Closure Operations on Ideals

by

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To my family.

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CHAPTER I

Introduction

The study of closure operations on ideals in commutative rings is a powerful tool for understanding the structure of those rings. One classic example is the integral closure of an ideal, a notion that has strong roots in algebraic geometry, and whose study goes back to the early 20th century (see [ZS2]). Another example is the tight closure of an ideal, defined first for ideals in rings of characteristic p, and then extended to the case of rings containing a field, by reduction to characteristic p. Using tight closure theory, Hochster and Huneke proved a generalized version of the Briançon-Skoda theorem, relating the notion of tight closure to that of integral closure (see [HH1]). There are various applications of tight closure theory in the recent literature (see [Hu]), and remarkable connections with the study of singularities of algebraic varieties (see [Sm1] and [Ha]).

In 1998, Hochster introduced a new closure operation on a set of ideals $I_1, \dots, I_n$ in a commutative Noetherian ring R of positive characteristic, named the *tight integral closure* (or *TI* closure). TI closure mixes the ideas of tight and integral closure: when all the ideals are principal, the TI closure is the tight closure of the sum of the ideals, and when there is only one ideal, i.e., when $n = 1$, it is the integral closure of that ideal. Using this notion, Hochster proved a Briançon-Skoda type theorem for sets of ideals in a regular ring (see [Ho2]), which significantly improved the original result. It however turned out to be very difficult to verify basic properties of *TI* closure, even though from the definition one would expect most of the properties of tight closure to generalize to TI closure.

This thesis explores two new closure operations on sets of ideals: the *blowup closure* and the *multiple closure* of a set of ideals. Blowup closure has the advantage that it allows us to reduce its study to the case where all the ideals are principal, which is the simplest case to work with. The multiple closure of a set of ideals in a ring, which can be a described as the blowup closure of some larger ideals in a larger ring, has certain other good properties; for example, it respects inclusions of ideals. In the thesis, we develop the theory of these two closure operations. Moreover, we show that in a Noetherian commutative ring of positive characteristic, under mild conditions, multiple closure agrees with TI closure for any set of ideals (see Theorem 4.1.3). In particular, Hochster's TI closure can be interpreted as a tight closure in an extension ring. This makes several basic features of TI closure transparent, and allows us to settle questions of Hochster posed in [Ho2].

Another part of the thesis deals with identifying classes of normal ideals in graded rings. In the theory of resolution of singularities, one wishes to be able to blow up a singular variety along a closed subscheme and obtain a smooth variety birational to the original one. One question that comes up is the existence of such resolutions; it is known for varieties over fields of characteristic zero [H], and is conjectured in the prime characteristic case. Another question is describing the closed subschemes that produce smooth blowups. Translating this problem into the language of algebra, one is interested in ideals I of a ring R such that Proj$R[It]$ is a smooth variety, where $R[It] = \oplus_{n \in \mathbf{N}} I^n t^n$ is the Rees ring of R along I. This leads one to focus on the Rees ring and its properties. In particular one would like to know when the Rees ring is normal. If R is a normal domain, $R[It]$ is normal if and only if I is a normal ideal, where by *normal ideal* we mean an ideal all of whose positive powers are integrally closed. Normal ideals have been studied in different cases, and in some special cases necessary and sufficient conditions have been given for an ideal to be normal; see for example [G], [O], and [HS].

In general, given a ring, finding an ideal whose blowup is regular has proven to be a difficult problem. A more attackable but still difficult problem is to find normal ideals. In this thesis we construct normal ideals for a general graded domain $R = k[x_0, \ldots, x_m]/J$ which depends only on the weights of the variables $x_0, \ldots, x_m$, and are thus very simple to construct.

The thesis is organized as follows:

In Chapter II we define the notions of tight closure, integral closure and tight integral closure. We give the basic definitions, state the main facts, and give references for the proofs. We also define tight closure and TI closure for finitely generated algebras over fields of characteristic zero, and give a brief overview of the method of reduction to characteristic p. We end the chapter by stating the questions of Hochster stated in [Ho2] about properties of TI closure. These questions arise naturally from the existence of similar properties for tight closure and for integral closure.

Chapter III introduces the blowup closure of a set of ideals. The idea is to extend the ideals to certain localizations of the Rees ring along their product. In this ring all the ideals are locally principal and therefore easier to handle. In Section 3.1 we define the blowup closure of a set of ideals in its original setting; this requires testing the tight closure of the sum of the ideals against all localizations of the Rees ring along their product. We then show that it suffices to carry the test for a finite cover of the blowup scheme, i.e., for finitely many localizations of the Rees ring. This observation makes blowup closure computable.

In Section 3.3, we show that the blowup closure of a set of ideals contains the TI closure of those ideals. Using properties of Rees rings, with mild conditions on the ring, we verify basic properties that one would expect from such an operation: that it is persistent under ring maps, and contracts from module finite ring extensions. In particular, blowup closure can be tested modulo minimal primes, which reduces the study to the case of domains.

In some cases we demonstrate that TI closure and blowup closure agree. This is not true in general: see Example 3.3.5. Section 3.4 is devoted to proving that for monomial ideals in a polynomial ring generated by distinct sets of variables, the blowup closure is equal to the TI closure, which is the sum of the integral closures of the ideals.

Although blowup closure seems to be similar in behavior to TI closure, we show, through an example, that it does not respect inclusions of ideals: if $I_1 \subseteq I_2$ and $J_1 \subseteq J_2$, then the inclusion $(I_1, J_1)^{\sim} \subseteq (I_2, J_2)^{\sim}$ does not necessarily hold. As a remedy to this problem, in Chapter IV we introduce the notion of the multiple closure of a set of ideals. The definition is similar to that of blowup closure, except that it checks tight closure against only one specific open affine set of some blowup scheme, rather than all affine sets in an open affine cover. It turns out that the multiple closure of some ideals in a ring is actually the contraction of the blowup closure of some larger ideals in an extension ring. Multiple closure therefore inherits all the properties of blowup closure.

The main result of Chapter IV is in Section 4.1. We prove that under mild conditions on the ring, the TI closure of a set of ideals is equal to the multiple closure of those ideals. This holds for example for finitely generated algebras over perfect fields, or for complete local rings with perfect residue field.

The strength of this result is in that it translates the TI closure of a set of ideals into the tight closure of one ideal in an extension ring of the original one. In Section 4.2, we apply this fact to address the questions on TI closure that were stated in Section 2.5. We settle the question of persistence of TI closure under ring maps, and we show that TI closure commutes with localization if and only if tight closure does (this property is not known for tight closure, but holds for integral closure).

We develop a theory of test elements for TI closure in Section 4.3. Test elements are the key ingredients for tight closure arguments, and the existence of test elements (see [HH2] or [HH3]) is one of the most important results in tight closure theory.

The existence of a test element in a ring makes it possible to determine that a given element of the ring is not in the tight closure of a given ideal: if the product of that element and the test element is not in the ideal, then that element is not in the tight closure of that ideal. However, the notion of such a uniform multiplier does not exist for integral closure (see Example 2.5.4), and so we alter the traditional definition of test elements in tight closure theory to make sense of test elements for TI closure. TI closure test elements, unlike those in tight closure, depend on the ideals that one works with.

In [HH2], Hochster and Huneke calculate specific tight closure test elements for affine algebras over fields of positive characteristic via a Jacobian theorem of Lipman and Sathaye. We use this result to describe specific test elements for the TI closure of a set of ideals. The main idea here is that if R is an affine algebra, so are the localization of its Rees rings, and so we can apply Hochster and Huneke's theorem to find tight closure test elements for the localized Rees ring, which yield TI closure test elements for the ideals in the original ring. It is worth pointing out that this result produces an easy method for calculating test elements, which involves a straightforward calculation when the defining set of equations for the algebra is known.

We treat the theory of TI closure for affine algebras over fields of characteristic zero in Section 4.5. We show that in fact, multiple closure and TI closure agree in this situation as well. Here again, we use properties of tight closure in equal characteristic zero to recover properties of TI closure in equal characteristic zero. We then define universal test elements for TI closure: these are the characteristic zero analogues of test elements in positive characteristic. With similar calculations to the positive characteristic case, we directly calculate universal test elements for a set of ideals in a finitely generated algebra over a field of characteristic p.

In Chapter V we introduce methods to calculate specific normal ideals for graded rings. The main theorem in Section 5.2 states that for a normal graded domain of the form $k[x_0, \ldots, x_m]/J$ where k is a domain and $x_0, \ldots, x_m$ are variables of positive weights $A_0, ..., A_m$, the ideal generated by all elements of degree larger than $mA - 1$ is a normal ideal. In Sections 5.3 and 5.4 we explore several examples concerning the effectiveness and the sharpness of the bound mA. Section 5.5 introduces another class of normal ideals using a lemma of [EGA]. We compare this class of ideals with those found in 5.2, and consider situations to make optimal use of each class.

CHAPTER II

Background Material

In all the discussions, we assume that all rings are commutative with identity. When we refer to a ring of characteristic p, we mean p is a positive prime integer.

Notation. If R is a ring, then $x \in R^o$ means that x is an element of R that is not in any minimal prime of R. When R has prime characteristic p, q is a power of p, and I is an ideal of R, $I^{[q]}$ denotes the ideal generated by the qth powers of the elements of I. In particular, if I is generated by $x_1, \dots, x_n$, then $I^{[q]}$ is generated by ${x_1}^q, \dots, {x_n}^q$. For a graded ring S and f a homogeneous element of S, by $S_{(f)}$ we mean the zeroth graded piece of the localized ring S_f, i.e., $S_{(f)} = (S_f)_0$.

2.1 Integral Closure

Definition 2.1.1. For rings $S \subseteq T$ the *integral closure* of S in T is the ring consisting of all $x \in T$ that satisfy an equation of the form $x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n = 0$, where $a_i \in S$ for $i = 1, ..., n$.

Definition 2.1.2. The *normalization* R' of a domain R is the integral closure of R in the field of fractions of R. If $R' = R$, R is said to be *normal.*

In the case when R is not a domain, we follow the definition in Section 9 of [M] for the normalization of R.

Definition 2.1.3. Let R be a Noetherian ring, and let $p_1, \dots, p_m$ be the minimal primes of R. Then we define the *normalization* of R, denoted by R', to be

$$(R/p_1)' \times \dots \times (R/p_m)',$$

where for $i = 1, \dots, m$, $(R/p_i)'$ is the normalization of the domain R/p_i in the field of fractions of R/p_i.

The following fact is well known; we record a proof since we were unable to find one in the literature.

Proposition 2.1.4. *Let R be a reduced Noetherian ring, and let $p_1, \ldots, p_m$ be the minimal primes of R. Then:*

(a) The minimal primes of R' are $p_1{}^\sharp, \ldots, p_m{}^\sharp$, where for $1 \leq i \leq m$,

$$p_i{}^\sharp = (R/p_1)' \times \ldots \times (R/p_{i-1})' \times (0) \times (R/p_{i+1})' \times \ldots \times (R/p_m)';$$

(b) $R'/p_i{}^\sharp \simeq (R/p_i)'$, for all i.

Proof. (a) The main point is that if $A_1, \ldots, A_n$ are rings, then the prime ideals of $A_1 \times \ldots \times A_n$ are of the form

$$p^\sharp = A_1 \times \ldots A_{i-1} \times p \times A_{i+1} \times A_n,$$

where $p \in \mathrm{Spec} A_i$, for $1 \leq i \leq n$.

The ideal $p^\sharp$ is a prime ideal, since $(A_1 \times \ldots \times A_n)/p^\sharp \simeq A_i/p$ is a domain.

Moreover, these are the only primes of $A_1 \times \ldots A_n$: Let $q \in \mathrm{Spec}(A_1 \times \ldots \times A_n)$. Observe that if $(a_1, \ldots, a_n) \in q$, then

$$(a_1, 0, \ldots, 0) = (1, 0, \ldots, 0)(a_1, \ldots, a_n) \in q.$$

Similarly $(0, \ldots, 0, a_i, 0, \ldots, 0) \in q$, $1 \leq i \leq n$. The converse of this statement is easy to see, and so we have

$$(a_1, \ldots, a_n) \in q \quad \text{iff} \quad (0, \ldots, 0, a_i, 0, \ldots, 0) \in q, \quad 1 \leq i \leq n. \tag{2.1}$$

Since q is a proper ideal of $A_1 \times \ldots \times A_n$, it follows from 2.1 that there is an element in one of the A_i, say there is an $a \in A_1$, such that $(a, 0, \ldots, 0) \notin q$.

Claim: For any $i = 2, \ldots, n$, if $b \in A_i$, then $(0, \ldots, 0, b, 0, \ldots, 0) \in q$ (where b is in the ith spot of the n-tuple $(0, \ldots, 0, b, 0, \ldots, 0)$).

Proof of Claim: Suppose for example that $b \in A_2$, but $(0, b, 0, \ldots, 0) \notin q$. Pick a nonzero $(a_1, \ldots, a_n) \in q$ (q is obviously not the zero ideal, since $A_1 \times \ldots \times A_n$ is not a domain). Then

$$(a_1, \ldots, a_n)(a, b, 0, \ldots, 0) \in q.$$

But

$$(a_1, \ldots, a_n)(a, b, 0, \ldots, 0) = (aa_1, ba_2, 0, \ldots, 0) = (a, a_2, 0, \ldots, 0)(a_1, b, 0, \ldots, 0),$$

and so $(a, a_2, 0, \ldots, 0) \in q$ or $(a_1, b, 0, \ldots, 0) \in q$. If $(a_1, b, 0, \ldots, 0) \in q$, by 2.1 we are done. Otherwise $(a, a_2, 0, \ldots, 0) \in q$, and it follows from 2.1 that $(a, 0, \ldots, 0) \in q$, which is a contradiction. This settles the claim.

From what we have shown so far, we can conclude that $q = p \times A_2 \times \ldots \times A_n$. We now need to show that p is a prime ideal of A_1. But this is clear, since $A_1/p \simeq (A_1 \times \ldots \times A_n)/q$ is a domain. The assertion in part (a) now follows.

(b) For each $i = 1, \ldots, n$, take the projection map

$$\psi_i : R' = R/{p_1}' \times \ldots \times R/{p_m}' \longrightarrow R/{p_i}',$$

where $\psi_i(a_1, \ldots, a_n) = a_i$. This is a surjective ring map, and the kernel of ψ_i is clearly ${p_i}^\sharp$. Hence the isomorphism in part (b) follows. $\square$

Definition 2.1.5. For a ring S and an ideal J of S, the *integral closure* of J in S, denoted by $\overline{J}$, is defined as

all $x \in S$ that satisfy an equation of the form $x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ where $a_i \in J^i$, $i = 1, ..., n$; or equivalently,

all $x \in S$ for which there exists a $c \in S$ that does not belong to any minimal prime of S, such that $cx^n \in J^n$ for all positive integers n ([Ho2] 1.2).

The integral closure of an ideal in a ring is closely related to the normalization of the Rees ring of that ideal; see Section 2.2 for more details.

We outline some basic properties of integral closure. Property 3 below shows that it is enough to study the integral closure of ideals in domains.

Theorem 2.1.6 (basic properties of integral closure). *Let R be a ring. Suppose that I is an ideal of R and $x \in R$.*

(1) $\overline{I}$ is an ideal of R, and $\overline{(\overline{I})} = \overline{I}$.

(2) If $h : R \longrightarrow S$ is a ring homomorphism, then

$$\overline{I}S \subseteq \overline{(IS)}.$$

If J is an integrally closed ideal of S, then the contraction of J is an integrally closed ideal of R.

(3) x is integral over I iff the image of x in R/p is integral over $(I + p)/p$ for every minimal prime p of R.

(4) Every prime ideal of R, and more generally, every radical ideal of R is integrally closed.

(5) If S is an integral extension of R, then $\overline{IS} \cap R = \overline{I}$.

(6) If R is a domain, then $x \in \overline{I}$ iff and only if for every valuation domain V containing R, $x \in IV$.

Proof. Parts 1, 2 and 4 follow easily from Definition 2.1.5. Part 3 follows from part 2 and Lemma 1.1 of [L2]. Part 5 is a special case of the lemma on page 795 of [L2]. For part 6, see page 353 of [ZS2]. $\square$

Below we state a proof of a well known feature of integral closure that we use often in the later chapters.

Proposition 2.1.7. *Let R be a domain, and let $I = (g_1, \dots, g_s)$ be an ideal in R. For every i, let S_i be the normalization of $R[g_1/g_i, \dots, g_s/g_i]$. Then $x \in \overline{I}$ if and only if $x \in IS_i$ for all $i = 1, \dots, s$.*

Proof. Suppose that $x \in \overline{I}$. Then from part 2 of Theorem 2.1.6 it follows that $x \in \overline{IS_i} = IS_i$ for all i, since $IS_i = (g_i)S_i$ is a principal ideal, and S_i is a normal ring, and so IS_i is integrally closed.

Now suppose that $x \in IS_i$ for all i. Let V be a valuation domain containing R. Since I is a finitely generated ideal, the image of I in V will be generated by one of the g_i. It follows that for some i, $S_i \subseteq V$. Then $x \in IS_i$ implies that $x \in IV$, and part 6 of Theorem 2.1.6 implies that $x \in \overline{I}$. $\square$

Corollary 2.1.8. *Let R be a Noetherian domain, and let X be a normal scheme, with a proper birational map $\pi : X \longrightarrow \mathrm{Spec}R$. Suppose that I is an ideal of R such that $I\mathcal{O}_X$ is an invertible sheaf of ideals on X. Then $I\mathcal{O}_X \cap R = \overline{I}$.*

Proof. See Proposition 6.2 of [L1] and the remark following it for the proof. $\square$

2.2 Integral Closure of Graded Rings and Rees Rings

Below we record some well known facts regarding the normalization of graded rings and Rees rings. In some cases that we could not find the proof in the literature, we outline a proof. For more facts about Rees rings we refer the reader to [V] or [HIO].

Definition 2.2.1. If R is a ring and I is an ideal of R, and t is an indeterminate over R, then the *Rees ring* $R[It]$ of I is the subring of $R[t]$ of the form

$$R[It] = R \oplus It \oplus I^2t^2 \oplus \dots .$$

Theorem 2.2.2 (integral closure of graded rings). *Let $A = \oplus_{i\in\mathbf{Z}} A_i$ be a graded ring, and B a graded A-algebra. Then the set A' of elements of B integral over A is a graded subalgebra of B.*

Proof. This is Proposition 20 in Chapter 5 of [B]. $\square$

A useful corollary that we shall use later immediately follows (this is discussed in Chapter 2 of [L1] and Proposition 4.13 of [E]).

Corollary 2.2.3. *Let $A = \oplus_{i\in\mathbf{Z}} A_i$ be a graded ring, and B a graded A-algebra. Let f be a nonzero element of A, and let $A_{(f)}$ and $B_{(f)}$ denote the zeroth graded pieces of the localized rings A_f and B_f, respectively. Then the integral closure of A_f in B_f is A'_f, and integral closure of $A_{(f)}$ in $B_{(f)}$ is $A'_{(f)}$. In other words,*

$$(A_f)' = A'_f \quad \text{and} \quad (A_{(f)})' = A'_{(f)}.$$

An immediate application of Theorem 2.2.2 is a well known description for the integral closure of Rees rings.

Theorem 2.2.4 (integral closure of Rees rings). *Let R be a ring, and I an be ideal of R, and t be an indeterminate over R. Then the integral closure of the Rees ring $R[It]$ in the ring $R[t]$ is the graded ring*

$$R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \dots ,$$

where $\overline{J}$ denotes the integral closure of the ideal J in R.

Proof. We know from Theorem 2.2.2 that the integral closure of $R[It]$ in $R[t]$ is a graded subring of $R[t]$. For a $z \in R$, suppose $zt^n \in R[t]$ is integral over $R[It]$. Then there are $a_1, \dots, a_m \in R[It]$ and a positive integer m such that

$$(zt^n)^m + a_1(zt^n)^{m-1} + \dots + a_m = 0.$$

We can rewrite this equation as a sum of homogeneous terms in $R[t]$, each of which is equal to zero. In particular, if we take the homogeneous term of degree mn, we have

$$(zt^n)^m + b_1(zt^n)^{m-1} + \dots + b_m = 0$$

where for $i = 1, \dots, m$, each b_i is a degree ni element of $R[It]$. In other words, $b_i = c_i t^{ni}$, where $c_i \in I^{ni}$ (c_i could be zero). We rewrite the equation as

$$z^m t^{mn} + c_1 z^{m-1} t^{mn} + \dots + c_m t^{mn} = 0.$$

Dividing this equation by t^{mn}, we obtain the equation

$$z^m + c_1 z^{m-1} + \dots + c_m = 0,$$

with $c_i \in I^{ni}$, which implies that $z \in \overline{I^n}$.

