Simplicial Trees are Sequentially Cohen-Macaulay

Sara Faridi*

August 27, 2003

Abstract

This paper uses dualities between facet ideal theory and Stanley-Reisner theory to show that the facet ideal of a simplicial tree is sequentially Cohen-Macaulay. The proof involves showing that the Alexander dual (or the cover dual, as we call it here) of a simplicial tree is a componentwise linear ideal. We conclude with additional combinatorial properties of simplicial trees.

The main result of the this paper is that the facet ideal of a simplicial tree is sequentially Cohen-Macaulay. Sequentially Cohen-Macaulay modules were introduced by Stanley [S] (following the introduction of nonpure shellability by Björner and Wachs [BW]) so that a nonpure shellable simplicial complex had a sequentially Cohen-Macaulay Stanley-Reisner ideal. Herzog and Hibi ([HH]) then defined the notion of a componentwise linear ideal, which extended a criterion of Eagon and Reiner ([ER]) for Cohen-Macaulayness of an ideal to a criterion for sequential Cohen-Macaulayness.

Simplicial trees, on the other hand, were introduced in [F1] in the context of Rees rings, and their facet ideals were studied further in [F2] for their Cohen-Macaulay properties, and in [Z] for their resolutions. The facet ideal of a given simplicial complex is a square-free monomial ideal where every generator is the product of the vertices of a facet of the complex. If the simplicial complex is a tree (Definition 3.5), it turns out that its facet ideal has many interesting algebraic and combinatorial properties.

Given a square-free monomial ideal, one could consider it as the facet ideal of one simplicial complex, and the Stanley-Reisner ideal of another. This in a sense gives two "languages" to study a square-free monomial ideal. Below we provide a dictionary which makes it easy to move from one language to the other. We use this dictionary to translate existing criteria for Cohen-Macaulayness and sequential Cohen-Macaulayness into the language of facet ideals, and then finally use these criteria to show that the facet ideal of a simplicial tree is sequentially Cohen-Macaulay (Corollary 5.6).

There are several byproducts. An immediate one is that the facet ideal of an unmixed simplicial tree (Definition 1.5) is Cohen-Macaulay (Corollary 5.8). This is discussed at length and proved independently in [F2], where we introduce the concept of "grafting" a simplicial complex. As it turns out, any unmixed tree is grafted, and any grafted simplicial complex is Cohen-Macaulay. This fact, in addition to proving the statement of Corollary 5.8, gives the precise combinatorial structure of a Cohen-Macaulay tree.

*Université du Québec à Montréal, Laboratoire de combinatoire et d'informatique mathématique, Case postale 8888, succursale Centre-Ville, Montréal, QC Canada H3C 3P8. email: *faridi@math.uqam.ca*.
This research was supported by a NSERC Postdoctoral Fellowship.

Another outcome is that the Stanley-Reisner complex corresponding to a Cohen-Macaulay tree is shellable. This was known in the case of graphs ([V]). In general, shellability is only a necessary condition for Cohen-Macaulayness.

The paper is organized as follows: Section 1 reviews the basics of facet ideal theory, introducing cover complexes. In Section 2 we discuss how facet ideal theory relates to Stanley-Reisner theory. In Section 3 we define simplicial trees and discuss their localization. In Section 4 we define sequentially Cohen-Macaulay and componentwise linear ideals, and introduce a criterion for an ideal to be sequentially Cohen-Macaulay, which we use in Section 5 to prove that trees are sequentially Cohen-Macaulay.

For the convenience of the reader, we have included a table of notation at the end of the paper (Figure 2).

We would like to thank Jürgen Herzog for raising the question of whether simplicial trees are sequentially Cohen-Macaulay, and for an earlier reading of this manuscript.

1 Basic Definitions

This section is a review of the basic definitions and notations in facet ideal theory. Much of the material here appeared in more detail in [F1] and [F2], except for the discussion on the cover complex.

Definition 1.1 (simplicial complex, facet, subcollection and more). A *simplicial complex* Δ over a set of vertices $V = \{v_1, \dots, v_n\}$ is a collection of subsets of V, with the property that $\{v_i\} \in \Delta$ for all i, and if $F \in \Delta$ then all subsets of F are also in Δ (including the empty set). An element of Δ is called a *face* of Δ, and the *dimension* of a face F of Δ is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. The faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively, and $\dim \emptyset = -1$. The maximal faces of Δ under inclusion are called *facets* of Δ. The dimension of the simplicial complex Δ is the maximal dimension of its facets.

We denote the simplicial complex Δ with facets $F_1, \dots, F_q$ by

$$\Delta = \langle F_1, \dots, F_q \rangle$$

and we call $\{F_1, \dots, F_q\}$ the *facet set* of Δ. A simplicial complex with only one facet is called a *simplex*. By a *subcollection* of Δ we mean a simplicial complex whose facet set is a subset of the facet set of Δ.

Definition 1.2 (connected simplicial complex). A simplicial complex $\Delta = \langle F_1, \dots, F_q \rangle$ is *connected* if for every pair i, j, $1 \leq i < j \leq q$, there exists a sequence of facets $F_{t_1}, \dots, F_{t_r}$ of Δ such that $F_{t_1} = F_i$, $F_{t_r} = F_j$ and $F_{t_s} \cap F_{t_{s+1}} \neq \emptyset$ for $s = 1, \dots, r-1$.

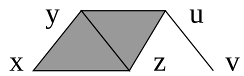
Definition 1.3 (facet ideal, facet complex). Let k be a field and $x_1, \dots, x_n$ be a set of indeterminates, and $R = k[x_1, \dots, x_n]$ be a polynomial ring.

- Let Δ be a simplicial complex over n vertices labeled $v_1, \dots, v_n$. We define the *facet ideal* of Δ, denoted by $\mathcal{F}(\Delta)$, to be the ideal of R generated by square-free monomials $x_{i_1} \dots x_{i_s}$, where $\{v_{i_1}, \dots, v_{i_s}\}$ is a facet of Δ.

