2,\dots,p_\alpha,q_1,q_2,\dots,q_\beta$, which satisfy the following conditions

$$\begin{array}{rcl} j &=& (t+1)(P+Q)+t(\alpha+\beta)+\beta \\ i &=& 2(P+Q)+2\beta+\alpha, \end{array}$$

where $P=\sum_{i=1}^{\alpha} p_i$ and $Q=\sum_{i=1}^{\beta} q_i$.

The reason for this terminology will become clear in the following statement.

Theorem 5.3. *Let $I=I(\Lambda)$ be the facet ideal of an induced subcollection Λ of $\Delta_t(C_n)$. Suppose i and j are integers with $i\le j<n$. Then the $\mathbb{N}$-graded Betti number $\beta_{i,j}(R/I)$ is the number of (i,j)-eligible subcollections of Λ.*

Proof. Since $\Delta(I)=\Lambda$ from Theorem 2.8 we have

$$\beta_{i,j}(R/I)=\sum_{\Gamma\subset\Lambda,|\mathrm{Vert}(\Gamma)|=j}\dim_K \widetilde{H}_{i-2}(\Gamma^c_{\mathrm{Vert}(\Gamma)})$$

where Vert(Γ) is the vertex set of Γ and the sum is taken over induced subcollections Γ of Λ.

Each induced subcollection of Λ is clearly an induced subcollection of $\Delta_t(C_n)$, and can therefore be written as a disjoint union of runs. So from Proposition 4.2 we can conclude the only Γ whose complements have nonzero homology are those corresponding to run sequences of the form (5.2). Such subcollections have j vertices where by Definition 3.5

$$\begin{array}{rcl} j &=& ((t+1)p_1+t)+\cdots+((t+1)p_\alpha+t)+((t+1)q_1+t+1)\cdots+((t+1)q_\beta+t+1) \\ &=& (t+1)(P+Q)+t(\alpha+\beta)+\beta. \end{array} \tag{5.3}$$

So

$$\Gamma^c_{\mathrm{Vert}(\Gamma)}=E((t+1)p_1+1,\dots,(t+1)p_\alpha+1,(t+1)q_1+2,\dots,(t+1)q_\beta+2)$$

and by Proposition 4.3 we have

$$\dim_K(\widetilde{H}_{i-2}(\Gamma^c_{\mathrm{Vert}(\Gamma)}))=\begin{cases} 1 & i=2(P+Q)+2\beta+\alpha \\ 0 & \text{otherwise.}\end{cases} \tag{5.4}$$

From (5.3) and (5.4) we see that each induced subcollection Γ corresponding to a run sequence as in (5.2) contributes 1 unit to $\beta_{i,j}$ if and only if

$$\begin{array}{rcl} j &=& (t+1)(P+Q)+t(\alpha+\beta)+\beta \\ i &=& 2(P+Q)+2\beta+\alpha. \end{array}$$

□

Theorem 5.3 holds in particular for $\Lambda=\Delta_t(L_m)$ and $\Lambda=\Delta_t(C_n)$ for any integers m,n.

Theorem 5.4. *Let i,j be integers and $i\le j<n$ and $j\le it$. Also suppose $n=(t+1)p+d$ and $d<t+1$. If $\beta_{i,j}(R/I_t(C_n))\neq 0$ we have $j-i\le(t-1)p$, and $i<2p$ for $d=0$ and $i\le 2p+1$ for $d\neq 0$.*

Proof. By using Theorem 5.3 we know $\beta_{i,j}(R/I_t(C_n)$ is equal to the number of (i,j)-eligible subcollections of $\Delta_t(C_n)$. So if we assume $\beta_{i,j}(R/I_t(C_n)) \neq 0$ we can conclude there exists a (i,j)-eligible subcollections C of $\Delta_t(C_n)$ which is composed of runs of lengths as described in (5.2). Therefore

$$j-i=(t-1)(P+Q+\alpha+\beta) \quad \text{and} \quad ti-j=(t-1)(P+Q+\beta). \tag{5.5}$$

It follows that $j-i \geq ti-j$ so

$$i(t+1) \leq 2j \Rightarrow i \leq 2\left(\frac{j}{t+1}\right) < 2\left(\frac{p(t+1)+d}{t+1}\right)$$

so if $d=0$ it follows that $i<2p$ and if $d \neq 0$ it follows that $i \leq 2p+1$.

On the other hand since $\Delta_t(C_n)$ has n facets and since there must be at least t facets between every two runs in C, we have

$$n \geq (t+1)P+(t+1)Q+\alpha+2\beta+t\alpha+t\beta \geq (t+1)(P+Q+\alpha+\beta) = \left(\frac{t+1}{t-1}\right)(j-i)$$

which implies that

$$\frac{j-i}{t-1} \leq p + \frac{d}{t+1}$$

and since from (5.5) we have $j-i/(t-1)$ is an integer the formula follows. □

We end the paper with the computation of the projective dimension and regularity of path ideals of cycles. The case $t=2$ is the case of graphs which appears in Jacques [7].

Corollary 5.5 (Projective dimension and regularity of path ideals of cycles). *Let n, t, p and d be integers such that $n \geq 2$, $2 \leq t \leq n$, $n=(t+1)p+d$, where $p \geq 0$, $0 \leq d \leq t$. Then*

i. *The projective dimension of the path ideal of a graph cycle C_n is given by*

$$pd(R/I_t(C_n)) = \begin{cases} 2p+1 & d \neq 0 \\ 2p & d=0 \end{cases}$$

ii. *The regularity of the path ideal of the graph cycle C_n is given by*

$$reg(R/I_t(C_n)) = (t-1)p+d-1.$$

Proof.

i. This follows from Theorem 5.1 and Theorem 5.4.

ii. By definition, the regularity of a module M is $\max\{j-i \mid \beta_{i,j}(M) \neq 0\}$. By Theorem 5.4, and the observation above, if $d=0$ then $reg(R/I_t(C_n))$ is

$$\max\{n-2p,(t-1)p\} = \max\{(t+1)p-2p,(t-1)p\} = (t-1)p$$

and if $d \neq 0$ then $reg(R/I_t(C_n))$ is

$$\max\{n-2p-1,(t-1)p\} = \max\{(t+1)p+d-2p-1,(t-1)p\} = (t-1)p+d-1.$$

The formula now follows.

□

References

[1] R. Bouchat, H. T. Ha, and A. O' Keefe. Path ideals of rooted trees and their graded betti numbers. 2411-2425.

[2] W. Bruns and J. Herzog. *Cohen-Macaulay rings*. Cambridge University Press, 1993.

[3] A. Conca and E. De Negri. M-sequences, graph ideals and ladder ideals of linear types. *J. Algebra*, 211:599–624, 1999.

[4] D. R. Grayson and M. E. Stillman. Macaulay 2, a software system for research in algebraic geometry. Technical report, http://www.math.uiuc.edu/Macaulay2/.

[5] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[6] J. He and A. Van Tuyl. Algebraic properties of path ideal of a tree. *Comm. Algebra*, 38:1725–1742, 2010.

[7] S. Jacques. Betti numbers of graph ideals. PhD thesis available on line *http://arxive.org/math/0410107*, 2004.

[8] E. Miller and B. Sturmfels. *Combinatorial Commutative Algebra*. Spring -Verlag, New York, 2005.

[9] CoCoA Team. Cocoa: a system for doing computations in commutative algebra. Technical report, Available at http://cocoa.dima.unige.it.

[10] R. H. Villarreal. *Monomial Algebras*. Marcel Dekker Inc, 2001.