The inverse of this argument shows that any homogeneous element of $R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \dots$ satisfies an integral equation over $R[It]$, and this finishes the proof. □

Corollary 2.2.5. *Let R be a normal domain, and I an be ideal of R, and t be an indeterminate over R. Then the normalization of $R[It]$ in its field of fractions is*

$$R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \dots ,$$

where $\overline{J}$ denotes the integral closure of the ideal J in R.

Proof. The field of fractions of $R[It]$ is the same as the field of fractions of $R[t]$, and since $R[It]$ is a subring of $R[t]$, the normalization of $R[It]$ is contained in the normalization of $R[t]$. But $R[t]$ is normal, and hence integrally closed in its field of fractions. The normalization of $R[It]$ will therefore be the integral closure of $R[It]$ in $R[t]$, which by Theorem 2.2.4 is equal to $R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \dots$. □

2.3 Tight Closure

Tight closure is an operation on ideals in Noetherian rings of positive prime characteristic p that contain a field. The main idea for characteristic p arguments is that when R is a ring of prime characteristic p, the Frobenius map (the map from R to R that takes an element r to its pth power r^p) will be a ring homomorphism. The theory is extended to rings of characteristic zero via the method of reduction to characteristic p. For details and references on tight closure, and for proofs of the facts posed below, see [HH1] or [Hu]. Another good source for an introduction to the topic and some applications is [Sm3].

Definition 2.3.1. Let R be a Noetherian ring of characteristic $p > 0$. Let I be an ideal of R. Then the *tight closure* of I , denoted by I^*, is the collection of all $z \in R$ for which there exists $c \in R^o$ such that $cz^q \in I^{[q]}$ for all large $q = p^e$. We say that an ideal I is *tightly closed* if $I = I^*$.

It is easy to check that the tight closure of an ideal is an ideal. Below we list some properties of tight closure that we use in the later sections.

Theorem 2.3.2 (basic properties of tight closure). *Let R be a Noetherian ring of prime characteristic p, and let I and J be ideals of R.*

(1) $(I^*)^* = I^*$, and $I \subseteq I^* \subseteq \overline{I}$.

(2) If $I \subseteq J$, then $I^* \subseteq J^*$.

(3) If I and J are tightly closed, then so is $I \cap J$.

(4) An element $x \in R$ is in I^* iff the image of x in R/p is in the tight closure of $(I + p)/p$ for every minimal prime p of R.

(5) If R is a regular ring, then for all ideals I of R, $I^* = I$.

Proof. See [HH1] Proposition 4.1 and Theorem 4.4, and [Hu] Theorem 1.3 for proofs. □

Property 4 above reduces the study of tight closure to the case of domains. In general the integral closure of an ideal is much larger than its tight closure.

Example 2.3.3. Let $R = k[x, y]$ be a polynomial ring over a field of characteristic p. Let $I = (x^2, y^2)$. Then $\overline{I} = (x^2, xy, y^2)$, while $I^* = I = (x^2, y^2)$ since R is a regular ring.

Theorem 2.3.4 (tight closure from contractions). *Let $R \subseteq S$ be a module-finite extension of Noetherian domains of positive characteristic p. Let I be an ideal of R. Then $(IS)^* \cap R \subseteq I^*$.*

Proof. See [Hu], Theorem 1.7 for a proof. □

In particular, if $R \subseteq S$ is an integral extension of a Noetherian domain of positive prime characteristic p, then every element in $(IS)^* \cap R$ will be in $(IT)^* \cap R$ for some finite extension T of R, and so it follows that $(IS)^* \cap R \subseteq I^*$.

A key element in tight closure theory is the existence of *test elements.*

Definition 2.3.5. Let R be a Noetherian ring of characteristic p. An element $c \in R^o$ is called a *test element* for R, if for every ideal I of R and every $x \in I^*$, $cx^q \in I^{[q]}$ for all $q = p^e$, $e \geq 0$. The ideal generated by the test elements for R is called the *test ideal* of R, and is denoted by $\tau(R)$.

Under certain mild conditions on the ring, Hochster and Huneke proved that test elements exist. Before stating this result, we introduce some terminology.

Note 2.3.6. If R is a ring of characteristic p, R is *essentially of finite type* over a ring S if it is a localization of a finitely generated algebra over S. R is *F-finite* if the Frobenius map is a finite map. Below, as well as in the next chapters, we frequently assume that our rings are F-finite or essentially of finite type over an excellent local ring. It is worth pointing out that these are not very restrictive conditions on a ring. For example, any reduced finitely generated algebra over a perfect field or any complete local ring with perfect residue field will satisfy these properties. Both these properties are preserved after localizing at a multiplicative set in R, and after taking finitely generated algebra extensions.

Theorem 2.3.7 (existence of test elements). *Let R be a Noetherian reduced ring of positive characteristic p, and suppose that R is essentially of finite type over an excellent local ring, or that R is F-finite. Let $c \in R^o$ be such that R_c is regular. Then c has a power that is a test element for R, and remains so after localizing or completing R.*

Proof. See [HH5] Theorem 3.4, or [Hu] Theorem 2.1 for the proof. □

A very useful property of tight closure follows from this theorem.

Theorem 2.3.8 (persistence of tight closure). *Let $R \longrightarrow S$ be homomorphism of Noetherian rings of positive characteristic p. Suppose that R is essentially of finite type over an excellent local ring, or that R_{red} (i.e. R modulo the nilradical of R) is F-finite. Then*

$$I^*S \subseteq (IS)^*.$$

Proof. See [HH3] Theorem 6.24, or [Hu] Theorem 2.3 for the proof. □

Remark 2.3.9. Persistence improves several of the results stated earlier: If tight closure persists for the map $f : R \longrightarrow S$ of Noetherian rings, then the contraction of a tightly closed ideal of S will be a tightly closed ideal of R. To see this, let $J \subseteq S$ be tightly closed in S, and take $u \in (f^{-1}(J))^*$. Then $f(u) \in (f^{-1}(J)S)^* \subseteq J^* = J$ (see Theorem 2.3.2). Hence $f(u) \in J$, and so $u \in f^{-1}(J)$.

Another useful observation is that if $f : R \longrightarrow S$ is any module finite extension of Noetherian domains of characteristic p, then tight closure persists under f. In this case, if I is an ideal of R, then $(IS)^* \cap R = I^*$: Theorem 2.3.4 implies that $(IS)^* \cap R \subset I^*$, and persistence ensures that $I^* \subseteq (IS)^* \cap R$.

A significant application of tight closure theory has been a simple proof to (in fact a generalization of) the Briançon-Skoda theorem.

Theorem 2.3.10 (tight closure Briançon-Skoda theorem). *Let R be a ring of characteristic p. Let I be any ideal generated by n elements. Then for all $w \geq 0$*

$$\overline{I^{n+w}} \subseteq (I^{w+1})^*.$$

Proof. This is [HH1] Theorem 5.4. □

2.3.1 Tight Closure in Equal Characteristic Zero

The method of reduction to characteristic p allows us to extend tight closure theory to rings of equal characteristic zero, that is, rings of characteristic zero containing a field. Consequently, tight closure results in characteristic p find analogous statements in characteristic zero. In this section, we briefly outline the method of reduction to characteristic p and give the definition of tight closure in characteristic zero for affine algebras over a field. For a brief but more detailed treatment of this topic see [Ho1] or [HH1]. The complete and general source on tight closure theory in equal characteristic zero is [HH2].

The definition of tight closure in equal characteristic zero is based on the existence of *descent data*, which provide us with a finitely generated subalgebra R_D of the ring over a finitely generated algebra D over the integers. We can then look at fibers of R_D over maximal ideals of D, which are rings of positive characteristic over finite (and therefore perfect) fields. We define tight closure for R via the (usual positive characteristic) definition of tight closure for these fibers.

Definition 2.3.11. Let R be a finitely generated algebra over a field K of characteristic zero, let I be an ideal of R and let $x \in R$. By *descent data* for R, I, and x we mean a triple (D, R_D, I_D), where D is a finitely generated $\mathbf{Z}$-subalgebra of K, R_D is a finitely generated D-subalgebra of R, and I_D is an ideal of R_D such that:

(a) I_D and R_D/I_D are D-free.

(b) The canonical map $K \otimes_D R_D \to R$ induced by the inclusions of K and R_D in R is a K-algebra isomorphism.

(c) $I = I_D R$.

(d) $x \in R_D$.

Descent data always exist: Let

$$R = K[x_1, \dots, x_n]/(g_1, \dots, g_s)$$

where K is a field of characteristic zero and $g_1, \dots, g_s$ are polynomials in $K[x_1, \dots, x_n]$. Let $h_1, \dots, h_t$ be elements of $K[x_1, \dots, x_n]$ whose images in R generate I, and let u be an element in $K[x_1, \dots, x_n]$ whose image in R is x. Then take D to be the algebra over $\mathbf{Z}$ generated by the coefficients of $g_1, \dots, g_s, h_1, \dots, h_t, u$, and let

$$R_D = D[x_1, \dots, x_n]/(g_1, \dots, g_s) .$$

Let I_D be the ideal generated be the images of $h_1, \dots, h_t$ under the map

$$D[x_1, \dots, x_n] \longrightarrow R_D$$

and let x_D be the image of x under this map. To ensure that condition (a) in Definition 2.3.11 is satisfied, one can replace D by its localization at a suitable nonzero element of D (This follows from the lemma of generic freeness due to Hochster and Roberts: Lemma 8.1 of [HR]). Conditions (b)-(d) easily follow.

Definition 2.3.12. Let R be a finitely generated algebra over a field K of characteristic zero and let I be an ideal of R. We say that an element x of R is in the *tight closure* of I, denoted by I^*, if there exist descent data (D, R_D, I_D) such that for every maximal ideal m of D, if $k = D/m$, then $x_k \in {I_k}^*$ in R_k, where the subscript k denotes images after applying $k \otimes_D$. If $I = I^*$, then we say that I is *tightly closed.*

In fact, by replacing D by a localization at a single element, one can see that it suffices to check that for *almost all* $m \in \mathrm{MaxSpec} D$, i.e. for all m in a Zariski dense open subset of $\mathrm{MaxSpec} D$, if $k = D/m$, then $x_k \in {I_k}^*$.

The following theorem follows from [HH2] 2.5.2 and 2.5.3:

Theorem 2.3.13 (independence of choice of descent and uniform multipliers). *Let K be a field of characteristic zero, let R be a finitely generated K-algebra, let I be an ideal of R and let $u \in R$. Let (D, R_D, I_D) be descent data for R, I, and u.*

(a) If $u \in I^*$, then for every maximal ideal m of D, if $k = D/m$, then $x_k \in {I_k}^*$ in R_k.

(b) There is an element c_D of R_D^o such that $u \in I^*$ iff for almost all maximal ideals m of D and $k = D/m$, $c_k {u_k}^q \in {I_k}^{[q]}$ for all positive powers q of p.

Part (b) involves the existence of *universal test elements*; these are the characteristic zero analogues of test elements.

Definition 2.3.14. Let $D \supseteq \mathbf{Z}$ be a domain finitely generated over $\mathbf{Z}$, and let R_D be a finitely generated D-algebra. We say that $c_D \in R_D$ is a *universal test element* for $D \to R_D$ if after replacing D by a suitable localization at a nonzero element of D the following conditions are satisfied:

(a) $c_D \in R_D^o$.

(b) For every homomorphism $D \to \Lambda$, where Λ is a regular domain of positive characteristic, c_Λ is a completely stable test element for R_Λ; or, equivalently, for every homomorphism $D \to \Lambda$, where Λ is a regular ring of positive characteristic, c_Λ is a completely stable test element for R_Λ (that is, c_Λ remains a test element after any localization R_Λ, and after completing a local ring of R_Λ at its maximal ideal).

Universal test elements are defined in this general way partially because the Jacobian theorem of Lipman and Sathaye (see [HH2]) produces elements with such a strong property. In practice, when R is reduced and equidimensional, $I \subseteq R$, $u \in I^*$ and $c_D \in R_D$ is a universal test element, then for almost all $m \in \mathrm{MaxSpec}D$, if $k = D/m$, we have $c_k {u_k}^q \in {I_k}^{[q]}$ for all positive powers q of p.

We will state a theorem due to Hochster and Huneke which enables us to explicitly calculate universal test elements in Section 4.5 (Theorem 4.5.5).

2.4 Tight Integral Closure

Tight integral closure (or TI closure) is an operation on a *set* of ideals in a Noetherian ring that generalizes the ideas of tight and integral closure. This notion was introduced by Hochster ([Ho2]) in 1998. Below, we give a brief introduction to TI closure, outlining its properties and the cases where it is computable. We also state the questions on TI closure, some of which will be answered later in this thesis.

Definition 2.4.1. Let R be a Noetherian commutative ring of prime characteristic p, and let $I_1, \ldots, I_n$ be ideals in R. We define $x \in R$ to be in $(I_1, \ldots, I_n)^{\underline{*}}$, called the *tight integral closure* (or *TI closure*) of $I_1, \ldots, I_n$, if and only if there exists an element $c \in R^o$ such that $cx^q \in I_1^q + \ldots + I_n^q$, for all large powers q of p.

Note that if $n = 1$, $(I)^{\underline{*}} = \overline{I}$, and if $I_1, \ldots, I_n$ are all principal ideals, then $(I_1, \ldots, I_n)^{\underline{*}} = (I_1 + \ldots + I_n)^*$.

Notation. If $\mathcal{I} = \{I_1, \ldots, I_n\}$ is a set of ideals, $\mathcal{I}^{\underline{*}}$ denotes $(I_1, \ldots, I_n)^{\underline{*}}$. For a positive integer k, $\mathcal{I}^k$ denotes the set consisting of all products of k elements of $\mathcal{I}$, and $\mathcal{I}^{[k]}$ will denote the ideal ${I_1}^k + \ldots + {I_n}^k$. For example, if $\mathcal{I} = \{I_1, I_2\}$, then $\mathcal{I}^2 = \{{I_1}^2, I_1 I_2, {I_2}^2\}$ and $\mathcal{I}^{[2]} = {I_1}^2 + {I_2}^2$.

Theorem 2.4.2 (basic properties of TI closure: [Ho2] 1.4). *Suppose that R is a Noetherian ring of prime characteristic p, and let $\mathcal{I} = \{I_1, \ldots, I_n\}$ and $\mathcal{J} = \{J_1, \ldots, J_n\}$ denote finite sets of ideals of R.*

(a) $\mathcal{I}^{\underline{*}}$ is a tightly closed ideal of R that contains $\overline{I_t}$ for every t, and is contained in $\overline{I_1+\ldots+I_n}$, i.e.,

$$(\overline{I_1}+\ldots+\overline{I_n})^* \subseteq \mathcal{I}^{\underline{*}} \subseteq \overline{I_1+\ldots+I_n}.$$

Moreover, there is a single element $c \in R^o$ such that $cu^q \in \mathcal{I}^{[q]}$ for all $u \in \mathcal{I}^{\underline{*}}$ and $q >> 0$.

(b) If $I_t \subseteq J_t$ for $1 \le t \le n$ then $\mathcal{I}^{\underline{*}} \subseteq \mathcal{J}^{\underline{*}}$.

(c) If $I_t \subseteq J_t \subseteq \overline{I_t}$ for $1 \le t \le n$ then $\mathcal{I}^{\underline{*}} = \mathcal{J}^{\underline{*}}$. In particular,

$$(\overline{I_1},\ldots,\overline{I_n})^{\underline{*}} = (I_1,\ldots,I_n)^{\underline{*}}.$$

(d) TI closure may be tested modulo minimal primes: Let $P_1,\ldots,P_s$ be minimal primes of R. Then $x \in \mathcal{I}^{\underline{*}}$ (in R) if and only if for every i, $1 \le i \le s$, the image of x in R/P_i is in $(I_1(R/P_i),\ldots,I_n(R/P_i))^{\underline{*}}$ in R/P_i.

(e) If J is a nilpotent ideal of R, then $x \in \mathcal{I}^{\underline{*}}$ if and only if the image of x in R/J is in $(I_1(R/J),\ldots,I_n(R/J))^{\underline{*}}$. If J is the nilradical of R, then $\mathcal{I}^{\underline{*}}$ is the inverse image in R of $(I_1(R/J),\ldots,I_n(R/J))^{\underline{*}}$. In particular, $\mathcal{I}^{\underline{*}}$ contains all nilpotents of R.

(f) If one of the ideals I_t in the set $\mathcal{I}$ is contained in the integral closure of another, I_t may be omitted without changing the TI closure of the set. This principle may be applied repeatedly. Thus, for example, if every I_t for $t > k$ is contained in the integral closure of one of the ideals $I_1,\ldots,I_k$, then $\mathcal{I}^{\underline{*}} = (I_1,\ldots,I_k)^{\underline{*}}$.

(g) If $\mathcal{I}$ is the union of several finite sets of ideals $\mathcal{I}_1,\ldots,\mathcal{I}_r$, then

$$\mathcal{I}^{\underline{*}} \subseteq (\mathcal{I}_1{}^{\underline{*}},\ldots,\mathcal{I}_r{}^{\underline{*}})^{\underline{*}}.$$

As is transparent above, in some special cases one can calculate TI closure using tight or integral closure. Moreover,

Theorem 2.4.3 ([Ho2] 1.9). *Let $R = D[x_1,\ldots,x_n]$ be a polynomial ring over a Noetherian ring D of positive prime characteristic p, and let $\mathcal{I} = \{I_1,\ldots,I_n\}$ be a finite set of ideals each of which is generated by monomials in the variables x_i. Then*

$$\mathcal{I}^{\underline{*}} = \overline{I_1}+\ldots+\overline{I_n}.$$

Theorem 2.4.4 ([Ho2] 3.1). *Let k be a field of characteristic p and let $R_1,\ldots,R_n$ denote a finite family of Noetherian k-algebras. We assume that we are in one of the two cases:*

(1) Every R_s is of finite type over k.

(2) Every (R_s,m_s) is a complete local ring, and for every s, $k \simeq R_s/m_s$.

Let R denote the tensor product of the R_s over k in case (1), or the complete tensor product of the R_s over k in case (2). Let I_s be an ideal of R_s, $1 \leq s \leq n$. Let $\mathcal{I} = \{I_1 R, \ldots, I_n R\}$. Then

$$\mathcal{I}^{\underline{*}} = \overline{I_1} R + \ldots + \overline{I_n} R.$$

Theorem 2.4.4 in particular calculates the TI closure of a set of ideals generated by disjoint sets of variables in a polynomial ring. The following theorem generalizes the Briançon-Skoda theorem.

Theorem 2.4.5 (TI closure Briançon-Skoda theorem: [Ho2] 2.3). *Let R be a Noetherian ring of characteristic $p > 0$, and let $\mathcal{I} = \{I_1, \ldots, I_n\}$ be a set of n ideals, let $k \geq 1$ be an integer, and let $I = I_1 + \ldots + I_n$. Then*

$$\overline{I^{n+k-1}} \subseteq \mathcal{I}^{k\underline{*}}.$$

In particular,

$$\overline{I^n} \subseteq \mathcal{I}^{\underline{*}}.$$

An application of Theorems 2.4.4 and 2.4.5 is

Theorem 2.4.6 ([Ho2] 0.1). *Let R be a polynomial ring $k[x_0, \ldots, x_m]$ where k is a field of arbitrary characteristic and the x_i are variables. Suppose that the variables are partitioned into n sets, $n \geq 1$ and that I_j is an ideal of R whose generators involve only variables from the jth set, $1 \leq j \leq n$. Then $\overline{(I_1 + \ldots + I_n)^n} \subseteq \overline{I_1} + \ldots + \overline{I_n}$.*

Example 2.4.7. Let R be the polynomial ring $k[x, y, u, v]$ over a field k of characteristic p, and let $I = (u^2, v^2)$ and $J = (x^2, y^2)$ be ideals of R. Then it follows from the TI closure version of the Briançon-Skoda theorem, that

$$(u, v, x, y)^4 = \overline{(u^2, v^2, x^2, y^2)^2} \subseteq \overline{(u^2, v^2)} + \overline{(x^2, y^2)} = (u^2, v^2, uv, x^2, y^2, xy).$$

This is an improvement of the tight closure Briançon-Skoda theorem, which would yield

$$(u, v, x, y)^8 = \overline{(u^2, v^2, x^2, y^2)^4} \subseteq (u^2, v^2, x^2, y^2)^* = (u^2, v^2, x^2, y^2).$$

2.4.1 Tight Integral Closure in Equal Characteristic Zero

TI closure can be defined for affine algebras over fields of characteristic zero by the method of reduction to characteristic p, described in Section 2.3.