- Let $I = (M_1, \dots, M_q)$ be an ideal in R, where $M_1, \dots, M_q$ are square-free monomials in $x_1, \dots, x_n$ that form a minimal set of generators for I. We define the *facet complex* of I, denoted by $\delta_{\mathcal{F}}(I)$, to be the simplicial complex over a set of vertices $v_1, \dots, v_n$ with facets $F_1, \dots, F_q$, where for each i, $F_i = \{v_j \mid x_j | M_i,\ 1 \leq j \leq n\}$.

Throughout this paper we often use a letter x to denote both a vertex of Δ and the corresponding variable appearing in $\mathcal{F}(\Delta)$, and $x_{i_1} \ldots x_{i_r}$ to denote a facet of Δ as well as a monomial generator of $\mathcal{F}(\Delta)$.

Example 1.4. If Δ is the simplicial complex $\langle xyz, yzu, uv \rangle$ drawn below,



then $\mathcal{F}(\Delta) = (xyz, yuz, uv)$ is its facet ideal.

Facet ideals give a one-to-one correspondence between simplicial complexes and square-free monomial ideals.

Next we define the notion of a vertex cover. The combinatorial idea here comes from graph theory. In algebra, it corresponds to prime ideals lying over the facet ideal of a given simplicial complex.

Definition 1.5 (vertex cover, vertex covering number, unmixed). Let Δ be a simplicial complex with vertex set V. A *vertex cover* for Δ is a subset A of V that intersects every facet of Δ. If A is a minimal element (under inclusion) of the set of vertex covers of Δ, it is called a *minimal vertex cover*. The smallest of the cardinalities of the vertex covers of Δ is called the *vertex covering number* of Δ and is denoted by $\alpha(\Delta)$.

A simplicial complex Δ is *unmixed* if all of its minimal vertex covers have the same cardinality.

Example 1.6. If Δ is the simplicial complex in Example 1.4, then the vertex covers of Δ are:

$$\{\mathbf{x}, \mathbf{u}\}, \{\mathbf{y}, \mathbf{u}\}, \{\mathbf{y}, \mathbf{v}\}, \{\mathbf{z}, \mathbf{u}\}, \{\mathbf{z}, \mathbf{v}\}, \{x, y, u\}, \{x, z, u\}, \{x, y, v\}, \ldots.$$

The first five vertex covers above (highlighted in bold), are the minimal vertex covers of Δ.

In all the arguments in this paper, unless otherwise stated, k denotes a field.

Given a square-free monomial ideal I in a polynomial ring $k[x_1, \ldots, x_n]$, the vertices of $\delta_{\mathcal{F}}(I)$ are those variables that divide a monomial in the generating set of I; this set may not include all elements of $\{x_1, \ldots, x_n\}$. The fact that some extra variables may appear in the polynomial ring has little effect on the algebraic or combinatorial structure of $\delta_{\mathcal{F}}(I)$. On the other hand, if Δ is a simplicial complex, being able to consider the facet ideals of its subcomplexes as ideals in the same ambient ring simplifies many of our discussions. For this reason we make the following definition.

Definition 1.7 (variable cover). Let I be a square-free monomial ideal in a polynomial ring $R = k[x_1, \ldots, x_n]$. A subset A of the variables $\{x_1, \ldots, x_n\}$ is called a *(minimal) variable cover* of $\Delta = \delta_{\mathcal{F}}(I)$ (or of I) if A is the generating set for a (minimal) prime ideal of R containing I.

If $x_1, \ldots, x_n$ are all vertices of $\Delta = \delta_{\mathcal{F}}(I)$, then a variable cover of Δ is exactly the same as a vertex cover of Δ. In general every variable cover of Δ contains a vertex cover of Δ. For example for the ideal $I = (xy, xz) \subseteq k[x, y, z, u]$, $\{x, u\}$ is a variable cover but not a vertex cover. The minimal vertex covers of Δ, however, are always the same as the minimal variable covers of Δ.

We now construct a new simplicial complex using the minimal vertex covers of a given simplicial complex.

Definition 1.8 (cover complex). Given a simplicial complex Δ, the simplicial complex Δ_M called the *cover complex of* Δ, is the simplicial complex whose facets are the minimal vertex covers of Δ.

Example 1.9. In Example 1.6, $\Delta = \langle xyz, yzu, uv\rangle$ and $\Delta_M = \langle xu, yu, yv, zu, zv\rangle$.

It is worth observing that Δ being unmixed is equivalent to Δ_M being *pure* (meaning that all facets of Δ_M are of the same dimension). This fact becomes useful in our discussions below. For example the simplicial complex Δ in Example 1.6 is unmixed, and Δ_M is pure.

The following fact is known in hypergraph theory (see, for example, [B]). We outline a proof below.

Proposition 1.10 (The cover complex is a dual). *If Δ is a simplicial complex, then Δ_M is a dual of Δ; i.e. $\Delta_{MM} = \Delta$.*

Proof. Suppose that $\Delta = \langle F_1, \dots, F_q\rangle$ and $\Delta_M = \langle G_1, \dots, G_p\rangle$. Suppose that for $i = 1, \dots, p$, q_i is the prime ideal generated by the elements of G_i, so that we have

$$I = \mathcal{F}(\Delta) = q_1 \cap \ldots \cap q_p.$$

We first show that every facet of Δ is a vertex cover of Δ_M. Consider the facet F_1. Since the monomial $F_1 \in q_i$ for all i, it follows that F_1 contains at least one vertex of each of the G_i. This proves that F_1 is a vertex cover of Δ_M.

Suppose now that F is any minimal vertex cover of Δ_M. Since F contains a vertex of each of the G_i, it belongs to all the ideals q_i (if we consider F as a monomial), and therefore $F \in I$. So some generator F_j of I must divide F. This means that $F_j \subseteq F$, but since F_j is already a vertex cover of Δ_M, it follows that $F = F_j$. This shows that $F_1, \dots, F_q$ are all the minimal vertex covers of Δ_M. $\square$

2 Relations to Stanley-Reisner theory

We begin by the basic definitions from Stanley-Reisner theory. For a detailed coverage of this topic, we refer the reader to [BH].

Definition 2.1 (nonface ideal, nonface complex). Let k be a field and $x_1, \dots, x_n$ be a set of indeterminates, and $R = k[x_1, \dots, x_n]$ be a polynomial ring.

