Chapter 4

Continuous Random Variables and Probability Distributions

4.1

Continuous Random Variables and Probability Distributions

Continuous Random Variables

A random variable X is *continuous* if its set of possible values is an entire interval of numbers (If $A < B$, then any number x between A and B is possible).

Probability Distribution

Let X be a continuous rv. Then a *probability distribution or probability density function (pdf)* of X is a function

$f(x)$ such that for any two numbers a and b,

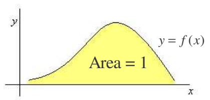
$$P(a \le X \le b) = \int_a^b f(x)dx$$

The graph of $f(x)$ is the *density curve*.

Probability Density Function

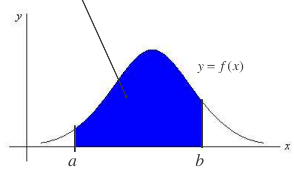
For $f(x)$ to be a pdf

1. $f(x) > 0$ for all values of x.
2. The area of the region between the graph of f and the x – axis is equal to 1.



Probability Density Function

$P(a \le X \le b)$ is given by the area of the shaded region.



Uniform Distribution

A continuous rv X is said to have a *uniform distribution* on the interval $[A, B]$ if the pdf of X is

$$f(x; A, B) = \begin{cases} \dfrac{1}{B-A} & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

Probability for a Continuous rv

If X is a continuous rv, then for any number c, $P(x = c) = 0$. For any two numbers a and b with $a < b$,

$$\begin{aligned} P(a \le X \le b) &= P(a < X \le b) \\ &= P(a \le X < b) \\ &= P(a < X < b) \end{aligned}$$

4.2

Cumulative Distribution Functions and Expected Values

The Cumulative Distribution Function

The cumulative distribution function, $F(x)$ for a continuous rv X is defined for every number x by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(y)dy$$

For each x, $F(x)$ is the area under the density curve to the left of x.

Using $F(x)$ to Compute Probabilities

Let X be a continuous rv with pdf $f(x)$ and cdf $F(x)$. Then for any number a,

$$P(X > a) = 1 - F(a)$$

and for any numbers a and b with $a < b$,

$$P(a \le X \le b) = F(b) - F(a)$$

Obtaining $f(x)$ from $F(x)$

If X is a continuous rv with pdf $f(x)$ and cdf $F(x)$, then at every number x for which the derivative $F'(x)$ exists, $F'(x) = f(x)$.

Percentiles

Let p be a number between 0 and 1. The *(100p)th percentile* of the distribution of a continuous rv X denoted by $\eta(p)$ is defined by

$$p = F(\eta(p)) = \int_{-\infty}^{\eta(p)} f(y)dy$$

Median

The *median* of a continuous distribution, denoted by ũ , is the 50th percentile. So

ũ satisfies 0.5 = F(ũ) That is, half the area under the density curve is to the left of ũ

Expected Value

The *expected* or *mean value* of a continuous rv X with pdf $f(x)$ is

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

Expected Value of $h(X)$

If X is a continuous rv with pdf $f(x)$ and $h(x)$ is any function of X, then

$$E[h(x)] = \mu_{h(X)} = \int_{-\infty}^{\infty} h(x) \cdot f(x)dx$$

Variance and Standard Deviation

The *variance* of continuous rv X with pdf $f(x)$ and mean is μ

$$\sigma_X^2 = V(x) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)dx$$
$$= E[(X-\mu)^2]$$

The *standard deviation* is $\sigma_X = \sqrt{V(x)}$.

Short-cut Formula for Variance

$$\boxed{V(X) = E\left(X^2\right) - [E(X)]^2}$$

Exponential Distribution

A continuous rv X has an *exponential distribution* with parameter λ if the pdf is

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Mean and Variance

The mean and variance of a random variable X having the exponential distribution

$$\mu = \alpha\beta = \frac{1}{\lambda} \qquad \sigma^2 = \alpha\beta^2 = \frac{1}{\lambda^2}$$

Cdf of exponential Distribution and its memoryless property

Let X have a exponential distribution
Then the cdf of X is given by

$$F(x;\lambda) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \ge 0 \end{cases}$$

Applications of the Exponential Distribution

Suppose that the number of events occurring in any time interval of length t has a Poisson distribution with parameter αt and that the numbers of occurrences in nonoverlapping intervals are independent of one another. Then the distribution of elapsed time between the occurrences of two successive events is exponential with parameter $\lambda = \alpha$.