Definition 2.4.8. Let R be a finitely generated algebra over a field K of characteristic zero, and let $\mathcal{I} = \{I_1, \ldots, I_n\}$ be a set of ideals of R. We say that an element x of R is in the *tight integral closure of* $\mathcal{I}$, denoted by $\mathcal{I}^{\underline{*}}$, if there exist descent data $(D, R_D, I_{1,D}, \ldots, I_{n,D})$, such that for every maximal ideal m of D, if $k = D/m$, then $x_k \in \mathcal{I}_k^{\underline{*}}$, where $\mathcal{I}_k$ denotes the set of ideals $\{I_{1,k}, \ldots, I_{n,k}\}$, with $I_{t,k} = k \otimes_k I_{t,D}$.

In [Ho2], characteristic zero analogues of several of the theorems mentioned in this section are stated.

2.5 Questions

As can be seen above, TI closure is able to generalize several statements, and to tie up tight closure and integral closure into one definition. However, there are many useful properties that both tight and integral closure satisfy, but have been turned out to be difficult to verify for TI closure using Definition 2.4.1. In [Ho2], Hochster stated the following questions:

Question 2.5.1. Does TI closure persist? That is, if $h : R \longrightarrow S$ is a homomorphism of Noetherian rings of characteristic p, and $I_1, \ldots, I_n$ are ideals of R, then is it true that $(I_1, \ldots, I_n)^{\underline{*}}\, S \subseteq (I_1 S, \ldots, I_n S)^{\underline{*}}$?

This property holds for both tight closure (with mild conditions on the ring, see Theorem 2.3.8) and for integral closure (see Theorem 2.1.6). It is easy to show that it also holds for for TI closure if $h(R^o) \subseteq S^o$; this happens for example when h is a flat map, or if h is any injective map of domains. In general, the lack of a test element theory for TI closure makes the problem obscure in its original setting. This question will be answered in Section 4.2.

Question 2.5.2. If R is a Noetherian ring of characteristic p, $I_1, \ldots, I_n$ are ideals of R and $c \in R^o$ and $x \in R$ are such that

$$cx^{p^e} \in {I_1}^{p^e} + \ldots + {I_n}^{p^e},$$

for infinitely many e (rather than *all large* e), then can one conclude that $x \in (I_1, \ldots, I_n)^{\underline{*}}$?

This is again a property that holds for both tight and integral closure, and it is reasonable to expect it for TI closure. In Section 4.2 we give an affirmative answer to this question.

Question 2.5.3. Can one develop a theory of test elements for TI closure (see Definition 2.3.5)?

Such a theory exists for tight closure (see Section 2.3), but it is not possible to define such a notion for integral closure that does not depend on the ideal:

Example 2.5.4. Let R be the polynomial ring $k[x, y]$, where k is a field. If $c \in R - \{0\}$ were an integral closure test element for R, then one would have $cz^m \in I^m$ for all ideals I of R, all $z \in \overline{I}$ and all positive integers m.

Now take the family of ideals $\{I_n\}_{n \in \mathbf{N}}$, where for each n, $I_n = (x^{2n}, y^{2n})$. Then $x^n y^n \in \overline{I_n}$ for all n, since $(x^n y^n)^2 = x^{2n} y^{2n} \in {I_n}^2$. As c is a test element, this implies that $cx^n y^n \in (x^{2n}, y^{2n})$ for all n. Since this holds in the polynomial ring $k[x, y]$, one can reduce to the case where c is a monomial in $R = k[x, y]$, which is not possible, since the degree of c will have to grow larger when n gets large.

Still, one could hope to find elements c that work for Definition 2.4.1, but depend on the ideals $I_1, \ldots, I_n$. Such specific elements of R are introduced in Section 4.3.

Question 2.5.5. Does TI closure commute with localization?

It is known that integral closure commutes with localization, and the same is conjectured for tight closure. In Section 4.2 we show that the question of TI closure commuting with localization is equivalent to the question of tight closure commuting with localization.

Question 2.5.6. Let R be an affine algebra over a field K of characteristic zero, and $\mathcal{I}$ be a set of ideals in R and $x \in \mathcal{I}^*$. Let $(D, R_D, \mathcal{I}_D)$ be descent data. Then can one find a $c \in R_D^o$ such that for all $m \in \mathrm{MaxSpec} D$ and $k = D/m$, $c_k {x_k}^q \in {\mathcal{I}_k}^{[q]}$ for all positive powers q of p?

The last question will be addressed in Section 4.5.

CHAPTER III

Blowup Closure

In this chapter we explore a new notion: the blowup closure of a set of ideals. Here, we are motivated by the fact that the extension of an ideal I to its blowup scheme $\mathrm{Proj}R[It]$ is locally principal. For a set of ideals, we consider the blowup scheme of their product, and each one of the original ideals will be locally principal there. This is the simplest situation to handle several ideals at the same time. We investigate properties of this operation, and compare it with tight, integral, and TI closure. Blowup closure is not in general equal to TI closure, but in some certain cases in which TI closure is explicitly computable, we show that blowup closure can be calculated as well. In particular, similar to TI closure, blowup closure generalizes tight and integral closure: if all the ideals are principal, their blowup closure is equal to the tight closure of their sum, and the blowup closure of one ideal is equal to the integral closure of that ideal.

Although the definition of blowup closure makes sense for any commutative ring of positive characteristic, we immediately restrict ourselves to rings R such that R is either essentially of finite type over an excellent local ring, or R_{red} is F-finite. The reason is that tight closure persists for maps from such rings (Theorem 2.3.8), and this property simplifies most blowup closure arguments. These conditions are not very restrictive, since the class of such rings includes most rings that one would normally encounter in commutative algebra and in algebraic geometry (see Note 2.3.6).

Notation. Here, if S is a ring, by $\mathcal{M}(S)$ we mean the set of all minimal primes of S. By S' we mean the normalization of S (see Definition 2.1.3). If S is graded and f is a homogeneous element of S, then by $S_{(f)}$ we mean the zeroth graded piece of the localized graded ring S_f, i.e. $S_{(f)} = (S_f)_0$.

3.1 Definition and Basic Facts

Definition 3.1.1. Let R be a Noetherian commutative ring of prime characteristic p, and let $I_1, \dots, I_n$ be ideals in R. We define $x \in R$ to be in $(I_1, \dots, I_n)^\sim$, called the *blowup closure* of $I_1, \ldots, I_n$, if and only if for every affine open set of the blowup of the product ideal $I = I_1 \ldots I_n$, if S is the coordinate ring of that affine set, then $x \in (JS)^*$, where $J = I_1 + \ldots + I_n$.

The following discussion shows how for certain rings, one can reduce the process of checking if an element x is in $(I_1, \ldots, I_n)^{\sim}$ to checking if it is in $(JS_i)^*$, $1 \leq i \leq m$, for any fixed open affine cover $\mathrm{Spec}S_1, \ldots, \mathrm{Spec}S_m$ of the blowup of I.

Lemma 3.1.2. *Let R be a ring of positive characteristic p, and let J be an ideal of R. Then for $x \in R$, $x \in J^*$ if and only if there are $g_1, \ldots, g_s \in R$ that generate the unit ideal in R, and for each j, $1 \leq j \leq s$, $x \in (J_{g_j})^*$.*

Proof. If $x \in J^*$, it is immediate that $x \in (JR_f)^*$ for all $f \in R$, therefore one direction is clear. Suppose we are given a sequence of elements $g_1, \ldots, g_s \in R$ such that $(g_1, \ldots, g_s) = R$, and for each j, $1 \leq j \leq s$, $x \in (J_{g_j})^*$.

For each j, let c_j be such that $c_j x^{p^e} \in {J_{g_j}}^{[p^e]}$ for all large enough e. One can replace each c_j by its product with a large enough power of g_j, so that $c_j \in R$. Let c be the product of all the c_j's for $j = 1, \ldots, s$. Then $c \in R$ and $cx^{p^e} \in {J_{g_j}}^{[p^e]}$ for all $j = 1, \ldots, s$, and all large enough e.

Fix p^e. Then for each j, there is some power N_j of g_j such that $g_j^{N_j} cx^{p^e} \in J^{[p^e]}$. Let q be a power of p that is larger than all the N_j for $j = 1, \ldots, s$. Then for all j, $g_j^q cx^{p^e} \in J^{[p^e]}$. On the other hand, since $(g_1, \ldots, g_s)$ is the unit ideal, so is $(g_1^q, \ldots, g_s^q)$, and so it follows that $cx^{p^e} \in J^{[p^e]}$. Therefore $x \in J^*$.

□

Now, let X be a scheme, and let $\mathcal{J}$ be a sheaf of ideals on X. Suppose that there is an open affine cover $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of X, such that for each i, $\mathcal{U}_i = \mathrm{Spec}B_i$, where B_i is essentially of finite type over an excellent local ring, or $(B_i)_{red}$ is F-finite. Suppose $x \in \mathcal{J}(\mathcal{U}_i)^*$ in $\mathcal{O}_X(\mathcal{U}_i)$, for $i = 1, \ldots, n$. Let $\mathcal{U}$ be any open affine set of X. We will show that $x \in \mathcal{J}(\mathcal{U})^*$.

One can write $\mathcal{U}$ as $(\mathcal{U} \cap \mathcal{U}_1) \cup \ldots \cup (\mathcal{U} \cap \mathcal{U}_n)$. If $\mathcal{U} = \mathrm{Spec}\mathcal{A}$, then one can refine this cover of $\mathcal{U}$ into

$$\mathrm{Spec}A_{f_1} \cup \ldots \cup \mathrm{Spec}A_{f_s},$$

where the elements $f_1, \ldots, f_s$ of A generate the unit ideal of A. So for every j, we have an inclusion $\mathrm{Spec}A_{f_j} \subseteq \mathcal{U}_i$, for some i, $1 \leq i \leq n$. This corresponds to a homomorphism of rings $B_i \longrightarrow A_{f_j}$. Since $x \in \mathcal{J}(\mathrm{Spec}B_i)^*$, from the persistence of tight closure it follows that $x \in \mathcal{J}(\mathrm{Spec}A_{f_j})^*$. Since the $f_1, \ldots, f_s$ generate the unit ideal, Lemma 3.1.2 implies that $x \in \mathcal{J}(\mathcal{U})^*$.

We have thus proved that:

Theorem 3.1.3. *Let R be a Noetherian commutative ring of prime characteristic p, such that R is either essentially of finite type over an excellent local ring, or R_{red} is F-finite. Let $I_1, \dots, I_n$ be ideals in R. If x is an element of R, then $x \in (I_1, \dots, I_n)^{\sim}$ if and only if for every element f of a fixed set of generators for the product ideal $I = I_1 \dots I_n$,*

$$x \in (JR[It]_{(ft)})^*,$$

where $J = I_1 + \dots + I_n$, and $R[It]_{(ft)}$ is the zeroth graded piece of the localized Rees ring $R[It]_{ft}$.

Note 3.1.4. Let R be a ring as above, and $I_1, \dots, I_n$ be ideals of R with $I = I_1 \dots I_n$ and $J = I_1 + \dots + I_n$. Let $\mathcal{G}$ be a fixed set of generators for the ideal I such that every f in $\mathcal{G}$ is of the form $f = f_1 \dots f_n$, where each f_i is a member of a fixed set of generators $f_1^i, \dots, f_{s_i}^i$ for I_i. For future reference we show what

$$(JR[It]_{(ft)})^*$$

looks like in practice.

$$\begin{aligned}
(JR[It]_{(ft)})^* &= ((I_1 + \dots + I_n)R[I_1 \dots I_n t]_{(f_1 \dots f_n t)})^* \\
&\simeq ((I_1 + \dots + I_n)R[\tfrac{I_1}{f_1}\tfrac{I_2}{f_2} \dots \tfrac{I_n}{f_n}])^* \\
&= ((f_1, \dots, f_n)R[\tfrac{I_1}{f_1}\tfrac{I_2}{f_2} \dots \tfrac{I_n}{f_n}])^* \\
&= ((f_1, \dots, f_n)R[\tfrac{f_1^1}{f_1}, \dots, \tfrac{f_{s_1}^1}{f_1}, \dots, \tfrac{f_1^n}{f_n}, \dots, \tfrac{f_{s_n}^n}{f_n}])^* \\
&= ((f_1, \dots, f_n)R[\tfrac{I_1}{f_1}, \tfrac{I_2}{f_2}, \dots, \tfrac{I_n}{f_n}])^*.
\end{aligned}$$

3.2 Blowup Closure Can Be Tested Modulo Minimal Primes

Theorem 3.2.1. *Let R be a Noetherian ring of positive characteristic such that R is either essentially of finite type over an excellent local ring, or R_{red} is F-finite, and let $I_1, \dots, I_n$ be ideals in R. Then $x \in (I_1, \dots, I_n)^{\sim}$ if and only if $\overline{x} \in (I_1R/p, \dots, I_nR/p)^{\sim}$, for all minimal primes p of R, where $\overline{x}$ is the image of x in R/p.*

We first specify the structure of the minimal primes of Rees rings, partially following an argument in [M].

Proposition 3.2.2 (minimal primes of Rees rings). *Let R be a Noetherian ring, and let $p_1, \dots, p_m$ be the minimal primes of R. If I is any ideal of R, then:*

(a) The minimal primes of $R[It]$ are $\hat{p_1}, \dots, \hat{p_m}$, where for $1 \le i \le m$,

$$\hat{p_i} = p_i R[t] \cap R[It] = p_i \oplus (I \cap p_i)t \oplus (I^2 \cap p_i)t^2 \oplus \dots ;$$

(b) For all primes p of R, $R[It]/\hat{p} \simeq (R/p)[I(R/p)t]$.

Proof. (a) Notice that if $p \in \operatorname{Spec}R$, then $\hat{p}$ is in $\operatorname{Spec}R[It]$. This is because $pR[t]$ is a prime ideal of $R[t]$, and $pR[It]$ is just a contraction of $pR[t]$ under the inclusion $R[It] \subseteq R[t]$. The same argument works for primary ideals.

If $(0) = q_1 \cap \ldots \cap q_s$ is a primary decomposition of (0) in the ring R, then $(0) = \hat{q_1} \cap ... \cap \hat{q_s}$ will be a primary decomposition of (0) in $R[It]$. So if $p_1, \ldots, p_m$ are minimal primes of R, then $\hat{p_1}, \ldots, \hat{p_m}$ will be the minimal primes of $R[It]$.

(b) For $p \in \operatorname{Spec}R$, construct the map

$$\phi : R[It] \longrightarrow R/p[I(R/p)t]$$

with $\phi(xt^n) = \overline{x}t^n$, where $\overline{x}$ is the image of x in R/p.

This is a surjective homomorphism of graded rings. To find the kernel, we observe that $\phi(xt^n) = 0$ if and only if $\overline{x}t^n = 0$. So xt^n is in the kernel of ϕ if and only if $x \in I^n \cap p$. So the kernel of ϕ is equal to $\hat{p}$, and hence we have an isomorphism $R[It]/\hat{p} \simeq (R/p)[I(R/p)t]$. □

Corollary 3.2.3. *Let R be a Noetherian ring, and let I be an ideal of R. Suppose ft^n is a homogeneous element of $R[It]$ for some positive integer n.*

(a) There is a one to one correspondence between the minimal primes of $R[It]$ that do not contain ft^n, and the minimal primes of $R[It]_{(ft^n)}$.

(b) Suppose $f \in I$ and p is a minimal prime of R not containing f. Let $\hat{p}$ be the minimal prime of $R[It]$ corresponding to p, and let p' be the minimal prime of $R[It]_{(ft)}$ corresponding to $\hat{p}$. Then

$$\frac{R[It]_{(ft)}}{p'} \simeq \left(\frac{R[It]}{\hat{p}}\right)_{(ft)} \simeq (R/p[(IR/p)t])_{(ft)}\,.$$

Proof. The statement of part (a) is equivalent to saying that there is a one to one correspondence between $\mathcal{M}(R[It]_{ft^n})$ and $\mathcal{M}(R[It]_{(ft^n)})$.

When $n = 1$, we have $R[It]_{ft} \simeq R[It]_{(ft)}[u, u^{-1}]$. To see this, notice that $R[It]_{(ft)} \simeq R[g_1/f, \ldots, g_s/f]$, where $g_1, \ldots, g_s$ is a fixed set of generators for I. We can then define the map

$$R[g_1/f, \ldots, g_s/f][u, u^{-1}] \longrightarrow R[It]_{ft}$$

by sending u^m to $(ft)^m$ for all nonzero integers m. It is easy to check that this map is an isomorphism, and it follows that all members of $\mathcal{M}(R[It]_{ft})$ are extensions of those in $\mathcal{M}(R[It]_{(ft)})$.

If $n > 1$, then $R[It]_{(ft^n)} \simeq R[\frac{I^n}{f}]$, where by $\frac{I^n}{f}$ we mean all elements of the form $\frac{x}{f}$, where $x \in I^n$. This is isomorphic to $(R[It]^{(n)})_{(ft)}$, where $R[It]^{(n)} = R[I^n t]$ is the nth Veronese subring of $R[It]$. By the previous paragraph, we know that that there is a one to one correspondence between $\mathcal{M}((R[It]^{(n)})_{ft})$ and $\mathcal{M}((R[It]^{(n)})_{(ft)})$. On the other hand, the homogeneous primes of $R[It]^{(n)}$ are contractions of the homogeneous primes of $R[It]$ (see [E]). Since all the minimal primes of $R[It]$ are homogeneous by Proposition 3.2.2, it follows again that $\mathcal{M}(R[It]_{ft^n})$ and $\mathcal{M}(R[It]_{(ft^n)})$ correspond. This settles part (a).

To prove part (b), from part (a) we notice that since $R[It]_{ft} \simeq R[It]_{(ft)}[u, u^{-1}]$, we have

$$\frac{R[It]_{(ft)}}{p'}[u, u^{-1}] \simeq \frac{R[It]_{ft}}{\hat{p}R[It]_{ft}} \simeq \left(\frac{R[It]}{\hat{p}}\right)_{ft} \simeq \left(\frac{R[It]}{\hat{p}}\right)_{(ft)}[u, u^{-1}],$$

where the second isomorphism is because localization is flat, and the third follows again from part (a) of this theorem along with part (b) of Proposition 3.2.2. Therefore

$$\frac{R[It]_{(ft)}}{p'} \simeq \left(\frac{R[It]}{\hat{p}}\right)_{(ft)},$$

and combining this with Proposition 3.2.2 part (b), we obtain the desired result. $\square$

Proof of Theorem 3.2.1. Let $I = I_1 \dots I_n$, $J = I_1 + \dots + I_n$, and $\mathcal{G}$ be a fixed set of generators for I.

Take $x \in (I_1, \dots, I_n)^{\sim}$. Then for any $f \in \mathcal{G}$,

$$x \in (JR[It]_{(ft)})^*. \tag{3.1}$$

Since tight closure can be tested modulo minimal primes (see Section 2.3), we see that Equation 3.1 is equivalent to

$$\overline{x} \in \left(J\frac{R[It]_{(ft)}}{p'}\right)^*,$$

for every minimal prime p' of $R[It]_{(ft)}$, which by part (a) of Corollary 3.2.3 and Proposition 3.2.2 corresponds to a minimal prime p of R that does not contain f. From Corollary 3.2.3 part (b) we see that

$$\overline{x} \in \left(J\frac{R[It]_{(ft)}}{p'}\right)^* \simeq \left(J\left(\frac{R[It]}{\hat{p}}\right)_{(ft)}\right)^* \simeq (JR/p[I(R/p)t]_{(ft)})^*.$$

This holds for all minimal primes p of R that do not contain f. Hence we equivalently have

$$\overline{x} \in (I_1R/p, \dots, I_nR/p)^{\sim}$$

for all minimal primes p of R. $\square$

3.3 Basic Properties of Blowup Closure

The following fact, which follows from the contraction property of tight closure, makes the computation of blowup closure simpler in several cases.

Proposition 3.3.1. *Let R be a Noetherian commutative ring of prime characteristic p, such that R is either essentially of finite type over an excellent local ring, or R_{red} is F-finite. Let $I_1, \dots, I_n$ be ideals in R, $I = I_1 \dots I_n$, $J = I_1 + \dots + I_n$, $f \in I$ and $x \in R$. Then*

$$x \in (JR[It]_{(ft)})^* \quad \textit{iff} \quad x \in (JR[It]'_{(ft)})^*,$$

where $R[It]'$ is the normalization of $R[It]$.

Proof. Let $p_1, \dots, p_m$ be the set of minimal primes of $R[It]$ that do not contain ft. These correspond to minimal primes $p'_1, \dots, p'_m$ of $R[It]_{(ft)}$ (Corollary 3.2.3) and minimal primes $p_1{}^\sharp, \dots, p_m{}^\sharp$ of $R[It]'_{(ft)}$ (Proposition 2.1.4 and Corollary 2.2.3).