- Let Δ be a simplicial complex over n vertices labeled $v_1, \dots, v_n$. We define the *nonface ideal* or the *Stanley-Reisner ideal* of Δ, denoted by $\mathcal{N}(\Delta)$, to be the ideal of R generated by square-free monomials $x_{i_1} \dots x_{i_s}$, where $\{v_{i_1}, \dots, v_{i_s}\}$ is not a face of Δ.

- Let $I = (M_1, \dots, M_q)$ be an ideal in R, where $M_1, \dots, M_q$ are square-free monomials in $x_1, \dots, x_n$ that form a minimal set of generators for I. We define the *nonface complex* or the *Stanley-Reisner complex* of I, denoted by $\delta_{\mathcal{N}}(I)$, to be the simplicial complex over a set of vertices $v_1, \dots, v_n$, where $\{v_{i_1}, \dots, v_{i_s}\}$ is a face of $\delta_{\mathcal{N}}(I)$ if and only if $x_{i_1} \dots x_{i_s} \notin I$.

Notation 2.2. To simplify notation, we use Δ_N to mean the nonface complex of $\mathcal{F}(\Delta)$ for a given simplicial complex Δ. In other words, we set

$$\Delta_N = \delta_{\mathcal{N}}(\mathcal{F}(\Delta)).$$

Given an ideal $I \subseteq k[V]$ where $V = \{x_1, \ldots, x_n\}$, if there is no reason for confusion, we use Δ and Δ_N to denote $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$, respectively. If F is a face of $\Delta = \langle F_1, \ldots, F_q \rangle$, we let the *complements* of F and Δ be

$$F^c = V \setminus F \text{ and } \Delta^c = \langle {F_1}^c, \ldots, {F_q}^c \rangle.$$

Definition 2.3 (Alexander dual). Let I be a square-free monomial ideal in the polynomial ring $k[V]$ with $V = \{x_1, \ldots, x_n\}$. Then the *Alexander dual* of Δ_N is the simplicial complex

$${\Delta_N}^{\vee} = \{F \subset V \mid F^c \notin \Delta_N\}.$$

It is easy to see that ${\Delta_N}^{\vee\vee} = \Delta_N$.

We now focus on the relations between Δ and Δ_N for a given square-free monomial ideal I. The first question we tackle is how to construct Δ_N from Δ.

Proposition 2.4. *Given a simplicial complex Δ, we have*

(a) $\Delta_N = {\Delta_M}^c$;

(b) ${\Delta_N}^{\vee} = \Delta_{MN} = \Delta^c$.

Proof. (a) This is easy to check. See, for example, [BH] Theorem 5.1.4.

(b) The last equality follows from Proposition 1.10 and Part (a), since

$$\Delta_{MN} = \Delta_{M{M}^c} = \Delta^c.$$

We translate both sides of the first equation using the notations in 2.2 and Definition 2.3:

$${\Delta_N}^{\vee} = \{F^c \mid F \notin \Delta_N\} \text{ and } \Delta^c = \langle F^c \mid F \text{ is a facet of } \Delta \rangle.$$

Suppose that $F^c \in {\Delta_N}^{\vee}$. Then $F \notin \Delta_N$, and therefore if f denotes the monomial that is the product of the vertices of F, and $I = \mathcal{N}(\Delta_N)$, then $f \in I$. It follows that for some generator g of I, $g|f$. If G is the facet of Δ corresponding to g, we have $G \subseteq F$, which implies that $F^c \subseteq G^c$; so $F^c \in \Delta^c$.

Conversely, let $G \in \Delta^c$. Then $G \subseteq F^c$, where F is a facet of Δ, so $f \in I$ which implies that $F \notin \Delta_N$. So $F^c \in {\Delta_N}^{\vee}$, which implies that $G \in {\Delta_N}^{\vee}$.

□

Proposition 2.4 is basically saying that the relationship between Δ_M and ${\Delta_N}^{\vee}$ is the same as the relationship between Δ and Δ_N. The example below clarifies this point.

Example 2.5. Let $I = (xyz, zu) \subseteq k[x, y, z, u]$. Then the *dual* ideal of I, which is the facet ideal of Δ_M, or equivalently the nonface ideal of ${\Delta_N}^{\vee}$, is the ideal $J = (xu, yu, z)$. The relationship between the four simplicial complexes and the two ideals is shown in Figure 1.

Proposition 2.4 justifies the following definition.

Definition 2.6 (dual of an ideal). Given a square-free monomial ideal I in a polynomial ring and $\Delta = \delta_{\mathcal{F}}(I)$, we define the *dual* of I, denoted by $I^{\vee}$, to be the facet ideal of Δ_M, or equivalently, the nonface ideal of ${\Delta_N}^{\vee}$. So

