Chapter 6

Point Estimation

6.1

General Concepts of Point Estimation

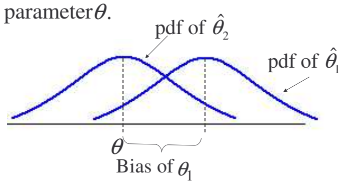
Point Estimator

A *point estimator* of a parameter θ is a single number that can be regarded as a sensible value for θ. A point estimator can be obtained by selecting a suitable statistic and computing its value from the given sample data.

Unbiased Estimator

A *point estimator* $\hat{\theta}$ is said to be an unbiased estimator of θ if $E(\hat{\theta}) = \theta$ for every possible value of θ. If $\hat{\theta}$ is biased, the difference $E(\hat{\theta}) - \theta$ is called the *bias* of $\hat{\theta}$.

The pdf's of a biased estimator $\hat{\theta}_1$ and an unbiased estimator $\hat{\theta}_2$ for a parameter θ.



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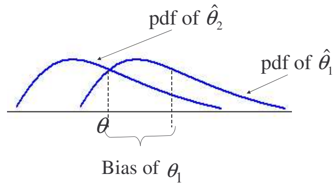


Unbiased Estimator

When X is a binomial rv with parameters n and p, the sample proportion $\hat{p} = X/n$ is an unbiased estimator of p.

Principle of Unbiased Estimation

When choosing among several different estimators of θ, select one that is unbiased.

Unbiased Estimator

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then the estimator

$$\hat{\sigma}^2 = S^2 = \frac{\sum \left(X_i - \overline{X} \right)^2}{n-1}$$

is an unbiased estimator.

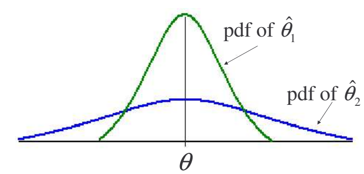
Unbiased Estimator

If $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with mean μ, then $\overline{X}$ is an unbiased estimator of μ. If in addition the distribution is continuous and symmetric, then the sample median and any trimmed mean are also unbiased estimators of μ.

Principle of Minimum Variance Unbiased Estimation

Among all estimators of θ that are unbiased, choose the one that has the minimum variance. The resulting $\hat{\theta}$ is called the *minimum variance unbiased estimator (MVUE)* of θ

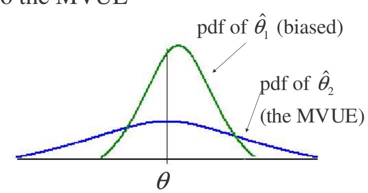
Graphs of the pdf's of two different unbiased estimators



MVUE for a Normal Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with parameters μ and σ. Then the estimator $\hat{\mu} = \overline{X}$ is the MVUE for μ.

A biased estimator that is preferable to the MVUE



Standard Error

The *standard error* of an estimator $\hat{\theta}$ is its standard deviation $\sigma_{\hat{\theta}} = \sqrt{V(\hat{\theta})}$. If the standard error itself involves unknown parameters whose values can be estimated, substitution into $\sigma_{\hat{\theta}}$ yields the *estimated standard error* of the estimator, denoted $\hat{\sigma}_{\hat{\theta}}$ or $s_{\hat{\theta}}$.

6.2

Methods of Point Estimation

Moments

Let $X_1, X_2, \ldots, X_n$ be a random sample from a pmf or pdf $f(x)$. For $k = 1, 2, \ldots$ the *kth population moment*, or *kth moment of the distribution f (x)* is $E(X^k)$. The *kth sample moment* is

$$\left(\frac{1}{n}\right)\sum_{i=1}^{n} X_i^k.$$

Moment Estimators

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pmf or pdf $f(x;\theta_1,\ldots,\theta_m)$, where $\theta_1,\ldots,\theta_m$ are parameters whose values are unknown. Then the *moment estimators* $\theta_1,\ldots,\theta_m$ are obtained by equating the first m sample moments to the corresponding first m population moments and solving for $\theta_1,\ldots,\theta_m$.

Likelihood Function

Let $X_1, X_2, \ldots, X_n$ have joint pmf or pdf

$$f(x_1,\ldots,x_n;\theta_1,\ldots,\theta_m)$$

where parameters $\theta_1,\ldots,\theta_m$ have unknown values. When $x_1,\ldots,x_n$ are the observed sample values and f is regarded as a function of $\theta_1,\ldots,\theta_m$, it is called the *likelihood function*.

Maximum Likelihood Estimators

The maximum likelihood estimates (mle's) $\hat{\theta}_1,\ldots,\hat{\theta}_m$ are those values of the θ_i's that maximize the likelihood function so that

$$f(x_1,\ldots,x_n;\hat{\theta}_1,\ldots,\hat{\theta}_m) \geq f(x_1,\ldots,x_n;\theta_1,\ldots,\theta_m)$$

for all $\theta_1,\ldots,\theta_m$

When the X_i's are substituted in the place of the x_i's, the *maximum likelihood estimators* result.

The Invariance Principle

Let $\hat{\theta}_1,\ldots,\hat{\theta}_m$ be the mle's of the parameters $\theta_1,\ldots,\theta_m$ Then the mle of any function $h(\theta_1,\ldots,\theta_m)$ of these parameters is the function $h(\hat{\theta}_1,\ldots,\hat{\theta}_m)$ of the mle's.

Desirable Property of the Maximum Likelihood Estimate

Under very general conditions on the joint distribution of the sample, when the sample size n is large, the maximum likelihood estimator of any parameter θ is approx.

unbiased $[E(\hat{\theta}) \approx \theta]$ and has variance that is nearly as small as can be achieved by any estimator.

$$\boxed{\text{mle}\,\hat{\theta} \approx \text{MVUE of } \theta}$$