Suppose $x \in (JR[It]'_{(ft)})^*$. Then by Theorem 2.3.2 we see that for $i = 1, \dots, m$,

$$\overline{x} \in \left(J\frac{R[It]'_{(ft)}}{p_i{}^\sharp}\right)^*,$$

where $\overline{x}$ is the image of x in the ring R/p_i. But Proposition 2.1.4 implies that

$$\frac{R[It]'_{(ft)}}{p_i{}^\sharp} \simeq \left(\frac{R[It]_{(ft)}}{p'_i}\right)',$$

so $\overline{x}$ (or rather the image of $\overline{x}$ under this isomorphism) belongs to

$$\left(J\left(\frac{R[It]_{(ft)}}{p'{}_i}\right)'\right)^*.$$

Therefore, by the contraction property of tight closure (see the discussion after Theorem 2.3.4), for all i,

$$\overline{x} \in \left(J\left(\frac{R[It]_{(ft)}}{p'_i}\right)'\right)^* \cap \frac{R[It]_{(ft)}}{p'_i} \subseteq \left(J\left(\frac{R[It]_{(ft)}}{p'_i}\right)\right)^*.$$

Applying Theorem 2.3.2 again, we see that $x \in (JR[It]_{(ft)})^*$.

The reverse inclusion follows because of the inclusion of $R[It]_{(ft)}$ in $R[It]'_{(ft)}$, and the persistence of tight closure (Theorem 2.3.8). $\square$

Theorem 3.3.2. *Let R be a Noetherian ring of prime characteristic p,such that R is either essentially of finite type over an excellent local ring, or R_{red} is F-finite, and let $I_1, \dots, I_n$ be ideals of R. Then:*

(a) The ideal $(I_1, \dots, I_n)^{\sim}$ is tightly closed, and

$$(I_1, \dots, I_n)^{\underline{*}} \subseteq (I_1, \dots, I_n)^{\sim}.$$

(b) If all $I_1, \dots, I_n$ are principal, then $(I_1, \dots, I_n)^{\sim} = (I_1 + \dots + I_n)^*$.

(c) If $n = 1$, then $(I_1)^{\sim} = \overline{I_1}$.

(d) $(\overline{I_1}, \dots, \overline{I_n})^{\sim} = (I_1, \dots, I_n)^{\sim}$.

Proof. Throughout the proof, we let $\mathcal{G}$ be a fixed set of generators for the ideal $I = I_1 \dots I_n$, such that every f in $\mathcal{G}$ is of the form $f = f_1 \dots f_n$, where each f_i is an element of a fixed set of generators for I_i. We let $J = I_1 + \dots + I_n$.

(a) We can assume that R is a domain (Theorem 3.2.1 and Theorem 2.4.2 part d).

For every member f in $\mathcal{G}$, tight closure persists under the map $R \longrightarrow R[It]_{(ft)}$, and so $(JR[It]_{(ft)})^* \cap R$ is tightly closed in R (see Remark 2.3.9). On the other hand, the intersection of tightly closed ideals is tightly closed (Theorem 2.3.2), and since $(I_1, \dots, I_n)^{\sim}$ is the intersection of finitely many ideals of the form $(JR[It]_{(ft)})^* \cap R$, it follows that $(I_1, \dots, I_n)^{\sim}$ is tightly closed in R.

Now let $z \in (I_1, \dots, I_n)^{\underline{*}}$. Then $cz^q \in I_1^q + \dots + I_n^q$, for all large $q = p^e$ and some nonzero $c \in R$. If $f \in \mathcal{G}$, and $f = f_1 \dots f_n$ where f_i is a generator of I_i as above, then this equation can be extended to

$$\begin{aligned} cz^q &\in I_1^q R[It]_{(ft)} + \dots + I_n^q R[It]_{(ft)} \\ &= (f_1)^q R[It]_{(ft)} + \dots + (f_n)^q R[It]_{(ft)} \quad \text{(see Note 3.1.4)} \\ &= (f_1, \dots, f_n)^{[q]} R[It]_{(ft)} \\ &= J^{[q]} R[It]_{(ft)} \\ &= (JR[It]_{(ft)})^{[q]} \end{aligned}$$

for all large powers q of p. It follows that $z \in (JR[It]_{(ft)})^*$. Since this holds for all $f \in \mathcal{G}$, by 3.1.3 we conclude that $z \in (I_1, \dots, I_n)^{\sim}$.

(b) We can again assume that R is a domain. If $I_1, \dots, I_n$ are principal, then so is their product I. So for each $f \in \mathcal{G}$, $R[It]_{(ft)}$ is the same as R. We therefore have

$$(I_1, \dots, I_n)^{\sim} = (I_1 + \dots + I_n)^*.$$

(c) Assume that R is a domain. In this case, $I = J = I_1$. We use the "normalized" definition of blowup closure, following Proposition 3.3.1, and we obtain

$$(I)^{\sim} = \bigcap_{f \in \mathcal{G}} IR[It]'_{(ft)} \cap R = \overline{I},$$

since the extension of I to the normal ring $R[It]'_{(ft)}$ is principal , and therefore tightly closed (see Proposition 2.1.7 and [HH1] Corollary 5.8).

(d) We use the "normalized" definition of blowup closure, following Proposition 3.3.1.

Let $J' = \overline{I_1} + \ldots + \overline{I_n}$ and $I' = \overline{I_1} \ldots \overline{I_n}$. Then $R[It]' = R[I't]'$, since $\overline{(I')^n} = \overline{I^n}$, and so $\mathrm{Proj}R[I't]'$ like $\mathrm{Proj}R[It]'$ is covered by affines of the form $\mathrm{Spec}R[It]'_{(ft)}$ for $f \in \mathcal{G}$. On the other hand, suppose $f = f_1 \ldots f_n \in \mathcal{G}$ where $f_i \in I_i$, $i = 1, \ldots, n$.

Then

$$\left(J'R[It]'_{(ft)}\right)^* = \left(J'R[I't]'_{(ft)}\right)^* = \left((f_1, \ldots, f_n)R[I't]'_{(ft)}\right)^* = \left(JR[It]'_{(ft)}\right)^*,$$

as pointed in Note 3.1.4. It follows that $(\overline{I_1}, \ldots, \overline{I_n})^{\sim} = (I_1, \ldots, I_n)^{\sim}$. □

Theorem 3.3.3 (blowup closure from contractions). *Suppose that R and S are Noetherian domains of prime characteristic p, such that R is either essentially of finite type over an excellent local ring, or R_{red} is F-finite. Let $I_1, \ldots, I_n$ be ideals of R. Suppose S is a module finite extension of R. Then*

$$(I_1S, \ldots, I_nS)^{\sim} \cap R \subseteq (I_1, \ldots, I_n)^{\sim}$$

.

Proof. Let I and J be the product and the sum of $I_1, \ldots, I_n$, respectively, and let $\mathcal{G}$ be a fixed set of generators for I in R.

Suppose S is generated as an R-module by elements $u_1, \ldots, u_m$. Then $u_1, \ldots, u_m$ also generate $S[(IS)t]$ as a module over $R[It]$. To see this, take a homogeneous element $zt^n \in S[(IS)t]$. Then $z \in I^nS$, and so if $I^n = (v_1, \ldots, v_r)$ in R, we can write $z = a_1v_1 + \ldots + a_rv_r$ for $a_1, \ldots, a_r \in S$. On the other hand for $i = 1, \ldots, r$, $a_i = a_{i1}u_1 + \ldots + a_{im}u_m$, where $a_{ij} \in R$ for $j = 1, \ldots, m$. Rewriting the equation describing z above, we have $z = b_1u_1 + \ldots + b_mu_m$, where $b_j \in I^n$ for $j = 1, \ldots, m$. It follows that $zt^n = (b_1t^n)u_1 + \ldots + (b_mt^n)u_m$. So $S[(IS)t]$ is module finite over $R[It]$. A similar argument shows that $S[(IS)t]_{(ft)}$ is module finite over $R[It]_{(ft)}$.

Now let $z \in (I_1S, \ldots, I_nS)^{\sim} \cap R$. Then by definition, z belongs to

$$\begin{aligned}
&\left(JS[(IS)t]_{(ft)}\right)^* \cap R \\
&= \left(JS[(IS)t]_{(ft)}\right)^* \cap R[It]_{(ft)} \cap R \\
&\subseteq \left(JR[It]_{(ft)}\right)^* \cap R && \text{by Theorem 2.3.4.}
\end{aligned}$$

This holds for all $f \in \mathcal{G}$, which implies that $z \in (I_1, \ldots, I_n)^{\sim}$. □

Theorem 3.3.4 (persistence of blowup closure). *Let $\phi : R \to S$ be a homomorphism of Noetherian rings of prime characteristic p, and let $I_1, \ldots, I_n$ be ideals of R. Suppose that either R is essentially of finite type over an excellent local ring, or that R_{red} is F-finite. Then:*

$$(I_1, \ldots, I_n)^{\sim}S \subseteq (I_1S, \ldots, I_nS)^{\sim}.$$

Proof. Let $I = I_1 \dots I_n$ and $J = I_1 + \dots + I_n$. Fix a finite set of generators for I, and let $f = f_1 \dots f_n$, $f_i \in I_i$, be in that set. If R is essentially of finite type over an excellent local ring or R_{red} is F-finite, then $R[It]_{(ft)}$ will have the same property since it is an algebra of finite type over R.

On the other hand, if $\phi(f)$ is nonzero, then ϕ induces a map

$$R[It]_{(ft)} \to S[(IS)t]_{(\phi(f)t)} \ ,$$

under which tight closure persists (see Theorem 2.3.8). It follows that when $\phi(f) \neq 0$,

$$\left(R \cap (JR[It]_{(ft)})^*\right) S \subseteq (JR[It]_{(ft)})^* S[(IS)t]_{(\phi(f)t)} \subseteq \left(JS[(IS)t]_{(\phi(f)t)}\right)^*.$$

If $g_1, \dots, g_r$ is a set of generators for I in R, then IS is generated by the $\phi(g_i)$, $i = 1, \dots, r$, that are nonzero. From the discussion above and Theorem 3.1.3, it follows that

$$(I_1, \dots, I_n)^{\sim} S \subseteq (I_1 S, \dots, I_n S)^{\sim}.$$

$\square$

Blowup closure satisfies most properties that TI closure does. However, in many cases these two operations do not produce the same ideal. Here is an example of ideals in a polynomial ring for which these two operations are not the same:

Example 3.3.5. Let $R = k[x, y]$ be a polynomial ring over a field k of characteristic p. Consider the ideals $I = (y^3)$ and $J = (x^3, x^2 y)$. Notice that I and J are both integrally closed ideals. We know from Theorem 2.4.3 that

$$xy^2 \notin (I, J)^{\underline{*}} = \overline{I} + \overline{J} = (x^3, y^3, x^2 y).$$

We show that, however, $xy^2 \in (I, J)^{\sim}$. To show this, we check the ideal $I + J$ against two localized Rees rings. With notation as in Note 3.1.4, if we take $f_1 = y^3$ and $f_2 = x^2 y$, we have

$$xy^2 \in (y^3, x^2 y) k[x, y, \frac{x}{y}],$$

since $xy^2 = y^3 (x/y)$.

If we take $f_1 = y^3$ and $f_2 = x^3$, we have

$$xy^2 \in (y^3, x^3) k[x, y, \frac{y}{x}],$$

since $xy^2 = x^3 (y/x)^2$.

Therefore $xy^2 \in (I, J)^{\sim}$, but $xy^2 \notin (I, J)^{\underline{*}}$.

Moreover, blowup closure fails to respect inclusions: If $J_1, \dots, J_n$ is a set of ideals such that $I_i \subseteq J_i$, for $i = 1, \dots, n$, then the inclusion $(I_1, \dots, I_n)^{\sim} \subseteq (J_1, \dots, J_n)^{\sim}$ does not necessarily hold. Here is an example:

Example 3.3.6. Let $R = k[x, y, u, v]$. Let $I_1 = (x^3, x^2y)$, $I_2 = (y^3)$, $J_1 = (x^3, x^2y, u)$ and $J_2 = (y^3, v)$. Then $xy^2 \in (I_1, I_2)^\sim$ as was shown in the previous example. But $xy^2 \notin (J_1, J_2)^\sim$, because looking at the affine patch corresponding to the generators u and v of J_1 and J_2, respectively, we can see that

$$xy^2 \notin (u, v)k[x, y, u, v, \frac{x^3}{u}, \frac{x^2y}{u}, \frac{y^3}{v}].$$

As a remedy to this problem, the notion of *multiple closure* is introduced in Section 4.1. Before introducing multiple closure, we discuss a case where blowup closure can be directly calculated.

3.4 The Case of Monomial Ideals in a Polynomial Ring

Theorem 3.4.1. *Let $R = k[x_1^1, \dots, x_{m_1}^1, \dots, x_1^n, \dots, x_{m_n}^n]$ be a polynomial ring in distinct variables x_j^i , $1 \le i \le n$ and $1 \le j \le m_i$, over an algebraically closed field k of prime characteristic p. Let $I_1, \dots, I_n$ be monomial ideals in R, where the generators of I_i are monomials in the variables $x_1^i, \dots, x_{m_i}^i$ for $1 \le i \le n$. Then $(I_1, \dots, I_n)^\sim = \overline{I_1} + \dots + \overline{I_n}$*

To prove this theorem, we will show that $(I_1, \dots, I_n)^\sim$ is a monomial ideal. Then, for a given monomial $M \in (I_1, \dots, I_n)^\sim$, we will show that for some $1 \le \alpha \le n$,

$$M \in R \cap I_\alpha R[It]' = \overline{I_\alpha},$$

and it will follow that $(I_1, \dots, I_n)^\sim = \overline{I_1} + \dots + \overline{I_n}$

We begin by fixing the notation. For each i, let $I_i = (f_1^i, \dots, f_{s_i}^i)$, where f_j^i is a monomial in the polynomial ring $S_i = k[x_1^i, \dots, x_{m_i}^i]$. Let $J = I_1 + \dots + I_n = (f_1^1, \dots, f_{s_1}^1, \dots, f_1^n, \dots, f_{s_n}^n)$, and $I = I_1 \dots I_n = \left(f_{i_1}^1 \dots f_{i_n}^n : 1 \le i_j \le s_j, 1 \le j \le n\right)$.

Our first goal is to show that

$$(I_1, \dots, I_n)^\sim = \bigcap_{\substack{1 \le i_j \le s_j \\ 1 \le j \le n}} (f_{i_1}^1, \dots, f_{i_n}^n)R[It]'_{(f_{i_1}^1 \dots f_{i_n}^n t)} \cap R.$$

By 3.1.4, for a fixed index set $i_1, \dots, i_n$, we are interested in the tight closure of the ideal $(f_{i_1}^1, \dots, f_{i_n}^n)$ in the ring:

$$k[x_1^1, \dots, x_{m_1}^1, \dots, x_1^n, \dots, x_{m_n}^n, \frac{f_1^1}{f_{i_1}^1}, \dots, \frac{f_{s_1}^1}{f_{i_1}^1}, \dots, \frac{f_1^n}{f_{i_n}^n}, \dots, \frac{f_{s_n}^n}{f_{i_n}^n}]. \tag{3.2}$$

Equivalently, by Theorem 3.3.1, we can study the tight closure of this ideal in the normalization of the ring described in 3.2. We claim that the tight closure of $(f_{i_1}^1, \dots, f_{i_n}^n)$ in the normalization of the ring in 3.2 is equal to $(f_{i_1}^1, \dots, f_{i_n}^n)$ itself.

To see this, let $f = f^1_{i_1} \dots f^n_{i_n}$. Since $R[It]$ is a monomial subring of $R[t]$, $R[It]'$ is also a monomial ring, and it is weakly F-regular (i.e., all ideals are tightly closed; see [Sm2]). Also, tight closure commutes with localization for $R[It]$ ([Sm2]). Since normalization also commutes with localization ([E] Proposition 4.13) it follows that

$$(JR[It]'_{ft})^* = JR[It]'_{ft}.$$

On the other hand,

$$\left(JR[It]'_{(ft)}\right)^* \subseteq \left((JR[It]'_{ft})^*\right)_0 = \left((JR[It]'_{ft})\right)_0 = JR[It]'_{(ft)},$$

and so $(JR[It]'_{(ft)})^* = JR[It]'_{(ft)}$.

We have therefore shown that if $I_1, \dots, I_n$ are monomial ideals in a polynomial ring R, then

$$(I_1, \dots, I_n)^{\sim} = \bigcap_{\substack{1 \le i_j \le s_j \\ 1 \le j \le n}} (f^1_{i_1}, \dots, f^n_{i_n}) R[It]'_{(f^1_{i_1} \dots f^n_{i_n} t)} \cap R. \tag{3.3}$$

Notice that this argument does not require the distinction of the sets of variables generating $I_1, \dots, I_n$.

Proposition 3.4.2. *Let $I_1, \dots, I_n$ be monomial ideals (not necessarily generated by distinct variables) in a polynomial ring $R = k[u_1, \dots, u_m]$ where k is an infinite field. Then $(I_1, \dots, I_n)^{\sim}$ is also a monomial ideal in R.*

Proof. We use the fact that an ideal I in a polynomial ring $R = k[u_1, \dots, u_m]$ over an infinite field k is generated by monomials, if and only if I is invariant under the action of the torus $(k^*)^m$, where k^* denotes $k - \{0\}$, and the torus action on R is defined as follows. If $\lambda = (\lambda_1, \dots, \lambda_m) \in (k^*)^m$, and x is a monomial ${u_1}^{c_1} \dots {u_m}^{c_m}$ of R, then

$$\lambda x = (\lambda_1 u_1)^{c_1} \dots (\lambda_m u_m)^{c_m} = {\lambda_1}^{c_1} \dots {\lambda_m}^{c_m} x,$$

and if x is a sum of monomials $M_1 + \dots + M_r$, then $\lambda x = \lambda M_1 + \dots + \lambda M_r$.

The action of the torus on monomials with negative powers is defined in a similar way. In the ring $R[{u_1}^{-1}, \dots, {u_m}^{-1}]$, where R is as above, if $\lambda = (\lambda_1, \dots, \lambda_m) \in (k^*)^m$ and $u = {u_1}^{c_1} \dots {u_m}^{c_m}$, where the c_i are integers, then

$$\lambda u = (\lambda_1 u_1)^{c_1} \dots (\lambda_m u_m)^{c_m} = {\lambda_1}^{c_1} \dots {\lambda_m}^{c_m} u,$$

and if $u = M_1 + \dots + M_r$, where M_i are monomials in the u_i with integer powers, then $\lambda u = \lambda M_1 + \dots + \lambda M_r$.

We show that $(I_1, \dots, I_n)^{\sim}$ is invariant under the action of the torus $(k^*)^m$. Suppose each I_i is generated by $f_1^i, \dots, f_{s_i}^i$. By the discussion preceding the theorem we only need to prove that for any given index set $i_1, \dots, i_n$, if

$$x \in (f_{i_1}^1, \dots, f_{i_n}^n) R[It]'_{(f_{i_1}^1 \cdots f_{i_n}^n t)}$$

and $\lambda \in (k^*)^m$, then

$$\lambda x \in (f_{i_1}^1, \dots, f_{i_n}^n) R[It]'_{(f_{i_1}^1 \cdots f_{i_n}^n t)}.$$

Take $\lambda = (\lambda_1, \dots, \lambda_m)$ and x as above. Then x can be written as

$$x = A_1 f_{i_1}^1 + \ldots + A_n f_{i_n}^n,$$

with $A_1, \dots, A_n \in R[It]'_{(f_{i_1}^1 \cdots f_{i_n}^n t)}$.

So

$$\begin{aligned} \lambda x &= \lambda(A_1 f_{i_1}^1) + \ldots + \lambda(A_n f_{i_n}^n) \\ &= (\lambda A_1)(\lambda f_{i_1}^1) + \ldots + (\lambda A_n)(\lambda f_{i_n}^n) \end{aligned}$$

Since each $f_{i_j}^j$ is a monomial in the u_i, $\lambda f_{i_j}^j$ will be just some scalar times $f_{i_j}^j$, and will therefore still belong to $(f_{i_1}^1, \dots, f_{i_n}^n)$.

As for the A_j, we claim that λA_j still remains in $R[It]'_{(f_{i_1}^1 \cdots f_{i_n}^n t)}$ for $j = 1, \dots, n$. To see this, fix some j. One can write A_j as

$$A_j = \frac{B_j}{(f_{i_1}^1 \cdots f_{i_n}^n)^r t^r},$$

where $B_j \in R[It]'$. We can then write B_j as

$$B_j = M_1 t^r + \ldots + M_{s_j} t^r,$$

where $M_1, \dots, M_{s_j}$ are monomials of R that belong to $\overline{I^r}$ (see Corollary 2.2.5). So if we set $f = f_{i_1}^1 \cdots f_{i_n}^n$, then $A_j = M_1 f^{-r} + \ldots + M_{s_j} f^{-r}$, and so

$$\lambda A_j = \alpha_1 M_1 f^{-r} + \ldots + \alpha_{s_j} M_{s_j} f^{-r},$$

where $\alpha_1, \dots, \alpha_{s_j} \in k$ are scalars. It follows that

$$\lambda A_j = \frac{\alpha_1 M_1 + \ldots + \alpha_{s_j} M_{s_j}}{f^r} \in R[It]'_{(ft)}.$$

Therefore $\lambda x \in (f_{i_1}^1, \dots, f_{i_n}^n) R[It]'_{(ft)}$, and so we are done.