$$I^{\vee} = \mathcal{F}(\Delta_M) = \mathcal{N}({\Delta_N}^{\vee}).$$

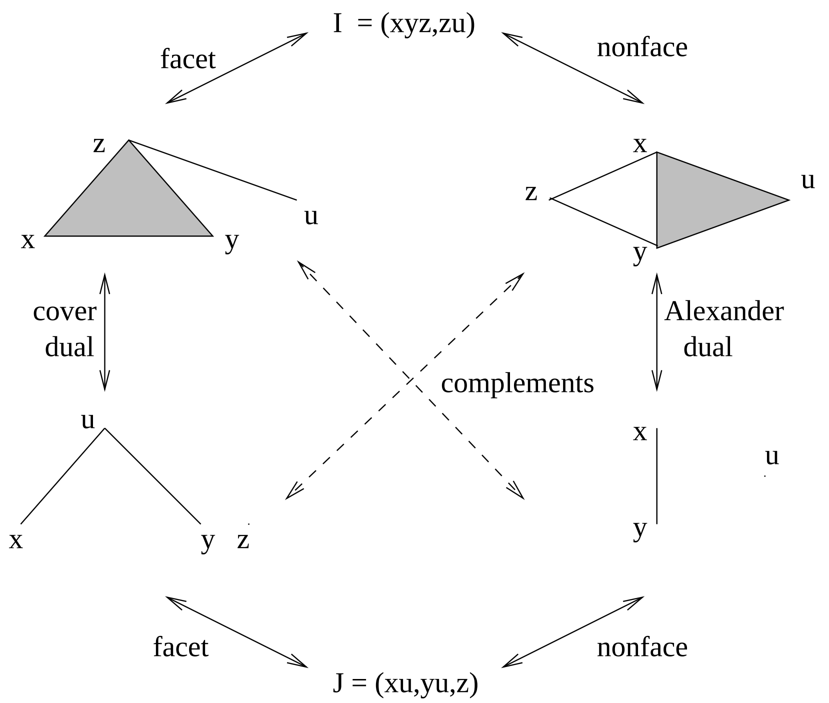


Figure 1: Diagram of Example 2.5

We now state a criterion for the Cohen-Macaulayness of a square-free monomial ideal that is due to Eagon and Reiner ([ER]) in the language stated above. First we define an ideal with a linear resolution.

Definition 2.7 (linear resolution). An ideal I in a polynomial ring $R = k[x_1, \dots, x_n]$ over a field k, with the standard grading $\deg(x_i) = 1$ for all i, is said to have a *linear resolution* if R/I has a minimal free resolution such that for all $j > 1$ the nonzero entries of the matrices of the maps $R^{\beta_j} \longrightarrow R^{\beta_{j-1}}$ are of degree 1.

Theorem 2.8 ([ER] Theorem 3). *Let I be a square-free monomial ideal in a polynomial ring R. Then R/I is Cohen-Macaulay if and only if $I^\vee$ has a linear resolution.*

3 Simplicial Trees

Considering simplicial complexes as higher dimensional graphs, one can define the notion of a *tree* by extending the same concept from graph theory. Simplicial trees were first introduced in [F1] in order to generalize results of [SVV] on facet ideals of graph-trees. The construction turned out to have interesting additional combinatorial and algebraic properties.

Before we define a tree, we determine what "removing a facet" from a simplicial complex means. We define this idea so that it corresponds to dropping a generator from its facet ideal.

Definition 3.1 (facet removal). Suppose Δ is a simplicial complex with facets $F_1, \dots, F_q$ and $\mathcal{F}(\Delta) = (M_1, \dots, M_q)$ its facet ideal in $R = k[x_1, \dots, x_n]$. The simplicial complex obtained by *removing the facet* F_i from Δ is the simplicial complex

$$\Delta \setminus \langle F_i \rangle = \langle F_1, \dots, \hat{F_i}, \dots, F_q \rangle.$$

Note that $\mathcal{F}(\Delta \setminus \langle F_i \rangle) = (M_1, \dots, \hat{M_i}, \dots, M_q)$.

Also note that the vertex set of $\Delta \setminus \langle F_i \rangle$ is a subset of the vertex set of Δ.

Example 3.2. Let Δ be a simplicial complex with facets $F = \{x, y, z\}$, $G = \{y, z, u\}$ and $H = \{u, v\}$. Then $\Delta \setminus \langle F \rangle = \langle G, H \rangle$ is a simplicial complex with vertex set $\{y, z, u, v\}$.

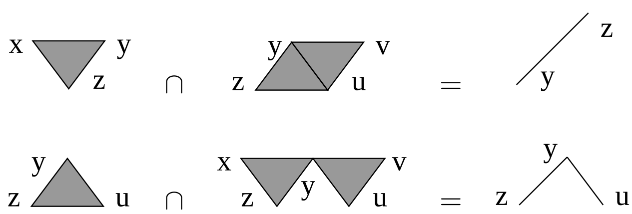
In graph theory, a tree is defined as a connected cycle-free graph. An equivalent definition is that a tree is a connected graph whose every subgraph has a *leaf*, where a leaf is a vertex that belongs to only one edge. We make an analogous definition for simplicial complexes by extending (and slightly changing) the definition of a leaf.

Definition 3.3 (leaf). A facet F of a simplicial complex is called a *leaf* if either F is the only facet of Δ, or for some facet $G \in \Delta \setminus \langle F \rangle$ we have

$$F \cap (\Delta \setminus \langle F \rangle) \subseteq G.$$

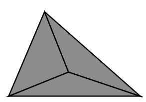
Equivalently, the facet F is a leaf of Δ if $F \cap (\Delta \setminus \langle F \rangle)$ is a face of $\Delta \setminus \langle F \rangle$.

Example 3.4. Let $I = (xyz, yzu, zuv)$. Then $F = xyz$ is a leaf, but $H = yzu$ is not, as one can see in the picture below.



Definition 3.5 (tree, forest). A connected simplicial complex Δ is a *tree* if every nonempty subcollection of Δ has a leaf. If Δ is not necessarily connected, but every subcollection has a leaf, then Δ is called a forest.

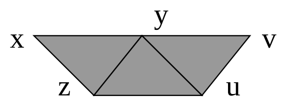
Example 3.6. The simplicial complexes in examples 1.4 and 3.4 are both trees, but the one below is not because it has no leaves. It is an easy exercise to see that a leaf must contain a free vertex, where a vertex is *free* if it belongs to only one facet.



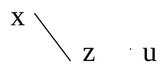
An effective way to make algebraic arguments on trees is using localization. It turns out that the minimal generating set of a localization of the facet ideal of a tree corresponds to a forest. As we shall see below, this fact makes it easy to use induction on the number of vertices of a tree.

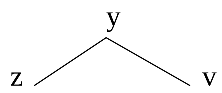
For details on the localization of s simplicial complex see [F2]. Here we give an example to clarify what we mean by localization.

Example 3.7. Let Δ be the simplicial complex below with $I = (xyz, yzu, yuv)$ its facet ideal in the polynomial ring $R = k[x, y, z, u, v]$.



Let $p = (x, u, z)$ be a prime ideal of R. Then $I_p = (xz, zu, u) = (xz, u)$ is the facet ideal of the forest below on the left. If $q = (y, z, v)$ then $I_q = (yz, yz, yv) = (yz, yv)$ corresponds to the tree on the right.

Localization at p: 

Localization at q: 

Example 3.7 is an example of the following general fact.