4.3
The Normal Distribution

Normal Distributions

A continuous rv X is said to have a normal distribution with parameters μ and σ, where $-\infty < \mu < \infty$ and $0 < \sigma$, if the pdf of X is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \qquad -\infty < x < \infty$$

Standard Normal Distributions

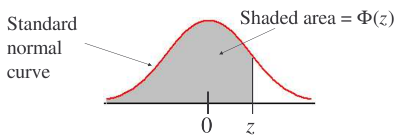
The normal distribution with parameter values $\mu = 0$ and $\sigma = 1$ is called a *standard normal distribution*. The random variable is denoted by Z. The pdf is

$$f(z;0,1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2} \qquad -\infty < z < \infty$$

The cdf is

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} f(y;0,1)dy$$

Standard Normal Cumulative Areas



Standard Normal Distribution

Let Z be the standard normal variable. Find (from table)

a. $P(Z \le 0.85)$

Area to the left of 0.85 = 0.8023

b. $P(Z > 1.32)$

$1 - P(Z \le 1.32) = 0.0934$

c. $P(-2.1 \le Z \le 1.78)$

Find the area to the left of 1.78 then subtract the area to the left of –2.1.

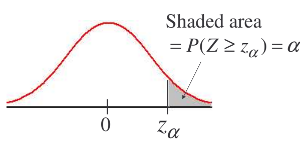
$= P(Z \le 1.78) - P(Z \le -2.1)$

$= 0.9625 - 0.0179$

$= 0.9446$

z_α Notation

z_α will denote the value on the measurement axis for which the area under the z curve lies to the right of z_α.



Ex. Let Z be the standard normal variable. Find z if a. $P(Z < z) = 0.9278$.

Look at the table and find an entry = 0.9278 then read back to find

$\boxed{z = 1.46.}$

b. $P(-z < Z < z) = 0.8132$

$P(z < Z < -z) = 2P(0 < Z < z)$
$= 2[P(z < Z) - ½]$
$= 2P(z < Z) - 1 = 0.8132$

$P(z < Z) = 0.9066$ $\boxed{z = 1.32}$

Nonstandard Normal Distributions

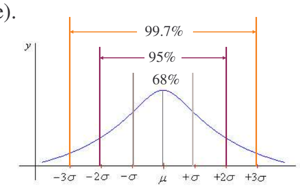
If X has a normal distribution with mean μ and standard deviation σ, then

$$Z = \frac{X-\mu}{\sigma}$$

has a standard normal distribution.

Normal Curve

Approximate percentage of area within given standard deviations (empirical rule).



Ex. Let X be a normal random variable with $\mu = 80$ and $\sigma = 20$.
Find $P(X \le 65)$.

$$P(X \le 65) = P\left(Z \le \frac{65-80}{20}\right)$$
$$= P(Z \le -.75)$$
$$= 0.2266$$

Ex. A particular rash shown up at an elementary school. It has been determined that the length of time that the rash will last is normally distributed with $\mu = 6$ days and $\sigma = 1.5$ days.

Find the probability that for a student selected at random, the rash will last for between 3.75 and 9 days.

$$P(3.75 \le X \le 9) = P\left(\frac{3.75-6}{1.5} \le Z \le \frac{9-6}{1.5}\right)$$
$$= P(-1.5 \le Z \le 2)$$
$$= 0.9772 - 0.0668$$
$$= 0.9104$$

Percentiles of an Arbitrary Normal Distribution

$$\begin{matrix}\text{(100}p\text{)th percentile} \\ \text{for normal } (\mu, \sigma)\end{matrix} = \mu + \begin{bmatrix}\text{(100}p\text{)th for} \\ \text{standard normal}\end{bmatrix} \cdot \sigma$$

Normal Approximation to the Binomial Distribution

Let X be a binomial rv based on n trials, each with probability of success p. If the binomial probability histogram is not too skewed, X may be approximated by a normal distribution with $\mu = np$ and $\sigma = \sqrt{npq}$.

$$P(X \le x) \approx \Phi\left(\frac{x + 0.5 - np}{\sqrt{npq}}\right)$$

Ex. At a particular small college the pass rate of Intermediate Algebra is 72%. If 500 students enroll in a semester determine the probability that at least 375 students pass.

$$\mu = np = 500(.72) = 360$$
$$\sigma = \sqrt{npq} = \sqrt{500(.72)(.28)} \approx 10$$
$$P(X \le 375) \approx \Phi\left(\frac{375.5-360}{10}\right) = \Phi(1.55)$$
$$= 0.9394$$