$\square$

Lemma 3.4.3. *Let $R = k[u_1, \dots, u_m, \frac{M_1}{N_1}, \dots, \frac{M_r}{N_r}]$, where $u_1, \dots, u_m$ are distinct variables, and $M_1, \dots, M_r, N_1, \dots, N_r$ are nonzero monomials in the polynomial ring $k[u_1, \dots, u_m]$ over a field k. Suppose $M, g_1, \dots, g_s$ are nonzero monomials in $k[u_1, \dots, u_m]$ such that $M \in (g_1, \dots, g_s)R$. Then for some i, $1 \le i \le s$, $M \in (g_i)R$.*

Proof. Since $M \in (g_1, \dots, g_s)R$, there are $A_1, \dots, A_s$ in R such that $M = A_1g_1 + \dots + A_sg_s$. After taking the common denominator A of the right hand side of the equation, and multiplying both sides of the equation by A, we end up with an equation of the form $AM = p_1g_1 + \dots + p_sg_s$, where $p_1, \dots, p_s$ are polynomials in $k[u_1, \dots, u_m]$, and for each i, $A_i = \frac{p_i}{A}$. Now notice that AM is a monomial, and so when you add the polynomials $p_1g_1, \dots, p_sg_s$ all terms that are not equal to a scalar multiple of AM cancel out with each other. So we can without loss of generality for each i, replace p_i with α_iQ_i, where $\alpha_i \in k$ is nonzero if the monomial AM appears as a term of p_ig_i, and Q_i is a monomial in $k[u_1, \dots, u_m]$ for which $Q_ig_i = AM$. So $AM = \alpha_1Q_1g_1 + \dots + \alpha_sQ_sg_s$, and $\alpha_1 + \dots + \alpha_s = 1$. Now it is clear that least one of the α_i must be nonzero; say α_1 is nonzero. Then α_1Q_1 is a term of p_1, and since $A_1 = \frac{p_1}{A} \in R$, $\frac{Q_1}{A} \in R$, and therefore $M = \frac{Q_1}{A}g_1 \in (g_1)R$.

□

A Noetherian ring R satisfies Serre's condition (R_n) if R_p is a regular local ring for $p \in \mathrm{Spec}R$ with $\dim R_p \leq n$. We say that R satisfies Serre's condition (S_n) if $\mathrm{depth}R_p \geq \min(n, \dim R_p)$ for all $p \in \mathrm{Spec}R$. For more on these conditions, see the first three chapters of [BH].

The following fact has also been partially discussed in [V]. Below, we give a different proof.

Proposition 3.4.4. *Let R and S be two domains, which are finitely generated k-algebras, where k is an algebraically closed field. Then:*

(a) For a positive integer n, if R and S satisfy (S_n), then so does $R \otimes_k S$.

(b) For a positive integer n, if R and S satisfy (R_n), then so does $R \otimes_k S$.

Proof. One can express R and S as

$$R = \frac{k[u_1, \dots, u_m]}{(g_1, \dots, g_s)} \quad \text{and} \quad S = \frac{k[v_1, \dots, v_r]}{(h_1, \dots, h_t)},$$

where $u_1, \dots, u_m$ and $v_1, \dots, v_r$ are distinct variables. Then

$$R \otimes_k S = \frac{k[u_1, \dots, u_m, v_1, \dots, v_r]}{(g_1, \dots, g_s, h_1, \dots, h_t)};$$

see [ZS1] for the tensor product of two rings.

(a) Suppose R and S are (S_n). The map $\phi : R \longrightarrow R \otimes_k S$ is a flat homomorphism of rings.

By Proposition 2.1.16 in [BH], to show that $R \otimes_k S$ is (S_n), it is sufficient to show that for all prime ideals p of R, $k(p) \otimes_R (R \otimes_k S) \simeq k(p) \otimes_k S$ is (S_n), where $k(p) = R_p/pR_p$ is a residue class field of R. By applying the same argument again to the flat homomorphism $\psi : S \longrightarrow k(p) \otimes_k S$, it is sufficient to show that $k(q) \otimes_S (k(p) \otimes_k S) \simeq k(p) \otimes_k k(q)$ is (S_n), where p and q are primes of R and S, respectively.

So fix p and q, and pick $f \in R/p$ such that $(R/p)_f$ is Cohen-Macaulay. Now $(R/p)_f$ is a finitely generated algebra over k, so by Theorem 2.1.10 of [BH], since $(R/p)_f$ is Cohen Macaulay, so is $k(q) \otimes_k (R/p)_f$. Hence $k(p) \otimes_k k(q)$ is Cohen Macaulay, since it is a localization of $k(q) \otimes_k (R/p)_f$.

(b) Suppose R and S satisfy (R_n), that means the defining ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ of the singular loci R and S, respectively, have heights larger than n. The defining ideal $\mathcal{J}$ of the singular locus of $R \otimes_k S$, is the ideal generated by the $d \times d$ minors of the Jacobian matrix

$$\begin{pmatrix} \frac{\delta g_1}{\delta u_1} & \cdots & \frac{\delta g_1}{\delta u_m} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\delta g_s}{\delta u_1} & \cdots & \frac{\delta g_s}{\delta u_m} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\delta h_1}{\delta v_1} & \cdots & \frac{\delta h_1}{\delta v_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{\delta h_t}{\delta v_1} & \cdots & \frac{\delta h_t}{\delta v_r} \end{pmatrix},$$

where d is the height of the ideal $(g_1, \ldots, g_s, h_1, \ldots, h_t)$ in the polynomial ring $k[u_1, \ldots, u_m, v_1, \ldots, v_r]$ (see Corollary 16.20 of [E]). On the other hand, $d = d_1 + d_2$, where d_1 is the height of $(g_1, \ldots, g_s)$ in $k[u_1, \ldots, u_m]$, and d_2 is the height of $(h_1, \ldots, h_t)$ in $k[v_1, \ldots, v_r]$ (see Chapter II of [Har]).

Now, it is an easy exercise to see that the only time that the determinant of a square matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

is nonzero is when A and B are both square matrices. So $\mathcal{J}$ is generated by determinants of the form

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = (\det A)(\det B),$$

where A and B are square matrices. In particular, it contains products of $d_1 \times d_1$ minors of the Jacobian of R with $d_2 \times d_2$ minors of the Jacobian of S. Therefore $\mathcal{J} \supseteq \mathcal{J}_1\mathcal{J}_2$, and hence $\operatorname{ht} \mathcal{J} \geq \min(\operatorname{ht} \mathcal{J}_1, \operatorname{ht} \mathcal{J}_2) > n$, since R and S are (R_n). Therefore $R \otimes_k S$ is smooth in codimension n.

$\square$

Corollary 3.4.5. *Let R and S be two normal domains, which are finitely generated k-algebras, where k is an algebraically closed field. Then $R \otimes_k S$ is also a normal domain.*

Proof. Serre's normality criterion says that a ring being normal is equivalent to it satisfying (S_2) and (R_1), and so by Proposition 3.4.4 $R \otimes_k S$ is normal. If K and K' are the quotient fields of R and S respectively, Theorem III.15.40 of [ZS1] proves that $K \otimes_k K'$ is a domain when k is algebraically closed, and since $R \otimes_k S$ is a subring of $K \otimes_k K'$, it follows that $R \otimes_k S$ is a domain. $\square$

Below, we adopt the following notation. If $R \otimes_k S$ is the tensor product of two finitely generated k-algebras R and S over the field k, and if $x \in R$ and $y \in S$, by $x \otimes_k y$ we mean the product of $x \otimes_k 1$ and $1 \otimes_k y$, where $x \otimes_k 1$ and $1 \otimes_k y$ are the images of x and y in $R \otimes_k S$ under the inclusions $R \longrightarrow R \otimes_k S$ and $S \longrightarrow R \otimes_k S$, respectively (see [ZS1]).

Lemma 3.4.6. *Let $R = k[x_1^1, \ldots, x_{m_1}^1, \ldots, x_1^n, \ldots, x_{m_n}^n]$, I and J be as in Theorem 3.4.1. Let M be a monomial of R that belongs to $(I_1, \ldots, I_n)^{\sim}$. Fix an index set $i_1, \ldots, i_n$, such that $1 \le i_v \le s_v$, for $1 \le v \le n$. Suppose that for some fixed β, $1 \le \beta \le n$, $M \in (f_{i_\beta}^\beta)R[It]'_{(f_{i_1}^1 \cdots f_{i_n}^n t)}$. Then for any index set $j_1, \ldots, j_n$, $1 \le j_v \le s_v$, $1 \le v \le n$, such that $j_\beta = i_\beta$ (so that $f_{j_\beta}^\beta = f_{i_\beta}^\beta$), we have $M \in (f_{j_\beta}^\beta)R[It]'_{(f_{j_1}^1 \cdots f_{j_n}^n t)}$.*

Proof. We know that $M \in (I_1, \ldots, I_n)^{\sim}$, so for all index sets $j_1, \ldots, j_n$ one has

$$M \in \left(JR[It]'_{(f_{j_1}^1 \cdots f_{j_n}^n t)}\right)^* = (f_{j_1}^1, \ldots, f_{j_n}^n)R[It]'_{(f_{j_1}^1 \cdots f_{j_n}^n t)},$$

as we proved earlier in this section.

Since the $f_{j_1}^1, \ldots, f_{j_n}^n$ are monomials in distinct sets of variables, Corollary 3.4.5 implies that

$$R[It]'_{(f_{j_1}^1 \cdots f_{j_n}^n t)} = S_1[I_1 t]'_{(f_{j_1}^1 t)} \otimes_k \ldots \otimes_k S_n[I_n t]'_{(f_{j_n}^n t)}, \tag{3.4}$$

where for each v, $1 \le v \le n$, $S_v = k[x_1^v, \ldots, x_{m_v}^v]$.

So, for the index set $i_1, \ldots, i_n$, following the structure in 3.4, we can write $M = M_1 \otimes \ldots \otimes M_n$, where each M_v is a monomial in S_v for $1 \le v \le n$, and $M_\beta \in (f_{i_\beta}^\beta)S_\beta[I_\beta t]'_{(f_{i_\beta}^\beta t)}$.

Let $j_1, \ldots, j_n$, $1 \le j_v \le s_v$ be any set of indices such that $j_\beta = i_\beta$, that is, $f_{j_\beta}^\beta = f_{i_\beta}^\beta$. Then since $M_\beta \in (f_{i_\beta}^\beta)S_\beta[I_\beta t]'_{(f_{i_\beta}^\beta t)}$, we still have $M \in (f_{j_\beta}^\beta)R[It]'_{(f_{j_1}^1 \cdots f_{j_n}^n t)}$. $\square$

Proof of Theorem 3.4.1. We want to show that $(I_1, \ldots, I_n)^{\sim} = \overline{I_1} + \ldots + \overline{I_n}$. Clearly $\overline{I_1} + \ldots + \overline{I_n} \subset (I_1, \ldots, I_n)^{\sim}$. We need to show that the other inclusion holds. From

Proposition 3.4.2 we know that since $I_1, \ldots, I_n$ are monomial ideals, $(I_1, \ldots, I_n)^{\sim}$ is a monomial ideal.

So we pick a monomial M of $R = k[x_1^1, \ldots, x_{m_1}^1, \ldots, x_1^n, \ldots, x_{m_n}^n]$ such that $M \in (I_1, \ldots, I_n)^{\sim}$. Our goal is to show that for some α, $1 \leq \alpha \leq n$, $M \in I_\alpha R[It]'$.

Fix an index set $i_1, \ldots, i_n$, and let $f = f_{i_1}^1 \ldots f_{i_n}^n$. Then from Equation 3.3 we see that

$$M \in \left(JR[It]'_{(ft)}\right)^* = (f_{i_1}^1, \ldots, f_{i_n}^n)R[It]'_{(ft)}.$$

From [EGA] Lemma 2.1.6, it follows that $\text{Proj}R[It]' = \text{Proj}R[\overline{I^h}t]$ for some $h \geq 1$, and we obtain

$$R[It]'_{(ft)} \simeq R[\overline{I^h}t]_{(f^h t)}.$$

Since I is a monomial ideal, $\overline{I^h}$ is a monomial ideal (see [E] Chapter 4), and we can write $\overline{I^h} = (H_1, \ldots, H_r)$, where $H_1, \ldots, H_r$ are monomials. So we have

$$M \in (f_{i_1}^1, \ldots, f_{i_n}^n)k[x_1^1, \ldots, x_{m_1}^1, \ldots .x_1^n, \ldots, x_{m_n}^n, H_1 t, \ldots, H_r t]_{(f^h t)},$$

which is isomorphic to

$$k[x_1^1, \ldots, x_{m_1}^1, \ldots, x_1^n, \ldots, x_{m_n}^n, \frac{H_1}{f^h}, \ldots, \frac{H_r}{f^h}],$$

and hence from Lemma 3.4.3 it follows that for some β, $1 \leq \beta \leq n$, $M \in (f_{i_\beta}^\beta)R[\overline{I^h}t]_{(f^h t)}$ which implies that $M \in (f_{i_\beta}^\beta)R[It]'_{(ft)}$.

We would like to prove that this choice is consistent for all the affine sets, that is, there is some β such that $M \in (f_{i_\beta}^\beta)R[It]'_{(f_{i_1}^1 \cdots f_{i_n}^n t)}$ for all choices of index sets $i_1, \ldots, i_n$.

Suppose that for each $\alpha = 1, ..., n-1$, there is some index γ_α, $1 \leq \gamma_\alpha \leq s_\alpha$, and some index set $i_1, \ldots, i_n$ with $i_\alpha = \gamma_\alpha$, for which $M \notin (f_{i_\alpha}^\alpha)R[It]'_{(f_{i_1}^1 \cdots f_{i_n}^n t)}$. Then by Lemma 3.4.6, for all j, $1 \leq j \leq s_n$, if one picks the index set $\gamma_1, \ldots, \gamma_{n-1}, j$,

$$M \notin (f_{\gamma_\alpha}^\alpha)R[It]'_{(f_{\gamma_1}^1 \cdots f_{\gamma_{n-1}}^{n-1} f_j^n t)}$$

for $1 \leq \alpha \leq n-1$. Therefore for all possible j,

$$M \in (f_j^n)R[It]'_{(f_{\gamma_1}^1 \cdots f_{\gamma_{n-1}}^{n-1} f_j^n t)}.$$

Applying Lemma 3.4.6 again, one gets that $M \in (f_{i_n}^n)R[It]'_{(f_{i_1}^1 \cdots f_{i_n}^n t)}$, for all index sets $i_1, \ldots, i_n$.

We have now proved that $M \in (I_1 + \ldots + I_n)R[It]'$, implies that for some β, $1 \leq \beta \leq n$, $M \in I_\beta R[It]'$. Therefore

$$M \in R \cap I_\beta R[It]' = \overline{I_\beta}$$

by Corollary 2.1.8. $\square$

CHAPTER IV

Multiple Closure

In this chapter we introduce the notion of multiple closure of a set of ideals in a ring of positive characteristic. The idea is to induce the property of respecting inclusions of ideals on blowup closure, without losing persistence. Under mild conditions on the ring, we show that the TI closure of a set of ideals is the same as their multiple closure (which is in fact the contraction of the blowup closure of certain extensions of those ideals in an extension of the original ring). We apply this result to answer the questions mentioned in Section 2.5 about TI closure. We develop a test element theory for TI closure based on tight closure test elements, and introduce specific TI closure test elements for ideals in affine algebras over perfect fields of positive characteristic. The last section treats TI closure in equal characteristic zero. We show that in this situation multiple closure and TI closure agree, and we prove the existence of universal test elements for TI closure in finitely generated algebras over a field.

4.1 Definition and Basic Facts

Definition 4.1.1. Let R be a Noetherian ring of positive characteristic p, and let $I_1, \dots, I_n$ be ideals in R. We define x in R to be in the *multiple closure* of $I_1, \dots, I_n$, denoted by $(I_1, \dots, I_n)^{\tilde{*}}$, if and only if the image of x is in:

$$\left((w_1, \dots, w_n) R[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}] \right)^*,$$

where $w_1, \dots, w_n$ are indeterminates.

Recall that the ring

$$S = R[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}]$$

above is obtained as follows: We take a ring extension $R[w_1, \dots, w_n]$ of R, and we consider the Rees ring $R[w_1, \dots, w_n][I't]$ ring of the product I' of the ideals

$I_1+(w_1), \dots, I_n+(w_n)$. We then localize this Rees ring at the element $w_1 \dots w_n t$, and take the zeroth graded piece of the localized ring to obtain S. The ideal $(w_1, \dots, w_n)$ in S is just the sum of the ideals $I_1 + (w_1), \dots, I_n + (w_n)$ extended to S. So the multiple closure of $I_1, \dots, I_n$ in the ring R is one of the affine patches to be considered to compute the blowup closure of $I_1 + (w_1), \dots, I_n + (w_n)$ in the ring $R[w_1, \dots, w_n]$ (see Theorem 3.1.3). It turns out that the multiple closure of $I_1, \dots, I_n$ *is* in fact the blowup closure of $I_1 + (w_1), \dots, I_n + (w_n)$ in $R[w_1, \dots, w_n]$, contracted back to R (Corollary 4.2.2). Multiple closure therefore enjoys all the basic properties of blowup closure.

We verify that multiple closure can be tested modulo minimal primes, and hence one can reduce most arguments to the case of domains.

Proposition 4.1.2 (multiple closure can be tested modulo minimal primes). *Let R be a Noetherian ring of positive characteristic p, and let $I_1, \dots, I_n$ be ideals in R. An element x of R is in $(I_1, \dots, I_n)^{\overset{*}{\sim}}$ if and only if for all minimal primes p of R, the image $\overline{x}$ of x in R/p is in $(I_1 R/p, \dots, I_n R/p)^{\overset{*}{\sim}}$.*

Proof. The proof follows from the description of minimal primes of localized Rees rings, as in Corollary 3.2.3.

Let $I = (I_1 + (w_1)) \dots (I_n + (w_n))$. Then by Proposition 3.2.2 and Corollary 3.2.3, a minimal prime p' of

$$S = R[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}] \simeq R[w_1, \dots, w_n, It]_{(w_1 \dots w_n t)}$$

corresponds to a minimal prime $p^\dagger$ of $R[w_1, \dots, w_n]$ that does not contain $w_1 \dots w_n$, which in turn corresponds to a minimal prime p of R. By the same results, we have the following isomorphisms

$$S/p' \simeq \big((R[w_1, \dots, w_n]/p^\dagger)[I(R[w_1, \dots, w_n]/p^\dagger)t]\big)_{(w_1 \dots w_n t)}$$

which is isomorphic to

$$((R/p)[w_1, \dots, w_n]\,[(I(R/p)[w_1, \dots, w_n])t])_{(w_1 \dots w_n t)}$$

which is isomorphic to

$$(R/p)\left[w_1, \dots, w_n, \frac{I_1(R/p)}{w_1}, \dots, \frac{I_n(R/p)}{w_n}\right].$$

Now take $x \in (I_1, \dots, I_n)^{\overset{*}{\sim}}$. By Definition 4.1.1, this is equivalent to

$$x \in ((w_1, \dots, w_n)S)^* \cap R.$$

From the isomorphisms above, equivalently for all $p \in \mathcal{M}(R)$

$$\overline{x} \in \left((w_1, \dots, w_n)S/p^{\dagger}\right)^* \cap R/p,$$

or, equivalently

$$\overline{x} \in \left((w_1, \dots, w_n)(R/p)[w_1, \dots, w_n, \frac{I_1(R/p)}{w_1}, \dots, \frac{I_n(R/p)}{w_n}]\right)^* \cap R/p$$

By Definition 4.1.1 this is equivalent to $\overline{x} \in (I_1R/p, \dots, I_nR/p)^{\tilde{*}}$. □

Theorem 4.1.3. *Let R be a Noetherian ring of prime characteristic p such that either R is essentially of finite type over an excellent local ring, or R_{red} is F-finite. Let $I_1, \dots, I_n$ be ideals of R. Then*

$$(I_1, \dots, I_n)^{\tilde{*}} = (I_1, \dots, I_n)^{\underline{*}} .$$

Proof. We can assume by propositions 4.1.2 and 2.4.2 that R is a domain. Let $S = R[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}]$, where $w_1, \dots, w_n$ are indeterminates. Pick $z \in (I_1, \dots, I_n)^{\underline{*}}$. Then there is a nonzero c in R such that $cz^q \in {I_1}^q + \dots + {I_n}^q$ for all large $q = p^e$. Since we have the inclusion $R \subset S$, it follows that for all large q

$$cz^q \in ({I_1}^q + \dots + {I_n}^q)S \subseteq ({w_1}^q, \dots, {w_n}^q)S = (w_1, \dots, w_n)^{[q]}S .$$

Hence

$$z \in ((w_1, \dots, w_n)S)^* \cap R = (I_1, \dots, I_n)^{\tilde{*}}.$$

To show the other inclusion, we choose an element c in R such that R_c is regular. Then

$$S_{cw_1 \dots w_n} = R_c[w_1, \dots, w_n, \frac{1}{w_1}, \dots, \frac{1}{w_n}]$$

is regular, and therefore $d = cw_1 \dots w_n$ has a power c' that is a test element for the ring S (Theorem 2.3.7). By multiplying c' with appropriate powers of elements in $\frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}$, we may assume that $c' \in R$.