Lemma 3.8 (Localization of a tree is a forest). *Let $I \subseteq k[x_1, \dots, x_n]$ be the facet ideal of a tree, where k is a field, and suppose that p is a prime ideal of $k[x_1, \dots, x_n]$. Then for any prime ideal p of R, $\delta_{\mathcal{F}}(I_p)$ is a forest.*

Proof. See [F2] Lemma 4.5. $\square$

4 Sequentially Cohen-Macaulay simplicial complexes

The notion of a sequentially Cohen-Macaulay ideal was introduced by Stanley following the introduction of nonpure shellability by Björner and Wachs [BW]. It was known that every shellable simplicial complex (which was by definition pure) was Cohen-Macaulay, but what about nonpure shellable simplicial complexes? As it turns out, "sequentially Cohen-Macaulay" is the correct notion to fill in the gap here. On the other hand, the criterion of Eagon and Reiner ([ER]) stated that a simplicial complex is Cohen-Macaulay if and only if its Alexander dual has a linear resolution. Herzog and Hibi ([HH]) developed the definition of a "componentwise linear ideal" so that the above criterion extended to sequentially Cohen-Macaulay ideals: a simplicial complex is sequentially Cohen-Macaulay if and only if its Alexander dual is componentwise linear.

In our setting, we use an equivalent characterization of sequentially Cohen-Macaulay given by Duval, along with Theorem 2.8 and the relationship between Alexander duality and cover complex duality discussed in Section 2, to prove that simplicial trees are sequentially Cohen-Macaulay. In fact, we show that if I is the facet ideal of a simplicial tree, then the dual $I^\vee$ of I has "square-free homogeneous components" with *linear quotients*. This property is slightly stronger than what we need, and it shows that if I is a Cohen-Macaulay ideal to begin with, then Δ_N is shellable (which was known for the case where Δ is a graph; Theorem 6.4.7 of [V]).

Another outcome is the fact that an unmixed tree is Cohen-Macaulay (Corollary 5.8), which was shown in [F2] using very different tools.

Definition 4.1 ([S] Chapter III, Definition 2.9). Let M be a finitely generated $\mathbb{Z}$-graded module over a finitely generated $\mathbb{N}$-graded k-algebra, with $R_0 = k$. We say that M is *sequentially Cohen-Macaulay* if there exists a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$$

of M by graded submodules M_i satisfying the following two conditions.

(a) Each quotient M_i/M_{i-1} is Cohen-Macaulay.

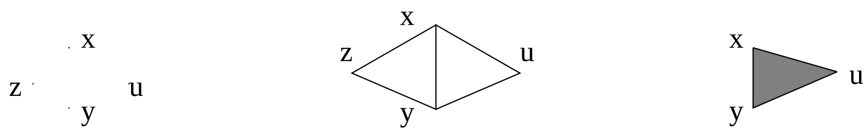
(b) dim $(M_1/M_0) <$ dim $(M_2/M_1) < \ldots <$ dim (M_r/M_{r-1}), where dim denotes Krull dimension.

A simplicial complex is said to be *sequentially Cohen-Macaulay* if its Stanley-Reisner ideal has a sequentially Cohen-Macaulay quotient.

The following characterization of a sequentially Cohen-Macaulay simplicial complex given by Duval ([D] Theorem 3.3) is what we use in this paper.

Theorem 4.2 ([D] sequentially Cohen-Macaulay). *Let I be square-free monomial ideal I in a polynomial ring R over a field k, and let $\Delta_N = \delta_{\mathcal{N}}(I)$. Then R/I is sequentially Cohen-Macaulay if and only if for every i, $-1 \leq i \leq \dim \Delta_N$, if $\Delta_{N,\ i}$ is the pure i-dimensional subcomplex of Δ_N, then $R/\mathcal{N}(\Delta_{N,\ i})$ is Cohen-Macaulay.*

Example 4.3. let $I = (xyz, zu)$ be the ideal of Example 2.5 in the diagram above. Then for $i = 0, 1, 2$, we have the following three simplicial complexes, respectively,



which are, respectively, the nonface complexes of the ideals $I_0 = (xy, xz, xu, yz, yu, zu)$, $I_1 = (xyz, xyu, zu)$ and $I_2 = (z)$. One can verify that all three of these ideals have Cohen-Macaulay quotients, so I is sequentially Cohen-Macaulay.

We define a componentwise linear ideal in the square-free case using [HH] Proposition 1.5.

Definition 4.4 (square-free homogeneous component, componentwise linear). Let I be a square-free monomial ideal in a polynomial ring R. For a positive integer k, the k-th *square-free homogeneous component* of I, denoted by $I_{[k]}$ is the ideal generated by all square-free monomials in I of degree k. The ideal I above is said to be *componentwise linear* if for all k, the square-free homogeneous component $I_{[k]}$ has a linear resolution.

Let

$$\Delta = \langle F_1, \ldots, F_q \rangle$$

be a simplicial complex with $\mathcal{F}(\Delta) \subseteq k[V]$, $V = \{x_1, \ldots, x_n\}$, and let

$$\Delta_M = \langle G_1, \ldots, G_p \rangle$$

be its cover complex . Then by Proposition 2.4 we know that

$$\Delta_N = \langle {G_1}^c, \ldots, {G_p}^c \rangle.$$

For a given i, consider the pure i-dimensional subcomplex of Δ_N

$$\Delta_{N,\ i} = \langle H_1, \ldots, H_u \rangle.$$

By Theorem 2.8 showing that $I_i = \mathcal{N}(\Delta_{N,\ i})$ is a Cohen-Macaulay ideal is equivalent to showing that ${I_i}^\vee$ has a linear resolution. By Proposition 2.4, ${I_i}^\vee$ is the facet ideal of ${\Delta_{N,\ i}}^c$.

So we focus on H^c, where H is a facet of $\Delta_{N,\ i}$. Since H belongs to a subcomplex of Δ_N, for some facet ${G_j}^c$ of Δ_N, $H \subseteq {G_j}^c$. This implies that $G_j = {G_j}^{cc} \subseteq H^c$; i.e. H^c contains a minimal vertex cover of Δ, and so H^c is a variable cover of Δ of cardinality $n-(i+1)$.