Now take $z \in (I_1, \dots, I_n)^{\tilde{*}}$. Then

$$c'z^q \in ({w_1}^q, \dots, {w_n}^q)S$$

for all $q = p^e$.

So for a given q, we can find $C_1, \dots, C_n$ in S, such that

$$c'z^q = C_1{w_1}^q + \dots + C_n{w_n}^q.$$

By taking common denominators, we can find a positive integer N, which we can take to be larger than q, such that $C_i = \frac{A_i}{(w_1 \ldots w_n)^N}$ for every $i = 1, \ldots, n$, where A_i is a polynomial in $R[w_1, \ldots, w_n]$. So we get

$$c' z^q (w_1 \ldots w_n)^N = A_1 {w_1}^q + \ldots + A_n {w_n}^q.$$

Since $R[w_1, \ldots, w_n]$ is a free module over R generated by the monomials in $w_1, \ldots, w_n$, and $c'z^q \in R$, we can without loss of generality take each A_i to be is a monomial of the form $B_i {w_1}^N \ldots {w_{i-1}}^N {w_i}^{N-q} {w_{i+1}}^N \ldots {w_n}^N$, where $B_i \in R$, for all $i = 1, \ldots, n$. So we can write $c'z^q$ as

$$\frac{A_1}{(w_1 \ldots w_n)^N} {w_1}^q + \ldots + \frac{A_n}{(w_1 \ldots w_n)^N} {w_n}^q = \frac{B_1}{{w_1}^q} {w_1}^q + \ldots + \frac{B_n}{{w_n}^q} {w_n}^q$$

which implies that $B_i \in {I_i}^q$ for all $i = 1, \ldots, n$. So

$$c' z^q \in {I_1}^q + \ldots + {I_n}^q.$$

This holds for all q, hence $z \in (I_1, \ldots, I_n)^{\underline{*}}$. $\square$

This equality translates the TI closure of a set of ideals in R into the tight closure of an ideal in an extension ring of R. In particular, most properties of tight closure can now be extended to TI closure.

4.2 Basic Properties of TI Closure Via Multiple Closure

Theorem 4.2.1 (persistence of TI closure). *Let R be a Noetherian ring of prime characteristic p that is either essentially of finite type over an excellent local ring or R_{red} is F-finite. Let $I_1, \ldots, I_n$ be ideals of R. Suppose $R \longrightarrow S$ is a homomorphism of rings. Then TI closure persists under this map:*

$$(I_1, \ldots, I_n)^{\underline{*}}\, S \subseteq (I_1 S, \ldots, I_n S)^{\underline{*}}\,.$$

Proof. The properties mentioned above for R are preserved when we pass to the ring $R^\dagger = R[w_1, \ldots, w_n, \frac{I_1}{w_1}, \ldots, \frac{I_n}{w_n}]$, since this is just a finitely generated algebra over R. Moreover, we have the obvious induced map

$$R^\dagger \longrightarrow S^\dagger = S[w_1, \ldots, w_n, \frac{I_1 S}{w_1}, \ldots, \frac{I_n S}{w_n}]$$

under which tight closure persists. Therefore, by Theorem 2.3.8,

$$(w_1, \ldots, w_n)^* S^\dagger \subseteq \left((w_1, \ldots, w_n) S^\dagger\right)^*$$

which implies that $(I_1, \ldots, I_n)^{\underline{*}}\, S \subseteq (I_1 S, \ldots, I_n S)^{\underline{*}}$. $\square$

An interesting corollary is that multiple closure (or TI closure) is indeed a blowup closure in a larger ring.

Corollary 4.2.2. *Let R be a Noetherian ring of prime characteristic p such that either R is essentially of finite type over an excellent local ring, or R_{red} is F-finite. Let $I_1, \dots, I_n$ be ideals of R. If $S = R[w_1, \dots, w_n]$, and for $i = 1, \dots, n$, $I'_i = I_i + (w_i)$ is an ideal of S, then*

$$(I_1, \dots, I_n)^{\underline{*}} = (I_1, \dots, I_n)^{\overset{*}{\sim}} = ((I'_1, \dots, I'_n)S)^{\sim} \cap R.$$

Proof. From Theorem 4.2.1, Theorem 2.4.2 part (b), and Theorem 3.3.2 part (a) we have

$$(I_1, \dots, I_n)^{\underline{*}} S \subseteq (I_1 S, \dots, I_n S)^{\underline{*}} \subseteq (I'_1, \dots, I'_n)^{\underline{*}} \subseteq (I'_1, \dots, I'_n)^{\sim},$$

and so

$$(I_1, \dots, I_n)^{\underline{*}} \subseteq (I'_1, \dots, I'_n)^{\sim} \cap R.$$

On the other hand, let I' and J' denote the product and sum of $I'_1, \dots, I'_n$, respectively, and let $\mathcal{G}'$ be a fixed set of generators for I' such that $w_1 \dots w_n \in \mathcal{G}'$. By Theorem 3.1.3,

$$(I'_1, \dots, I'_n)^{\sim} = \bigcap_{f \in \mathcal{G}'} \left(J'S[I't]_{(ft)}\right)^* \cap S.$$

From the previous paragraph, for every $f \in \mathcal{G}'$, $(I_1, \dots, I_n)^{\underline{*}} \subseteq \left(J'S[I't]_{(ft)}\right)^* \cap R$. On the other hand, by definition of multiple closure, we know that if $f = w_1 \dots w_n$, then $(I_1, \dots, I_n)^{\underline{*}} = \left(J'S[I't]_{(ft)}\right)^* \cap R$. So we have

$$(I_1, \dots, I_n)^{\underline{*}} = ((I'_1, \dots, I'_n)S)^{\sim} \cap R.$$

□

One can see that tight closure commuting with localization and TI closure commuting with localization are equivalent properties.

Theorem 4.2.3. *Let $\mathcal{R}$ denote the class of all Noetherian rings R, such that R is either essentially of finite type over an excellent local ring or R_{red} is F-Finite. Then TI closure commutes with localization for all rings in $\mathcal{R}$ if and only if tight closure commutes with localization for all rings in $\mathcal{R}$.*

Proof. It is clear that if TI closure commutes with localization for any ring, then so will tight closure, since the tight closure of an ideal is equal to the TI closure of a set of principal ideals (Theorem 2.4.2).

Now suppose that tight closure commutes with localization, and take a set of ideals $I_1, \dots, I_n$ in a ring R described above. Let U be a multiplicative set in R. Then

$$\begin{aligned} U^{-1}(I_1, \dots, I_n)^{\underline{*}} &= U^{-1}\left[\left((w_1, \dots, w_n)R[w_1, \dots, w_n, \tfrac{I_1}{w_1}, \dots, \tfrac{I_n}{w_n}]\right)^* \cap R\right] \\ &= \left((w_1, \dots, w_n)U^{-1}R[w_1, \dots, w_n, \tfrac{I_1}{w_1}, \dots, \tfrac{I_n}{w_n}]\right)^* \cap U^{-1}R \\ &= (U^{-1}I_1, \dots, U^{-1}I_n)^{\underline{*}}. \end{aligned}$$

□

In Section 4.4 we strengthen the statement of Theorem 4.2.3 using the notion of test exponents.

Theorem 4.2.4. *Let R be a Noetherian ring of prime characteristic p such that either R is essentially of finite type over an excellent local ring, or R_{red} is F-Finite. Let $I_1, \dots, I_n$ be ideals of R. Suppose that $c \in R^o$ and $z \in R$ are such that $cz^q \in {I_1}^q + \dots + {I_n}^q$ for infinitely many powers q of p. Then $z \in (I_1, \dots, I_n)^{\underline{*}}$.*

Proof. Let

$$S = R[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}].$$

Since $c \in R^o$, it immediately follows that $c \in S^o$; see Corollary 3.2.3. Also, $cz^q \in {I_1}^q + \dots + {I_n}^q$ implies that $cz^q \in (w_1, \dots, w_n)^{[q]}$ in S for infinitely many q. Therefore

$$z \in \left((w_1, \dots, w_n)R[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}]\right)^* \cap R = (I_1, \dots, I_n)^{\underline{*}}.$$

□

4.3 Test Elements for TI Closure

Theorem 4.1.3 allows us to develop a theory of test elements for TI closure (see Section 2.3). Test elements are useful since they help us decide whether a given element of a ring is in the TI closure of a given set of ideals in that ring. Test elements do not exist for integral closure (see Example 2.5.4), and since the TI closure of one ideal is the integral closure of that ideal (see Section 2.4), we expect the test elements for TI closure to depend on the ideals. We therefore first specify what we mean by test elements for TI closure.

Definition 4.3.1. Let R be a Noetherian ring of prime characteristic p. Let $I_1, \dots, I_n$ be ideals in R. We say that $c \in R^o$ is a *TI closure test element for* $I_1, \dots, I_n$, or in short, a *test element for* $I_1, \dots, I_n$, if for every $z \in (I_1, \dots, I_n)^{\underline{*}}$, $cz^q \in I_1^q + \dots + I_n^q$, for all nonnegative powers q of p. We call the ideal generated by the test elements for $I_1, \dots, I_n$ the *test ideal for* $I_1, \dots, I_n$, and denote it by $\tau(I_1, \dots, I_n)$.

From the proof of Theorem 4.1.3 it immediately follows that:

Corollary 4.3.2. *Let R be a Noetherian ring of prime characteristic p such that either R is essentially of finite type over an excellent local ring, or R_{red} is F-finite. Let $I_1, \dots, I_n$ be ideals of R, and let $S = R[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}]$. Then*

$$\tau(S) \cap R \subseteq \tau(I_1, \dots, I_n).$$

Here, $\tau(S)$ is the usual tight closure test ideal for S. We also obtain *locally stable TI closure test elements for a set of ideals*; these are TI closure test elements of a set of ideals $I_1, \dots, I_n$ that remain test elements for these ideals after we localize the ring at any multiplicative set.

Theorem 4.3.3. *Let R be a Noetherian ring of prime characteristic p such that either R is essentially of finite type over an excellent local ring, or R_{red} is F-finite. Let $I_1, \dots, I_n$ be ideals of R. Let c be an element of R^o such that R_c is regular. Then for some positive integer N and all choices $a_1 \in I_1, \dots, a_n \in I_n$, $(ca_1 \dots a_n)^N$ is a locally stable TI closure test element for the ideals $I_1, \dots, I_n$.*

Proof. Let U be a multiplicative set in R. Suppose $z \in (U^{-1}I_1, \dots, U^{-1}I_n)^{\underline{*}}$. Therefore

$$z \in \left((w_1, \dots, w_n)U^{-1}R[w_1, \dots, w_n, \frac{U^{-1}I_1}{w_1}, \dots, \frac{U^{-1}I_n}{w_n}]\right)^* = \left((w_1, \dots, w_n)U^{-1}S\right)^*,$$

where

$$S = R[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}],$$

and U denotes the multiplicative set U in R, as well as the image of U in S.

Since R_c is regular, so is $S_{cw_1 \dots w_n}$, and so $cw_1 \dots w_n$ to some positive power N is a locally stable test element for S (see Theorem 2.3.7). This means that $(cw_1 \dots w_n)^N$ is a test element for $U^{-1}S$, and therefore $(ca_1 \dots a_n)^N = (cw_1 \dots w_n)^N (\frac{a_1}{w_1} \cdots \frac{a_n}{w_n})^N$ is a test element for $U^{-1}S$ for any $a_i \in I_i$.

For any such test element $d = (ca_1 \dots a_n)^N$ we have

$$dz^q \in (w_1, \dots, w_n)^{[q]} U^{-1}S$$

for all powers q of p, and following the proof of Theorem 4.1.3, we get

$$dz^q \in U^{-1}{I_1}^q + \dots + U^{-1}{I_n}^q$$

for all nonnegative powers q of p. This implies that d is a TI closure test element for $U^{-1}I_1, \dots, U^{-1}I_n$, and since this holds for all U, we conclude that d is a locally stable TI closure test element for $I_1, \dots, I_n$ in R. $\square$

For ideals in a finitely generated algebra over a field of characteristic p, we are able to compute explicit TI closure test elements. Using a theorem of Lipman and Sathaye, Hochster and Huneke described specific tight closure test elements for such rings:

Theorem 4.3.4 (Corollary 1.5.5 [HH2]). *Let k be a field of characteristic p and let R be a d-dimensional geometrically reduced domain over k that is finitely generated as a k-algebra. Let $R = k[u_1, \dots, u_m]/(g_1, \dots, g_s)$ be a presentation of R as a homomorphic image of a polynomial ring. Then the $(m-d) \times (m-d)$ minors of the Jacobian matrix $(\delta g_i/\delta u_j)$ are contained in the test ideal of R, and remain so after localization and completion. Thus, any element of the Jacobian ideal generated by all these minors that is in R^o is a completely stable test element.*

Here the term *geometrically reduced* means the following: If k is a field and $\overline{k}$ is the algebraic closure of k, and R is a k-algebra such that $\overline{k} \otimes_k R$ is reduced, then R is *geometrically reduced* over k.

In practice, k being of characteristic zero or perfect and R being reduced ensures that R is geometrically reduced (see tensor products over fields in [ZS1]).

Theorem 4.3.5. *Let k be a field of characteristic p and let R be a d-dimensional geometrically reduced domain over k that is finitely generated as a k-algebra. Let $R = k[u_1, \dots, u_m]/(g_1, \dots, g_s)$ be a presentation of R as a homomorphic image of a polynomial ring. Take ideals $I_1, \dots, I_n$ of R where for each $i = 1, \dots, n$, I_i is minimally generated by the elements $f_1^i, \dots, f_{m_i}^i$ of R. Then*

$$I_1{}^{m_1-1} \dots I_n{}^{m_n-1} \mathcal{J}_{m-d} \subseteq \tau(I_1, \dots, I_n) \ ,$$

where $\mathcal{J}_{m-d}$ is the ideal generated by the $(m-d) \times (m-d)$ minors of the Jacobian matrix $\mathcal{J}(R) = (\frac{\delta g_i}{\delta u_j})$.

Proof. Let S be the ring

$$S = R[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}].$$

Then there is a surjective map to S from the polynomial ring

$$H = k[u_1, \dots, u_m, w_1, \dots, w_n, x_1^1, \dots, x_{m_1}^1, \dots, x_1^n, \dots, x_{m_n}^n]$$

mapping x_i^j to $\frac{f_i^j}{w_j}$. If for $1 \leq i \leq n$ and $1 \leq j \leq m_i$, F_j^i is an element of the polynomial ring $k[u_1, \dots, u_m]$ whose image in R is f_j^i, the kernel $\wp$ of this map is generated by

$$g_1, \dots, g_s, w_1 x_1^1 - F_1^1, \dots, w_1 x_{m_1}^1 - F_{m_1}^1, \dots, w_n x_1^n - F_1^n, \dots, w_n x_{m_n}^n - F_{m_n}^n$$

$$
\begin{pmatrix}
\begin{matrix} \frac{\delta g_1}{\delta u_1} & \cdots & \frac{\delta g_1}{\delta u_m} \\ \vdots & \ddots & \vdots \\ \frac{\delta g_s}{\delta u_1} & \cdots & \frac{\delta g_s}{\delta u_m} \end{matrix} & \text{\Large 0} \\
\text{\Large A} &
\begin{matrix}
x_1^1 & 0 & \cdots & 0 & w_1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_{m_1}^1 & 0 & \cdots & 0 & 0 & \cdots & w_1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & x_1^2 & \cdots & 0 & 0 & \cdots & 0 & w_2 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & x_{m_2}^2 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & w_2 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & x_1^n & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & w_n & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & x_{m_n}^n & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & w_n
\end{matrix}
\end{pmatrix}
$$

Figure 4.1: Part of the Jacobian matrix

and possibly other polynomials. So S is isomorphic under this map to $H/\wp$. A part of the Jacobian matrix for this presentation of S will then look like the matrix shown in Figure 4.1.

Now $\dim R = d$, and so $\dim S = d + n$. If $m' = m_1 + \ldots + m_n$, then $m + n + m' - (n + d) = m + m' - d$, and so we take the ideal $\mathcal{I}$ of the $(m + m' - d) \times (m + m' - d)$ minors of this matrix. We note that $\mathcal{I}$ is contained in the ideal generated by the $(m + m' - d) \times (m + m' - d)$ minors of the Jacobian matrix of S, and so from Theorem 4.3.4 we can conclude that $\mathcal{I}$ is in the (tight closure) test ideal for S. Corollary 4.3.2 then implies that $\mathcal{I} \cap R \subseteq \tau(I_1, \ldots, I_n)$.

To find elements of $\mathcal{I}$ that are in R, we make the following partition of the matrix:

Step 1. We take the $m' \times m'$ minor of the lower right corner of the matrix by taking the first n columns, and for each i, $i = 1, \ldots, n$, removing one of the columns (say the s_ith column) involving a w_i. The outcome is $x^1_{s_1} \ldots x^n_{s_n} {w_1}^{m_1-1} \ldots {w_n}^{m_n-1}$.

Step 2. The upper left corner of this matrix is just $\mathcal{J}(\mathcal{R})$. By taking the $(m - d) \times (m - d)$ minors of the upper left corner we obtain $\mathcal{J}_{m-d}$.

So

$$x^1_{s_1} \ldots x^n_{s_n} {w_1}^{m_1-1} \ldots {w_n}^{m_n-1} \mathcal{J}_{m-d} \subseteq \mathcal{I}.$$

It follows that:

$$\left(\frac{I_1}{w_1}\right)^{m_1-2} \ldots \left(\frac{I_n}{w_n}\right)^{m_n-2} x^1_{s_1} \ldots x^n_{s_n} {w_1}^{m_1-1} \ldots {w_n}^{m_n-1} \mathcal{J}_{m-d} \subseteq \mathcal{I},$$

and so

$$F^1_{s_1} \ldots F^n_{s_n} {I_1}^{m_1-2} \ldots {I_n}^{m_n-2} \mathcal{J}_{m-d} \subseteq \mathcal{I}, \tag{4.1}$$

since for each i, $x^i_{s_i} w_i = F^i_{s_i} \in I_i$.

By repeating Step 1, and for each $i = 1, \ldots, n$ allowing s_i to vary between 1 and m_i, we see that the inclusion in 4.1 will hold for $1 \leq s_i \leq m_i$ and $i = 1, \ldots, n$. It follows that

$$I_1 \ldots I_n {I_1}^{m_1 - 2} \ldots {I_n}^{m_n - 2} \mathcal{J}_{m-d} = {I_1}^{m_1 - 1} \ldots {I_n}^{m_n - 1} \mathcal{J}_{m-d} \subseteq \mathcal{I}.$$

$\square$

Example 4.3.6. Let $R = k[x, y, z]$ be a polynomial ring over a perfect field k of characteristic p. Let $I = (f, g)$ and $J = (h)$ be ideals of R. Then every element of I is a TI closure test element for the ideals I and J.

4.4 Test Exponents and the Localization of TI Closure

A recent advance in tight closure theory is the development of *test exponents*. Hochster and Huneke show in [HH4] that the existence of test exponents for tight closure is roughly equivalent to tight closure commuting with localization. This result suggests that the study of test exponents could provide a breakthrough in the localization problem for tight closure.

The situation is similar for TI closure: below we define TI closure test exponents, and establish that TI closure commuting with localization is related to the existence of test exponents, although the result we get for TI closure is somewhat weaker than the corresponding result for tight closure in [HH4]. Nevertheless, Theorem 4.4.4 below strengthens our previous statement (Theorem 4.2.3) on the equivalence of tight closure commuting with localization with TI closure commuting with localization.

We begin by stating some relevant facts from tight closure theory.

Definition 4.4.1 ([HH4]). Let R be a reduced Noetherian ring of positive prime characteristic p. Let c be a fixed test element for R, and let I be an ideal of R. Then $q = p^e$ is called a *test exponent for c and I*, if whenever $cu^Q \in I^{[Q]}$ for some $u \in R$ and $Q \geq q$, then $u \in I^*$.