Similarly, if G is a variable cover of cardinality $n-(i+1)$ of Δ, then one can see that G^c is a facet of $\Delta_{N,\ i}$.

The discussion above shows that ${I_i}^\vee$ is generated by monomials corresponding to variable covers of cardinality $n-i-1$ of Δ. In other words

$${I_i}^\vee = {I^\vee}_{[n-i-1]},$$

where ${I^\vee}_{[j]}$ denotes the j-th square-free homogeneous component of $I^\vee$, and showing that $\Delta_{N,\ i}$ is Cohen-Macaulay is equivalent to showing that ${I^\vee}_{[n-i-1]}$ has a linear resolution.

We have thus shown that:

Proposition 4.5 (Criterion for being sequentially Cohen-Macaulay). *Let I be a square-free monomial ideal in a polynomial ring. Then I is a sequentially Cohen-Macaulay ideal if and only if $I^\vee$ is componentwise linear.*

5 Simplicial trees are Sequentially Cohen-Macaulay

This section contains the main results of the paper. Our goal here is to show that the facet ideal I of a simplicial tree is sequentially Cohen-Macaulay. By Proposition 4.5 this is equivalent to showing that the facet ideal $I^\vee$ of the cover complex of a tree is componentwise linear (Definition 4.4). In fact, we show that $I^\vee$ satisfies a stronger property: for every i, we show below that ${I^\vee}_{[i]}$ has linear quotients. This property, defined by Herzog and Takayama in [HT], implies that ${I^\vee}_{[i]}$ has a linear resolution. It also implies additional combinatorial properties for I (see Corollary 5.9).

Definition 5.1 (linear quotients ([HT])). If $I \subset k[x_1,\dots,x_n]$ is a monomial ideal and $G(I)$ is its unique minimal set of monomial generators, then I is said to have *linear quotients* if there is an ordering $M_1,\dots,M_q$ on the elements of $G(I)$ such that for every $i=2,\dots,q$, the quotient ideal

$$(M_1,\dots,M_{i-1}) : M_i$$

is generated by a subset of the variables $x_1,\dots,x_n$.

The following is a well-known fact. We reproduce an argument (almost identical to one given in [Z] for the case of trees).

Lemma 5.2. *If $I=(M_1,\dots,M_q)$ is a monomial ideal in the polynomial ring $R=k[x_1,\dots,x_n]$ over the field k that has linear quotients and all the M_i are of the same degree, then I has a linear resolution.*

Proof. The proof is by induction on q. The case $q=1$ is clear. Given that the ideal $I'=(M_1,\dots,M_{q-1})$ has linear quotients and therefore a linear resolution, and that the degree of all the M_i is equal to d, we have that (see Section 5.5 of [BH]) for all i:

$\mathrm{Tor}_i^R\ (k, R/I')_a = 0$ unless $a=i+d$;

$\mathrm{Tor}_i^R\ (k, R/I' : I)_a = 0$ unless $a=i+1$ ($I' : I$ is generated by degree 1 monomials).

Consider the short exact sequence:

$$0 \longrightarrow R/(I' : I)(-d)) \xrightarrow{.M_q} R/I' \longrightarrow R/I \longrightarrow 0$$

We obtain the long exact homology sequence:

$$\cdots \longrightarrow \mathrm{Tor}_i^R\ (k, R/(I' : I)(-d)) \longrightarrow \mathrm{Tor}_i^R\ (k, R/I') \longrightarrow \mathrm{Tor}_i^R\ (k, R/I)$$
$$\longrightarrow \mathrm{Tor}_{i-1}^R\ (k, R/(I' : I)(-d)) \longrightarrow \cdots$$

For a given i, $\mathrm{Tor}_i^R\ (k, R/I)_a = 0$ unless

$$\mathrm{Tor}_i^R\ (k, R/I')_a \neq 0 \text{ or } \mathrm{Tor}_{i-1}^R\ (k, R/(I' : I))(-d)_a \neq 0.$$

Either way, this means that for any i, if $\mathrm{Tor}_i^R\ (k, R/I)_a \neq 0$, then $a = i + d$. This implies that R/I has a linear resolution. $\square$

We now set out to prove if $I \subseteq k[V]$, $V = \{x_1, \ldots, x_n\}$, is the facet ideal of a tree (in fact, a forest) Δ, and i, $\alpha(\Delta) \leq i \leq n$, is a given integer, then ${I^\vee}_{[i]}$ has linear quotients.

We use induction on n. If $n = 1$, Δ can only be the vertex $\langle x_1 \rangle$, and so the only thing to check is if ${I^\vee}_{[1]} = (x_1)$ has linear quotients, which is obvious.

Suppose that $n > 1$. We first deal with some special cases. If Δ is a forest of singletons of the form

$$\Delta = \langle x_1, \ldots, x_j \rangle$$

where $j < n$, then we can consider $I' = \mathcal{F}(\Delta)$ as an ideal in the polynomial ring $R' = k[x_1, \ldots, x_{n-1}]$ (I and I' have the same generating set, they only live in two different rings). By the induction hypothesis, for every i, ${I'^\vee}_{[i]}$ has linear quotients.

It is easy to see that for every i,

$${I^\vee}_{[i]} = {I'^\vee}_{[i]} + x_n {I'^\vee}_{[i-1]}.$$

Suppose that

$${I'^\vee}_{[i]} = (A_1, \ldots, A_a) \text{ and } {I'^\vee}_{[i-1]} = (B_1, \ldots, B_b)$$

where the generators of both ideals are written in the correct order for linear quotients (recall that we are using the notation xA to mean $\{x\} \cup A$, since generally we are always thinking of sets as monomials). To see that

$${I^\vee}_{[i]} = (A_1, \ldots, A_a) + x_n(B_1, \ldots, B_b)$$

has linear quotients, we consider the case where for some monomial m in $k[x_1, \ldots, x_n]$ (we can without loss of generality assume here that the products are square-free),

$$m x_n B_j \in (A_1, \ldots, A_a, x_n B_1, \ldots, x_n B_{j-1}).$$

If $m x_n B_j \in (x_n B_1, \ldots, x_n B_{j-1})$, since ${I'^\vee}_{[i-1]}$ has linear quotients, it follows that for some variable z dividing the monomial m, we have $z x_n B_j \in (x_n B_1, \ldots, x_n B_{j-1})$ (note that $m \neq 1$).