Thus the existence of test exponents reduces the process of checking whether an element u of the ring is in the tight closure of an ideal I, to just checking if $cu^Q \in I^{[Q]}$ for *some* large Q, rather than *all* large Q. It is easy to see that the existence of a test exponent for an ideal forces that ideal to commute with localization (see Proposition 2.3 of [HH4]). The converse of this statement is however difficult to prove. Here, we produce a parallel definition for TI closure test exponents. It will easily follow that the existence of such an exponent for a set of ideals will force the TI closure of those ideals to commute with localization. For the converse, we exploit the fact that

TI closure can be described in terms of tight closure and apply results of [HH4] to achieve the desired statement.

Definition 4.4.2. Let R be a Noetherian of prime characteristic p, and let $I_1, \dots, I_n$ be a set of ideals in R. Let c be a fixed test element for $I_1, \dots, I_n$ in R. Then $q = p^e$ is called a *test exponent for* $c, I_1, \dots, I_n$, if whenever $cu^Q \in {I_1}^Q + \dots + {I_n}^Q$ for some $u \in R$ and $Q \geq q$, then $u \in (I_1, \dots, I_n)^{\underline{*}}$.

Theorem 4.4.3. *Let R be a Noetherian ring of prime characteristic p, and let $I_1, \dots, I_n$ be ideals of R. Suppose that c is a locally stable test element for $I_1, \dots, I_n$, and suppose that c, $I_1, \dots, I_n$ have a test exponent q. Then TI closure commutes with localization for $I_1, \dots, I_n$.*

Proof. Let U be a multiplicative set in R, and let $z/1 \in (U^{-1}I_1, \dots, U^{-1}I_n)^{\underline{*}}$. We have

$$cz^q/1 \in U^{-1}{I_1}^q + \dots + U^{-1}{I_n}^q,$$

and so for some $u \in U$, $ucz^q \in {I_1}^q + \dots + {I_n}^q$, hence $c(uz)^q \in {I_1}^q + \dots + {I_n}^q$. Since q is a test exponent, this implies that $uz \in (I_1, \dots, I_n)^{\underline{*}}$, and so

$$z/1 \in U^{-1}(I_1, \dots, I_n)^{\underline{*}}.$$

□

Theorem 4.4.4. *Let R be a Noetherian ring of prime characteristic p that is either essentially of finite type over an excellent local ring or R_{red} is F-Finite. Let $I_1, \dots, I_n$ be ideals of R. Suppose that the tight closure of the ideal $(w_1, \dots, w_n)$ in the ring $S = R[w_1, \dots, w_n, I_1/w_1, \dots, I_n/w_n]$ commutes with localization at all primes ideals in $\operatorname{Ass}(S/(w_1, \dots, w_n)^*)$. Let c be a locally stable test element for $I_1, \dots, I_n$ in R, which is also a locally stable test element for the ring S (such test elements exist; see the statement and proof of Theorem 4.3.3). Then $c, I_1, \dots, I_n$ have a test exponent.*

Proof. Since tight closure of $(w_1, \dots, w_n)$ commutes with localization at all primes in $\operatorname{Ass}(S/(w_1, \dots, w_n)^*)$, Theorem 2.4 of [HH4] implies that $c, (w_1, \dots, w_n)$ have a test exponent q.

Now suppose $cu^Q \in {I_1}^Q + \dots + {I_n}^Q$ for some $u \in R$ and $Q \geq q$. Then cu^Q belongs to the ideal $(w_1, \dots, w_n)^{[Q]}$ in S, which implies that $u \in (w_1, \dots, w_n)^*$. Hence

$$u \in (w_1, \dots, w_n)^* \cap R = (I_1, \dots, I_n)^{\underline{*}},$$

which proves that q is a test exponent for $c, I_1, \dots, I_n$. □

We are now able to strengthen one direction of the statement of Theorem 4.2.3.

Corollary 4.4.5. *If R, S, $I_1, \dots, I_n$ are as in Theorem 4.4.4, then the TI closure of $I_1, \dots, I_n$ commutes with localization at any multiplicative set in R, if the tight closure of $(w_1, \dots, w_n)$ in S commutes with localization at all primes that belong to* $\operatorname{Ass}(S/(w_1, \dots, w_n)^*)$.

4.5 TI Closure in Equal Characteristic Zero

The translation of TI closure into tight closure enables us to extend TI closure to rings of characteristic zero in an effective way. The definition of TI closure in characteristic zero was introduced in [Ho2], but due to the difficulty of working with the positive characteristic definition, several questions remained unanswered (see Section 2.4). In this section, we address some of those questions for affine algebras over fields of characteristic zero. We show that TI closure in equal characteristic zero has similar features to tight closure in equal characteristic zero. We also introduce *universal test elements* for TI closure based on universal test elements for tight closure.

We begin by recalling the definition of TI closure in characteristic zero:

Definition 4.5.1. Let R be a finitely generated algebra over a field K of characteristic zero, and let $\mathcal{I} = \{I_1, \dots, I_n\}$ be a set of ideals of R. We say that an element x of R is in the *tight integral closure of* $\mathcal{I}$, denoted by $\mathcal{I}^{\underline{*}}$, if there exist descent data $(D, R_D, I_{1,D}, \dots, I_{n,D})$, such that for every maximal ideal m of D, if $k = D/m$, then $x_k \in \mathcal{I}_k{}^{\underline{*}}$, where $\mathcal{I}_k$ denotes the set of ideals $\{I_{1,k}, \dots, I_{n,k}\}$, with $I_{t,k} = k \otimes_k I_{t,D}$.

See Definition 2.3.11 for the definition of descent data. Similar to the positive characteristic situation, we can describe TI closure in equal characteristic zero in terms of tight closure.

Theorem 4.5.2. *Let R be an affine algebra over a field K of characteristic zero, which can be presented as*

$$R = K[u_1, \dots, u_m]/(g_1, \dots, g_s)$$

where $u_1, \dots, u_m$ are indeterminates and $g_1, \dots, g_s$ are polynomials in $K[u_1, \dots, u_m]$. Suppose $I_1, \dots, I_n$ are ideals of R, and $x \in R$. Then $x \in (I_1, \dots, I_n)^{\underline{*}}$ if and only if

$$x \in \left((w_1, \dots, w_n)R[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}]\right)^*$$

where $w_1, \dots, w_n$ are indeterminates.

Proof. The main point of the proof is that one can construct descent data that work for the ideals $I_1, \dots, I_n$ in R as well as for the ideal $(w_1, \dots, w_n)$ in

$$S = R[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}].$$

Suppose that for each i, $I_i = (f_1^i, \dots, f_{s_i}^i)$ in R and for every i and j, $1 \le i \le n$ and $1 \le j \le s_i$, F_j^i is a polynomial in $K[u_1, \dots, u_m]$ whose image in R is f_j^i. Pick $u \in K[u_1, \dots, u_m]$ whose image in R is x.

We can represent S as a polynomial ring

$$K[u_1, \dots, u_m, w_1, \dots, w_n, \alpha_1^1, \dots, \alpha_{s_1}^1, \dots, \alpha_1^n, \dots, \alpha_{s_n}^n]$$

modulo the ideal J which is generated by $g_1, \dots, g_s$, polynomials of the form $w_i\alpha_j^i - F_j^i$ for $1 \le i \le n$ and $1 \le j \le s_i$, and possibly other polynomials (see the proof of Theorem 4.3.5 for a more detailed description of this isomorphism).

We can now construct descent data D for $(w_1, \dots, w_n)$ and x in S by adjoining the coefficients of all the generators of J in K and the coefficients of u in K to the ring of integers $\mathbf{Z}$. We then replace D by a localization at a single element to assure that it satisfies the properties of descent data (Lemma of Generic Freeness, [HR]). This D will also work as descent data for $I_1, \dots, I_n$ and x in R, since it is an enlargement of some basic descent data that one would construct, and once some descent data work, every enlargement of them work as well (see Section 2.3).

With this construction, we have that $x \in (I_1, \dots, I_n)^{\underline{*}}$ if and only if for every maximal ideal m of D, setting $k = D/m$

$$x_k \in (I_{1,k}, \dots, I_{n,k})^{\underline{*}}$$

in R_k. Since k is a finite and therefore perfect field, R_k will be F-finite, and so equivalently

$$x_k \in \left((w_1, \dots, w_n)R_k[w_1, \dots, w_n, \frac{I_{1,k}}{w_1}, \dots, \frac{I_{n,k}}{w_n}]\right)^*.$$

This is equivalent to $x \in ((w_1, \dots, w_n)S_k)^*$, and since this holds for all residue class fields k of D, we conclude that equivalently $x \in ((w_1, \dots, w_n)S)^*$. □

The description of TI closure as tight closure in characteristic zero yields the following:

Theorem 4.5.3 (independence of choice of descent). *Let K be a field of characteristic zero, R a finitely generated K-algebra, $I_1, \dots, I_n$ ideals of R and $u \in R$. Let $(D, I_{1,D}, \dots, I_{n,D})$ be descent data for R, $I_1, \dots, I_n$ and u. If $u \in (I_1, \dots, I_n)^{\underline{*}}$, then for almost all maximal ideals m of D (i.e. for m in a dense open subset of* MaxSpec*D), if $k = D/m$, then $u_k \in (I_{1,k}, \dots, I_{n,k})^{\underline{*}}$ in R_k.*

Proof. By joining finitely many elements of K to D and localizing at an element, we obtain descent data (A, S_A, γ_A) for

$$S = R[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}],$$

$\gamma = (w_1, \dots, w_n)$ and u. We can localize both A and D at an element of D so that A will be free and hence faithfully flat over D ([HR]).

Let $m \in \mathrm{MaxSpec} D$ and let $k = D/m$. Take $m' \in \mathrm{MaxSpec} A$ that lies over m, and let $k' = A/m'$. Since tight closure is independent of the choice of descent, $u_{k'} \in {\gamma_{k'}}^*$ in $S_{k'}$. On the other hand $S_{k'} = k' \otimes_k S_k$ (where $S_k = R_k[w_1, \dots, w_n, \frac{I_{1,k}}{w_1}, \dots, \frac{I_{n,k}}{w_n}]$). The map $k \longrightarrow k'$ is a finite separable extension of fields, since both k and k' are finite. It follows now from [HH3] Theorem $7.29a^o$ that $u_k \in {\gamma_k}^*$ in S_k, which implies that $u_k \in (I_{1,k}, \dots, I_{n,k})^{\underline{*}}$ in R_k. □

We now introduce the notion of universal test element for TI closure in the case where R is an affine domain over a field of characteristic zero.

Definition 4.5.4. Let R be an affine algebra over a field of characteristic zero that is a domain and let $I_1, \dots, I_n$ be ideals of R. Let $(D, R_D, I_{1,D}, \dots, I_{n,D})$ be descent data. Then an element $c_D \in R_D^o$ is called a *universal test element for $I_{1,D}, \dots, I_{n,D}$* if for every $u \in (I_1, \dots, I_n)^{\underline{*}}$, and almost all $m \in \mathrm{MaxSpec} D$, if $k = D/m$, then

$$c_k {u_k}^q \in {I_{1,k}}^q + \dots + {I_{n,k}}^q,$$

for all positive powers q of p, where p is the characteristic of k.

Similar to the situation in the positive characteristic case, we can explicitly calculate universal test elements for the TI closure of a set of ideals. We first state the analogous theorem for tight closure.

Theorem 4.5.5 ([HH2] 2.4.10). *Let $A \supseteq \mathbf{Z}$ be a domain finitely generated over $\mathbf{Z}$ with fraction field $\mathcal{F}$, and let R_A be a finitely generated A-algebra. Suppose that $R_{\mathcal{F}}$ is an absolute domain of dimension d, that is, $\overline{\mathcal{F}} \otimes_{\mathcal{F}} R_{\mathcal{F}}$ (where $\overline{\mathcal{F}}$ is the algebraic closure of the field $\mathcal{F}$) is a domain. Let*

$$R_A = A[u_1, \dots, u_m]/(g_1, \dots, g_s).$$

Then every element of the ideal generated by the $(m-d) \times (m-d)$ minors of the Jacobian matrix $(\delta g_i / \delta u_j)$ is a universal test element of R_A over A.

Similar to the positive characteristic situation, we can deduce

Theorem 4.5.6. *Let K be a field of characteristic zero and let R be an equidimensional finitely generated reduced algebra over K. Take ideals $I_1, \dots, I_n$ of R where for each $i = 1, \dots, n$, I_i is minimally generated by the elements $f^i_1, \dots, f^i_{m_i}$ of R. Let $(D, R_D, I_{1,D}, \dots, I_{n,D})$ be descent data such that if $\mathcal{F}$ is the fraction field of D, then $R_{\mathcal{F}}$ is an absolute domain of dimension d. Suppose that R_D is presented as*

$$R_D = D[u_1, \dots, u_m]/(g_1, \dots, g_s).$$

Then every nonzero element of

$$I_{1,D}^{m_1-1}\dots I_{n,D}^{m_n-1}\mathcal{J}_{m-d}$$

is a universal test element for $I_{1,D},\dots,I_{n,D}$*, where* $\mathcal{J}_{m-d}$ *is the ideal generated by the* $(m-d)\times(m-d)$ *minors of the Jacobian matrix* $(\delta g_i/\delta u_j)$.

Proof. Let $u \in (I_1,\dots,I_n)^{\underline{*}}$. Then for almost all maximal ideals m of R_D, if $k = D/m$, $u_k \in (I_{1,k},\dots,I_{n,k})^{\underline{*}}$. Equivalently, by Theorem 4.5.2, $u_k \in ((w_1,\dots,w_n)S_k)^*$, where

$$S_D = R_D[w_1,\dots,w_n,\frac{I_1}{w_1},\dots,\frac{I_n}{w_n}].$$

If c is a universal test element for S_D over S, then c_k is a test element for S_k for almost all $m \in \mathrm{MaxSpec}D$, $k = D/m$ (see [HH2] or Theorem 2.3.13). If c happens to be in R_D, it will follow that c_k is a TI closure test element for the ideals $I_{1,k},\dots,I_{n,k}$ for almost all $m \in \mathrm{MaxSpec}D$, $k = D/m$, and so c will be a universal test element for $I_{1,D},\dots,I_{n,D}$.

As in Theorem 4.3.5 we use the D-algebra structure of S_D and construct part of its Jacobian matrix, and take appropriate minors of the Jacobian to generate universal test elements for $I_{1,D},\dots,I_{n,D}$ in R_D.

With the presentation of R_D as a finitely generated D-algebra given above, S_D will be isomorphic to the polynomial ring

$$D[u_1,\dots,u_m,w_1,\dots,w_n,x_1^1,\dots,x_{m_1}^1,\dots,x_1^n,\dots,x_{m_n}^n]$$

modulo the ideal generated by

$$g_1,\dots,g_s,w_1x_1^1-F_1^1,\dots,w_1x_{m_1}^1-F_{m_1}^1,\dots,w_nx_1^n-F_1^n,\dots,w_nx_{m_n}^n-F_{m_n}^n$$

and possibly other polynomials, where F_j^i is an element of the polynomial ring $D[u_1,\dots,u_m]$ whose image in R_D is f_j^i, $1 \le i \le n$, $1 \le j \le m_i$. A part of the Jacobian matrix of S_D will then look like the matrix shown in Figure 4.1 on page 43.

We are given that $\dim R_{\mathcal{F}} = d$, and so $\dim S_{\mathcal{F}} = d+n$. If $m' = m_1+\dots+m_n$, then $m+n+m'-(n+d) = m+m'-d$, and so to obtain universal test elements via Theorem 4.5.5, we are interested in the ideal generated by the $(m+m'-d)\times(m+m'-d)$ minors of this matrix, which is contained in the ideal generated by the $(m+m'-d)\times(m+m'-d)$ minors of the Jacobian matrix of S_D.

Following the exact same steps as in the proof of Theorem 4.3.5, we can see that this ideal will contain the ideal

$$\mathcal{H} = I_{1,D}^{m_1-1}\dots I_{n,D}^{m_n-1}\mathcal{J}_{m-d}.$$

Since $\mathcal{H} \subseteq R_D$, every element of $\mathcal{H}$ gives a universal test element for $I_{1,D},\dots,I_{n,D}$.

$\square$

CHAPTER V

Normal Ideals of Graded Rings

In this chapter, we explore different methods to construct classes of normal ideals for graded rings. These are ideals that produce normal Rees rings for normal rings. In most practical cases, this means that they produce blowups that are smooth in codimension one. The method we introduce to construct such ideals is an effective one: given a graded ring, we can easily produce several normal ideals. We will begin by recalling some necessary definitions. For more on the construction of Rees rings see [V].

5.1 Basic Facts

Suppose that R is any $\mathbf{N}$-graded domain, which is a quotient of a polynomial ring $k[x_0, \ldots, x_m]$ modulo a homogeneous ideal J, where $k = R_0$ is an arbitrary domain and $x_0, \ldots, x_m$ are variables of positive weights $A_0, ..., A_m$. In practice, k is usually a field. Let $\mathbf{m} = (x_0, ..., x_m)$ be the irrelevant ideal of R. By abuse of notation, by $x_0, \ldots, x_m$ we will mean the images of the variables $x_0, \ldots, x_m$ in R. Throughout, $R_{\geq\alpha}$ refers to the ideal of R generated by the elements of degree at least α in the graded ring R.

We begin our search for normal ideals by reviewing some basic facts about them.

Theorem 5.1.1. *If S is a normal domain, then $S[It]$ is normal iff I^n is integrally closed for every positive integer n.*

Proof. This follows immediately from Corollary 2.2.5, since the normalization of $S[It]$ has the form

$$S \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \ldots .$$

$\square$

Theorem 5.1.2. *In an $\mathbf{N}$-graded domain $(S, \mathbf{m})$, for any positive integer α, the $\mathbf{m}$-primary ideal $I = S_{\geq\alpha}$ is integrally closed.*

Proof. If x is a homogeneous element of S which lies in $\bar{I}$, then there is some nonzero $c \in S$ such that $cx^n \in I^n$ for all positive integers n. It follows that $\deg cx^n \geq n\alpha$. Hence $\deg c + n \deg x \geq n\alpha$, and so $\deg x \geq \alpha + \frac{\deg c}{n}$ for all n. As n gets very large, we see that $\deg x \geq \alpha$ and so $x \in I = S_{\geq \alpha}$. □

Using these two facts, we will be looking for an ($\mathbf{m}$-primary) ideal I of the form $R_{\geq \alpha}$, with the property that $I^n = R_{\geq n\alpha}$ for all integers $n \geq 1$. It is worth pointing out that Lemma 2.1.6 in [EGA] guarantees the exists of such an α; we discuss this in Section 5.5.

5.2 Main Theorem

Theorem 5.2.1 (Main Theorem). *Let R be a graded domain, which is a quotient of a polynomial ring $k[x_0, \dots, x_m]$ modulo a homogeneous ideal J, where k is an arbitrary domain and $x_0, \dots, x_m$ are variables of positive weights $A_0, ..., A_m$. Let A be the least common multiple of $A_0, ..., A_m$. Then the ideal $I = R_{\geq mA}$ is a normal ideal. In particular, if R is normal, the Rees ring $R[It]$ is normal.*

We will show that for all positive integers p, $I^p = R_{\geq pmA}$. By Theorem 5.1.2, this will complete the proof.

Theorem 5.2.2 (Inductive Step). *Let R, I, and A be as in Theorem 3. For $p \geq 2$, if $I^{p-1} = R_{\geq (p-1)mA}$ then $I^p = R_{\geq pmA}$.*

It is obvious that $I^p \subseteq R_{\geq pmA}$. We have to show that the other inclusion holds. Also, observe that elements in $R_{\geq pmA}$ are sums of monomials in the x_i with coefficients in k, whose degrees are larger than or equal to pmA, and therefore we notice that $R_{\geq pmA}$ is generated by such monomials. Hence it is enough to show that every monomial of $k[x_0, \dots, x_m]$ which lies in $R_{\geq pmA}$ also lies in I^p.

Lemma 5.2.3. *With notation as above, let $x_0^{c_0} ... x_m^{c_m} \in R_{\geq pmA}$, where $c_0, ..., c_m$ are nonnegative integers, and let $a_0, ..., a_m$ be positive integers such that $a_i A_i = A$ for all i. Assume $I^{p-1} = R_{\geq (p-1)mA}$. Fix n such that $1 \leq n \leq m$, and suppose that $a_i \leq c_i$ for $0 \leq i < n$. For each i smaller than n, let k_i be the unique positive integer such that $k_i a_i \leq c_i < (k_i + 1) a_i$. Then:*

1) if $k_0 + ... + k_{n-1} \leq m - 1$, then for some a_j $(n \leq j \leq m)$, $a_j \leq c_j$;

2) if $k_0 + ... + k_{n-1} \geq m$, then $x_0^{c_0} ... x_m^{c_m} \in I^p$.