If $m x_n B_j \in (A_1, \ldots, A_a)$, then since B_j is already a variable cover of Δ, for any variable z not in B_j, zB_j covers Δ and is of cardinality i, and hence $zB_j \in \{A_1, \ldots, A_a\}$. Therefore for any z dividing m we can again conclude that $z x_n B_j \in (A_1, \ldots, A_a)$.

This argument settles the case where $\Delta = \langle x_1, \ldots, x_j \rangle$, and $j < n$.

If $\Delta = \langle x_1, \ldots, x_n \rangle$, then the only ideal to consider is ${I^\vee}_{[n]} = (x_1 \ldots x_n)$ which by definition has linear quotients.

So now we can assume that Δ is a forest containing a facet with more than one vertex.

We begin our discussion with the following simple observation.

Lemma 5.3. *Let Δ be a simplicial complex with $\mathcal{F}(\Delta) \subseteq k[V]$, k a field, and $V = \{x_1, \ldots, x_n\}$. Suppose that $x \in V$ is such that $V \setminus \{x\}$ is a variable cover for Δ, and let p_x be the prime ideal generated by the set $V \setminus \{x\}$. Then localizing Δ at p_x corresponds, via the cover duality, to removing all facets of Δ_M that contain x. In other words, if $\Delta' = \delta_{\mathcal{F}}(\mathcal{F}(\Delta)_{p_x})$ and $A_1, \ldots, A_t$ are the facets of Δ_M that contain x, then*

$$\Delta'_M = \Delta_M \setminus \langle A_1, \ldots, A_t \rangle.$$

Proof. Note that a facet of Δ'_M is the generating set for a minimal prime of $I = \mathcal{F}(\Delta)$ not containing x, and therefore belongs to Δ_M as well. Conversely, if A is a facet of the right-hand-side, then it corresponds to a minimal prime of I not containing x and hence to a minimal prime of I_{p_x}. $\square$

Now assume that the forest Δ has a leaf F with positive dimension and a free vertex (see Example 3.6) $x = x_1$. We can write:

$$\Delta_{M,\ [i]} = \delta_{\mathcal{F}}({I^{\vee}}_{[i]}) = \langle A_1, \ldots, A_t \rangle \cup \langle xB_1, \ldots, xB_s \rangle$$

where $A_1, \ldots, A_t$ are all the variable covers of Δ that have cardinality i and do not contain x, and $xB_1, \ldots, xB_s$ are all the other variable covers of cardinality i.

Now let

$$\Delta' = \delta_{\mathcal{F}}(\mathcal{F}(\Delta)_{p_x}) \ \text{ and } \ \Delta'' = \Delta \setminus \langle F \rangle.$$

Both Δ' and Δ'' are forests (by the definition of a tree, and by Lemma 3.8) whose vertex sets are contained in $\{x_2, \ldots, x_n\}$. Also note that Δ' is a nonempty simplicial complex.

With notation as above, by Lemma 5.3

$$\Delta'_{M,\ [i]} = \langle A_1, \ldots, A_t \rangle.$$

Also notice that

$$\Delta''_{M,\ [i-1]} = \langle B_1, \ldots, B_s \rangle.$$

To see this last equation, note that since for $j = 1, \ldots, s$, xB_j covers Δ, B_j has to cover Δ'' (as x is a free vertex of F and hence only covers F). On the other hand, if A is any variable cover of Δ'' of cardinality $i - 1$, then xA is in $\Delta_{M,\ [i]}$, and so $xA \in \{xB_1, \ldots, xB_s\}$.

Applying the induction hypothesis to the forests Δ' and Δ'' we see that the ideals

$${I'^{\vee}}_{[i]} = (A_1, \ldots, A_t) \text{ and } {I''^{\vee}}_{[i-1]} = (B_1, \ldots, B_s)$$

of $k[x_2, \ldots, x_n]$ both have linear quotients. Without loss of generality assume that the given orders on the A's and the B's are appropriate for taking quotients. We show that the ideal

$${I^{\vee}}_{[i]} = (A_1, \ldots, A_t) + x(B_1, \ldots, B_s)$$

also has linear quotients. Here we assume that $1 < i < n$, since ${I^{\vee}}_{[n]} = (x_1 \ldots x_n)$ has linear quotients by definition, as does ${I^{\vee}}_{[1]}$ which is, if nonzero, generated by a subset of $\{x_1, \ldots, x_n\}$.

The first case of interest is the ideal

$$(A_1, \ldots, A_t) : xB_1.$$

Now B_1 is a variable cover of $\Delta'' = \Delta \setminus \langle F \rangle$, so $yB_1 \in I^{\vee}{}_{[i]}$ for any vertex y of F not in B_1. So if m is any monomial such that $mxB_1 \in I'^{\vee}{}_{[i]}$, then for some monomial n and some j, assuming without loss of generality that both products below are square-free, we have

$$mxB_1 = nA_j.$$

If B_1 already contains a vertex of F, then it is a variable cover of cardinality $i-1$ for Δ', and so for any $y|m$, $yB_1 \in \{A_1, \dots, A_t\}$. Otherwise, since there is some vertex y of F in A_j, y has to divide m, which again implies that $yxB_1 \in (A_1, \dots, A_t)$.

In general, for the ideal

$$(A_1, \dots, A_t, xB_1, \dots, xB_{j-1}) : xB_j$$

if for some monomial m, $mxB_j \in (xB_1, \dots, xB_{j-1})$, then by the induction hypothesis on $I''^{\vee}{}_{[i-1]}$ there is a variable y that divides m such that $yxB_j \in (xB_1, \dots, xB_{j-1})$.

If $mxB_j \in (A_1, \dots, A_t)$, then it follows from an argument identical to the case $j = 1$ above that there is a variable y dividing m such that $yxB_j \in (A_1, \dots, A_t)$.

We have thus proved that:

Theorem 5.4. *If $I \subseteq k[x_1, \dots, x_n]$ is the facet ideal of a simplicial tree (forest) Δ, then $I^{\vee}{}_{[i]}$ has linear quotients for all $i = \alpha(\Delta), \dots, n$.*

Theorem 5.4 along with Lemma 5.2 result in the following statement.

Corollary 5.5. *If Δ is a simplicial tree (forest), then $\mathcal{F}(\Delta)^{\vee}$ is a componentwise linear ideal.*

Putting Corollary 5.5 together with Proposition 4.5, we arrive at our final goal.

Corollary 5.6 (Trees are sequentially Cohen-Macaulay). *The facet ideal of a simplicial tree (forest) is sequentially Cohen-Macaulay.*

Example 5.7. The ideal I in Example 4.3 is sequentially Cohen-Macaulay because it is the facet ideal of a tree.

It follows easily that if the tree Δ is unmixed to begin with, then it must be Cohen-Macaulay. This is because in this case $\mathcal{F}(\Delta)^{\vee}$ itself is a square-free homogeneous component, which has a linear resolution. So by applying Theorem 2.8 we have

Corollary 5.8 (An unmixed tree is Cohen-Macaulay). *If Δ is an unmixed simplicial tree, then $\mathcal{F}(\Delta)$ has a Cohen-Macaulay quotient.*

Corollary 5.8 was proved in [F2] using very different tools. In particular, in [F2] we show that a tree is unmixed if and only if it is "grafted". The notion of grafting is what gives a Cohen-Macaulay tree its definitive combinatorial structure.

Another interesting fact that follows is that in the case of a simplicial tree Δ, if Δ is Cohen-Macaulay, then Δ_N is shellable (see [BH] for the definition). Given a square-free monomial ideal I, if $\delta_{\mathcal{N}}(I)$ is shellable, then I is Cohen-Macaulay (see [BH]), but the converse is not true in general.

Corollary 5.9. *If Δ is a Cohen-Macaulay simplicial tree, then Δ_N is shellable.*

Proof. If $I = \mathcal{F}(\Delta)$ is Cohen-Macaulay, then by Theorem 5.4, $I^{\vee}$ has linear quotients (since it has generators of the same degree). The rest follows directly from the definitions of shellability and linear quotients; see [HHZ] Theorem 1.4, part (c). $\square$

Notation	Meaning	First appearance
$\mathcal{F}(\Delta)$	facet ideal of Δ	Definition 1.3
$\delta_{\mathcal{F}}(I)$	facet complex of I	Definition 1.3
$\alpha(\Delta)$	vertex covering number of Δ	Definition 1.5
Δ_M	cover complex of Δ	Definition 1.8
$\mathcal{N}(\Delta)$	nonface ideal of Δ	Definition 2.1
$\delta_{\mathcal{N}}(I)$	nonface complex of I	Definition 2.1
Δ_N	$\delta_{\mathcal{N}}(\mathcal{F}(\Delta))$	Notation 2.2
F^c, Δ^c	complements of F and Δ	Notation 2.2
$\Delta^\vee$	Alexander dual of Δ	Definition 2.3
$I^\vee$	dual of I	Definition 2.6
$\Delta \setminus \langle F \rangle$	removal of facet F from Δ	Definition 3.1
$\Delta_{N,\ i}$	pure i-dimensional subcomplex of Δ_N	Theorem 4.2
$I_{[i]}$	i-th square-free homogeneous component of I	Definition 4.4
p_x	ideal generated by all variables but x	Lemma 5.3
$\Delta_{M,\ [i]}$	facet complex of ${I^\vee}_{[i]}$	following Lemma 5.3

Figure 2: Index of Notation

References

[B] Berge, C. *Hypergraphs, Combinatorics of finite sets*, North-Holland Mathematical Library, 45. North-Holland Publishing Co., Amsterdam, 1989.

[BH] Bruns, W., Herzog, J. *Cohen-Macaulay rings*, vol. 39, Cambridge studies in advanced mathematics, revised edition, 1998.

[BW] Björner, A., Wachs, M.L. *Shellable nonpure complexes and posets*, I. Trans. Amer. Math. Soc. 348 (1996), no. 4, 1299–1327.

[D] Duval, A.M. *Algebraic shifting and sequentially Cohen-Macaulay simplicial complexes*, Electron. J. Combin. 3 (1996), no. 1, Research Paper 21

[ER] Eagon J.A., Reiner, V. *Resolution of Stanley-Reisner rings and Alexander duality*, J. Pure Appl. Algebra 130 (1998), no. 3, 265–275.

[F1] Faridi, S. *The facet ideal of a simplicial complex*, Manuscripta Mathematica 109 (2002), 159-174.

[F2] Faridi, S. *Cohen-Macaulay properties of square-free monomial ideals*, Preprint.

[HH] Herzog, J., Hibi, T. *Componentwise linear ideals*, Nagoya Math. J. 153 (1999), 141–153.

[HHZ] Herzog, J., Hibi, T., Zheng, X. *Dirac's theorem on chordal graphs and Alexander duality*, Preprint.

[HRW] Herzog, J., Reiner, V., Welker, V. *Componentwise linear ideals and Golod rings*, Michigan Math. J. 46 (1999), no. 2, 211–223.

[HT] Herzog, J., Takayama, Y. *Resolutions by mapping cones*, The Roos Festschrift volume, 2. Homology Homotopy Appl. 4 (2002), no. 2, part 2, 277–294 (electronic).

[S] Stanley, R.P. *Combinatorics and commutative algebra*, Second edition. Progress in Mathematics, 41. Birkhuser Boston, Inc., Boston, MA, 1996. x+164 pp. ISBN: 0-8176-3836-9.

[SVV] Simis A., Vasconcelos W., Villarreal R., *On the ideal theory of graphs*, J. Algebra 167 (1994), no. 2, 389–416.

[V] Villarreal R., *Monomial algebras*, Monographs and Textbooks in Pure and Applied Mathematics, 238. Marcel Dekker, Inc., New York, 2001.

[Z] Zheng, X. *Resolutions of facet ideals*, Preprint.