Proof of Lemma. 1) If $c_i < a_i$ for all $n \leq i \leq m$, then

$$\begin{aligned} pmA &\leq c_0A_0 + ... + c_mA_m \\ &< (k_0+1)a_0A_0 + ... + (k_{n-1}+1)a_{n-1}A_{n-1} + a_nA_n + ... + a_mA_m \\ &= (k_0 + ... + k_{n-1})A + nA + (m-n+1)A \\ &\leq (m-1)A + nA + (m-n+1)A \\ &= 2mA. \end{aligned}$$

It follows that $p < 2$, contrary to the hypothesis.

2) Choose nonnegative integers $s_0, ..., s_{n-1}$ such that $s_i \leq k_i$ for all $0 \leq i \leq n-1$, and $s_0 + ... + s_{n-1} = m$. We see that

$$x_0^{c_0}...x_m^{c_m} = x_0^{s_0a_0}...x_{n-1}^{s_{n-1}a_{n-1}} \cdot x_0^{c_0-s_0a_0}...x_{n-1}^{c_{n-1}-s_{n-1}a_{n-1}}x_n^{c_n}...x_m^{c_m}.$$

Now

$$\begin{aligned} &\deg x_0^{s_0a_0}...x_{n-1}^{s_{n-1}a_{n-1}} \\ &= s_0A + ... + s_{n-1}A \\ &= mA, \end{aligned}$$

which implies that $x_0^{s_0a_0}...x_{n-1}^{s_{n-1}a_{n-1}} \in I$.

On the other hand

$$\begin{aligned} &\deg x_0^{c_0-s_0a_0}...x_{n-1}^{c_{n-1}-s_{n-1}a_{n-1}}x_n^{c_n}...x_m^{c_m} \\ &= \deg x_0^{c_0}...x_m^{c_m} - mA \\ &\geq pmA - mA \\ &= (p-1)mA. \end{aligned}$$

Therefore $x_0^{c_0-s_0a_0}...x_{n-1}^{c_{n-1}-s_{n-1}a_{n-1}}x_n^{c_n}...x_m^{c_m} \in I^{p-1}$, so $x_0^{c_0}...x_m^{c_m} \in I^p$.

□

Proof of Inductive Step. We have $x_0^{c_0}...x_m^{c_m} \in R_{\geq pmA}$, where $c_0, ..., c_m$ are nonnegative integers, and we want to show that it lies in I^p as well. Let $a_0, ..., a_m$ be positive integers such that $a_iA_i = A$ for all i.

If $c_i \geq ma_i$ for any i, say if $c_0 \geq ma_0$, then we will get

$$x_0^{c_0}...x_m^{c_m} = x_0^{ma_0} \cdot x_0^{c_0-ma_0}...x_m^{c_m}.$$

Now, $\deg x_0^{ma_0} = ma_0A_0 = mA$, and so $x_0^{ma_0} \in I$. On the other hand,

$$\deg x_0^{c_0-ma_0}...x_m^{c_m} = \deg x_0^{c_0}...x_m^{c_m} - mA \geq pmA - mA = (p-1)mA.$$

So $x_0^{c_0-ma_0}...x_m^{c_m} \in I^{p-1}$ by our assumption. Therefore $x_0^{c_0}...x_m^{c_m} \in I^p$, and we are done.

So let us look at the case where $c_i < ma_i$ for all i.

Now, we cannot have $c_i < a_i$ for all i, because in that case we get

$$pmA \leq c_0A_0 + ... + c_mA_m < a_0A_0 + ... + a_mA_m = (m+1)A,$$

and so

$$pm < m+1 \Rightarrow p < 1 + \frac{1}{m} \leq 2,$$

which is a contradiction.

So without loss of generality let $c_0 \geq a_0$, and let k_0 be the positive integer ($0 < k_0 < m$) such that $k_0a_0 \leq c_0 < (k_0+1)a_0$.

Hereafter, we proceed by induction: Having chosen $k_0, ..., k_{n-1}$, if $k_0 + ... + k_{n-1} \geq m$, then $x_0^{c_0}...x_m^{c_m} \in I^p$ by the above lemma, and hence we are done.

However, if $k_0 + ... + k_{n-1} \leq m-1$, by part 1 of the lemma there exists a $j \geq n$ for which $a_j \leq c_j$. We can without loss of generality assume that $j = n$, and repeat the same cycle. Thus we are reduced to considering the case where $a_i \leq c_i$ for $i = 0, ..., m$. Let the $k_0, ..., k_m$ be the unique positive integers for which $k_ia_i \leq c_i < (k_i+1)a_i$, $(0 \leq i \leq m)$. We can easily see that $k_0 + ... + k_m \geq m$, because otherwise

$$\begin{aligned} pmA &\leq c_0A_0 + ... + c_mA_m \\ &< (k_0+1)a_0A_0 + ...(k_m+1)a_mA_m \\ &= (k_0 + ... + k_m)A + (m+1)A \\ &\leq (m-1)A + (m+1)A \\ &= 2mA, \end{aligned}$$

which implies that $p < 2$, contrary to our assumption. It follows then from part 2 of the same lemma that $x_0^{c_0}...x_m^{c_m} \in I^p$. This completes the proof of the inductive step. □

Conclusion of Proof of Main Theorem. We need I^p to be integrally closed for all $p \geq 1$. By Theorem 5.1.2, it suffices to have $I^p = R_{\geq pmA}$ for $p \geq 1$. We prove this by induction. The case $p = 1$ is the definition of I. If $I^k = R_{\geq kmA}$ for all $1 \leq k \leq p-1$, then from the inductive step it follows that $I^p = R_{\geq pmA}$. If R is normal, it follows from Theorem 5.1.1 that $R[It]$ is a normal ring. □

5.3 Examples

Example 5.3.1. Let k be a field, and $R = k[x,y,z]/(x^2+y^3-z^5)$, where x, y, z have weights 15,10, and 6, respectively. the least common multiple of these variables is 30, and therefore by theorem 3,

$$\begin{aligned} I &= R_{\geq 60} = (x^4, x^3y^2, x^3yz, x^3z^3, x^2y^3, x^2y^2z^2, x^2yz^4, x^2z^5, xy^5, \\ &xy^4z, xy^3z^3, xy^2z^5, xyz^6, xz^8, z^{10}, yz^9, y^2z^7, y^3z^5, y^4z^4, y^5z^2, y^6) \end{aligned}$$

is a normal ideal for this ring.

Example 5.3.2. In general, if k is a field, and $R = k[x, y, z]/(x^a + y^b + z^c)$ is a domain, where the variables x, y, z have weights bc, ac, and ab, respectively, the ideal $I = R_{\geq 2abc}$ will be a normal ideal of R.

Example 5.3.3. For the polynomial ring $R = k[x, y, z]$ over a field k, one can find many normal ideals by assigning different weights to the variables x, y and z. For example, if we set $\deg x = 1$, $\deg y = 1$ and $\deg z = 2$ we find that the ideal $I = R_{\geq 4} = (x^4, x^3y, x^2y^2, x^2z, xy^3, xyz, y^4, y^2z, z^2)$ is a normal ideal.

Remark 5.3.4. The method described in Example 5.3.3 above for finding normal ideals of polynomial rings is not as interesting in the case of two variables. It is a theorem due to Zariski (see [ZS2]), that the set of integrally closed ideals in a regular local ring of dimension two is closed under multiplication. Therefore, in the ring $R = k[x, y]$, all integrally closed ideals are normal. In particular, all ideals of the form $I = R_{\geq \alpha}$, where α is a positive integer, are normal (all powers of I are integrally closed).

Remark 5.3.5. Looking at Theorems 5.1.1 and 5.1.2, it is natural to wonder whether for an $\mathbf{N}$-graded normal ring R and all positive integers α, one can say that $\overline{(R_{\geq \alpha})^n} = R_{\geq n\alpha}$ for $n \in \mathbf{N}$. The answer in general is no. For example, consider the ring $R = k[x, y]$ where k is a field, $\deg x = 2$, and $\deg y = 3$. Let $I = (R_{\geq 7})^3$. We will check if $\overline{I}$ is equal to $R_{\geq 21}$. Since $R_{\geq 7} = (x^4, yx^2, xy^2, y^3)$, we can calculate $I = (x^{12}, x^{10}y, x^8y^2, x^6y^3, x^5y^4, x^4y^5, x^3y^6, x^2y^7, xy^8, y^9)$. If Γ is the set of pairs (a, b) corresponding to generators x^ay^b of I, we plot the points in Γ on the $\mathbf{R^2}$ plane, and we see that there are no pairs of positive integers (c, d) in the convex hull of the region $\Gamma + \mathbf{R^{2+}}$, besides those in $\Gamma + \mathbf{R^{2+}}$ itself. This implies that I is an integrally closed ideal (see [E]). On the other hand $x^{11} \in R_{\geq 21}$, but x^{11} does not belong to $I = \overline{I}$. Therefore $R_{\geq 21}$ cannot be the integral closure of I. (Note that it also follows from Zariski's theorem mentioned in Remark 5.3.4 that I is an integrally closed ideal.)

Remark 5.3.6. One might ask if all ideals of the form $I = R_{\geq \alpha}$, $\alpha \in \mathbf{N}$, in a normal graded ring are normal. The answer is no. A counterexample is the ring $R = k[x, y, z]/(x^2 + y^3z + z^4)$, where x, y and z have degrees 2, 1 and 1, respectively. The ideal $I = R_{\geq 1} = (x, y, z)$ is integrally closed, but $I^2 = (x^2, xy, xz, y^2, yz, z^2)$ is not: $(x)^2 + (y^2)(yz) + (z^2)^2 = 0$, and hence $x \in \overline{I^2}$, but x is not in I^2. This, by the way, is an example of a ring whose test ideal is not normal, given by Hara and Smith in [HSm].

5.4 How Effective is this Bound?

A natural question for a graded ring of the form

$$R = k[x_0, \dots, x_m]/J$$

that we have been studying in this chapter is: What is the smallest positive integer α such that $R_{\geq\alpha}$ is a normal ideal? We do not know the answer to this question. However, we know that the bound $\alpha = mA$ is the sharpest one can get up to multiples of A (with notation as in Theorem 5.2.1). That is, if $n < m$ and $\alpha = nA$, $R_{\geq\alpha}$ is not necessarily a normal ideal.

Example 5.4.1. Take a polynomial ring $R = k[x, y, z]$ over a field k, such that

$$\deg x = 12 \quad \deg y = 15 \quad \deg z = 20.$$

$A = \operatorname{lcm}(12, 15, 20) = 60$. We know that $I = R_{\geq 120}$ is a normal ideal. The ideal $J = R_{\geq A} = R_{\geq 60}$ is however not a normal ideal. This is because J^2 is not integrally closed:

$$x^3y^3z^2 \in \overline{J^2}$$

since

$$(x^3y^3z^2)^2 = (x^5z^3)(xy^6z) \in (J^2)^2,$$

but

$$x^3y^3z^2 \notin J^2.$$

An interesting question is whether in general, given a ring $R = k[x_0, \dots, x_m]/J$ as above, $R_{\geq mA-1}$ is a normal ideal. We do not know the answer to this question.

5.5 Another Class of Normal Ideals

Let R be a graded ring of the form $k[x_0, \dots, x_m]/J$, where k is a Noetherian domain. By applying Lemma 2.1.6 of [EGA] to

$$R^\natural = R_{\geq 0} \oplus R_{\geq 1} \oplus R_{\geq 2} \oplus \dots,$$

one can identify a class of normal ideals for R, different from those described earlier in this chapter.

Theorem 5.5.1. *Let $R = k[x_0, \dots, x_m]/J$ be a graded domain, where k is a Noetherian normal domain. Suppose that the variables $x_0, \dots, x_m$ have positive weights $A_0, \dots, A_m$, respectively. Then the ideal $I = R_{\geq h}$ is a normal ideal, where*

$$h = \operatorname{lcm}(1, 2, \dots, \max(A_0, \dots, A_m)).$$

Proposition 5.5.2. *Let $R = k[x_0, \dots, x_m]/J$ be a graded domain over a Noetherian domain k, and let the variables $x_0, \dots, x_m$ have positive weights $A_0, \dots, A_m$, respectively. Then the graded blowup $R^\natural = R_{\geq 0} \oplus R_{\geq 1}t \oplus R_{\geq 2}t^2 \oplus \dots$ is generated as an $(R^\natural)_0 = R_{\geq 0}$ algebra by the homogeneous elements*

$$x_0 t, \dots, x_0 t^{A_0}, x_1 t, \dots, x_1 t^{A_1}, \dots, x_m t, \dots, x_m t^{A_m} \tag{5.1}$$

of $R^\natural$.

Proof. For a nonnegative integer α, the elements of $R_{\geq \alpha}$ can be written as sums of monomials in $k[x_0, \dots, x_m]$. We consider a homogeneous element $\mathbf{x} = {x_0}^{c_0} \dots {x_m}^{c_m} t^n$ of $R^\natural$, where $c_i \geq 0$ for $i = 0, \dots, m$; so $c_0 A_0 + \dots + c_m A_m \geq n$. If all the c_i are 0, then $n = 0$ and $\mathbf{x} \in (R^\natural)_0$, and so there is nothing to show. Suppose, without loss of generality, that $c_0, \dots, c_s$ are nonzero for some $s \leq m$, and that $c_{s+1} = c_{s+2} = \dots = c_m = 0$. Also, suppose that $e \leq s$ is such that

$$A_0 c_0 + \dots + A_{e-1} c_{e-1} \leq n \leq A_0 c_0 + \dots + A_e c_e. \tag{5.2}$$

We can now write

$$\mathbf{x} = x_{e+1}^{c_{e+1}} \dots x_s^{c_s} (x_0 t^{A_0})^{c_0} \dots (x_{e-1} t^{A_{e-1}})^{c_{e-1}} ({x_e}^{c_e} t^{n - A_0 c_0 - \dots - A_{e-1} c_{e-1}}).$$

From the inequality in 5.2 it follows that $0 \leq n - A_0 c_0 - \dots - A_{e-1} c_{e-1} \leq A_e c_e$. Let $0 \leq c \leq c_e$ be such that

$$A_e c \leq n - A_0 c_0 - \dots - A_{e-1} c_{e-1} < A_e (c+1).$$

Set $A = n - A_0 c_0 - \dots - A_{e-1} c_{e-1} - A_e c$. We see that $0 \leq A \leq A_e$, where $A \leq A_e$ if $c < c_e$, and $A = 0$ if $c = c_e$. So we can write

$$\mathbf{x} = x_{e+1}^{c_{e+1}} \dots x_s^{c_s} (x_0 t^{A_0})^{c_0} \dots (x_{e-1} t^{A_{e-1}})^{c_{e-1}} (x_e t^{A_e})^c ({x_e}^{c_e - c} t^A),$$

which is a monomial generated over $k[x_0, \dots, x_m]$ by the terms appearing in 5.1. □

Proof of Theorem 5.5.1. By the arguments given in Section 5.1, the ideal $I = R_{\geq h}$ is normal if $I^n = R_{\geq nh}$ for all positive integers n. By Proposition 5.5.2, $R^\natural$ is an algebra of finite type over $(R^\natural)_0$, and so $R^\natural$ is Noetherian. Now, Lemma (2.1.6) of [EGA] implies that there exists a positive integer h, for which $(R^\natural)_{nh} = ((R^\natural)_h)^n$ for all $n > 0$. So there is an h for which $(R_{\geq h})^n = R_{\geq nh}$ for all $n > 0$. Moreover, the proof of this lemma shows that h is the least common multiple of the weights of the $(R^\natural)_0$-algebra generators of $R^\natural$. So by Proposition 5.5.2, $h = \mathrm{lcm}(1, 2, \dots, \max(A_0, \dots, A_m))$. Now if we set $I = R_{\geq h}$, we have $I^n = R_{\geq nh}$ for all positive n, and hence I is a normal ideal. □

Remark 5.5.3. The number h given in Theorem 5.5.1 is very useful mostly in the cases where the weights of the variables are small. Theorem 5.2.1 says that in a polynomial ring $k[x_0, \dots, x_m]$, where $x_0, \dots, x_m$ have weights $A_0, \dots, A_m$, respectively, the ideal $J = R_{\geq \alpha}$ is a normal ideal where $\alpha = m\,\mathrm{lcm}(A_0, \dots, A_m)$. So let us compare these two given numbers α and h:

If $R = k[x, y, u, v]$, where the variables x, y, u and v have weights 1,2,2 and 1, respectively, then $h = 2$, whereas $\alpha = 3 \times 2 = 6$, and so h gives a better normal ideal.

But if $R = k[x, y, z]$, where the variables x, y and z have weights 12, 15 and 20, respectively, then $\alpha = 2 \times 60 = 120$, while $h = \mathrm{lcm}(1, 2, \dots, 20) = 232792560$, which is such a large number that it is almost impossible to work with it.

BIBLIOGRAPHY

BIBLIOGRAPHY

[B] Bourbaki, N. *Elements of Mathematics, Commutative Algebra*, Springer Verlag, 1989.

[BH] Bruns, W. and Herzog, J. *Cohen-Macaulay Rings*, vol. 39, Cambridge studies in advanced mathematics, 1993.

[E] Eisenbud, D. *Commutative Algebra with a View Towards Algebraic Geometry*, Graduate Texts in Mathematics, No. 150, Springer Verlag, New York 1995.

[EGA] A. Grothendieck. *Èlèments de Gèomètrie Algèbrique* II. I.H.E.S Publ. math. **4** (1965).

[G] Goto, S. *Integral closedness of complete-intersection ideals*, J. Algebra vol. 108 (1987), 151-160

[H] Hironaka, H. *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. Math. vol. 79 (1964), 109-326.

[Ha] Hara, H. *A characterization of rational singularities in terms of injectivity of Frobenius maps*, Amer. J. Math. 120 (1998), no. 5, 981-996.

[Har] Harris, J. *Algebraic geometry. A first course*, Graduate Texts in Mathematics, No. 133. Springer-Verlag, New York, 1992.

[Hart] Hartshorne, R. *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[HH1] Hochster, M. and Huneke, C. *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. vol. 3 (1990), 31-116.

[HH2] Hochster, M. and Huneke, C. *Tight closure in equal characteristic zero*, preprint.

[HH3] Hochster, M. and Huneke, C. *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. 346 (1994), 1-62.

[HH4] Hochster, M. and Huneke, C. *Localization and test exponents for tight closure*, preprint.

[HH5] Hochster, M. and Huneke, C. *Tight closure and strong F-regularity*, Mm. Soc. Math. France (N.S.) No. 38, (1989), 119–133.

[HHu] Huckaba, S. and Huneke, C. *Normal ideals in regular rings*, J. Reine Angew. Math. 510 (1999), 63-82.

[HIO] Herrmann, M., Ikeda, S. and Orbanz, U. *Equimultiplicity and blowing up. An algebraic study*, With an appendix by B. Moonen, Springer-Verlag, Berlin-New York, 1988.

[Ho1] Hochster, M. Appendix in *Tight closure and its applications* by C. Huneke, CBMS Regional Conference Series in Mathematics, 88.

[Ho2] Hochster, M. *The tight integral closure of a set of ideals*, to appear in J. Algebra.

[HR] Hochster, M. and Roberts, J.L. *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math. 13 (1974), 115-175.

[HS] Huneke, C. and Sally, J.D. *Birational extensions in dimension two and integrally closed ideals*, J. Algebra vol. 115 (1988), 481-500.

[HSm] Hara, N. and Smith, K.E. *On the strong test ideal*, preprint.

[Hu] Huneke, C. *Tight closure and its applications*, CBMS Regional Conference Series in Mathematics, 88.

[L1] Lipman, J. *Rational singularities, with applications to algebraic surfaces and unique factorization*, Inst. Hautes Études Sci. Publ. Math. No. 36 1969 195–279.

[L2] Lipman, J. *Relative Lipschitz-saturation*, Amer. J. Math. 97 (1975), no. 3, 791–813.

[M] Matsumura, H. *Commutative ring theory*, second edition, Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge-New York, 1989.

[O] O'Carroll, L. *A generic approach to the structure of certain normal ideals*, Compositio Math. vol. 104 (1996), 217-225.

[Sm1] Smith, K.E. *F-Rational rings have rational singularities*, American Journal, 119, 159-180 (1997).

[Sm2] Smith, K.E. *Tight Closure commutes with localization in binomial rings*, to appear in Proceedings of AMS.

[Sm3] Smith, K.E. *An introduction to tight closure*, to appear in the proceedings of the Conference in Commutative Algebra and Algebraic Geometry held in Messina, Italy, June 1999.

[V] Vasconcelos, W. *The Arithmetic of Blowup Algebras*, Cambridge University Press, 1994.

[ZS1] Zariski, O. and Samuel, P. *Commutative Algebra*, Vol. 1, GTM 28, Springer Verlag, New York 1958.

[ZS2] Zariski, O. and Samuel, P. *Commutative Algebra*, Vol. 2, Van Nostrand, Princeton 1960.