Math 2790: Assignment 4
Due on Friday, November 30$^{\text{th}}$

Answer both A1 and A2. Answer *either* B1 or B2. Both A1 and A2 are worth five marks, and your question from part B will be worth thirty marks.

A1. There were n people invited to a banquet. All but one person arrived for the reception, and during the reception, each person shook hands with everyone else. However, Karin strolled in twenty minutes late, and so she was only able to shake hands with some of the other guests. If there were exactly 73 handshakes in total, determine the number of hands Karin shook.

A2. Three men, A, B, and C are in separate cells under a death sentence. One of these is to be reprieved, but the announcement is not yet to be made. Prisoner A tries to extract the information from the warden, but to no avail. "Then at least", he says, "give me the name of one of the other two who will be executed - that can't hurt, as I can't communicate with them. If B is to be pardoned, give me C's name; if C is to be pardoned, give me B's name; and if I am to be pardoned, give me either B or C by flipping a coin." The warden went off to think about it, and returned to announce that B would be executed. After he left, A smiled to himself: since one of A and C would be spared, his chances of being spared had just risen to one-half. The warden also did not know that A had a way of communicating with C by tapping on the wall, and A immediately shared the news with C who was overjoyed to discover that his chances of survival had also risen to one-half. Did the two men reason correctly?

B1. In the "Price is Right" clip, you saw my friend Ryan advance to the Showcase Showdown. In this question, you will determine the probability that Ryan wins the "Big Wheel". Let's assume there are two contestants (Jacob and Ryan), rather than three, and that Ryan spins second. There are twenty numbers on the wheel, namely 5, 10, 15, 20,..., 100. The objective is to get closest to 100 without exceeding 100, and a player may spin the wheel once or twice. Whoever comes closest wins. After a player's first spin, he may elect to spin the wheel a second time, but if he exceeds 100 on the total of his two spins, he will be eliminated.

Jacob spins first, and Ryan follows. You can assume that Jacob will spin the wheel a second time if he gets a score of 50 or less on his first spin, but will elect to stay otherwise. (He wants to get a respectable score but does not want to get greedy and risk going over 100, so you can assume that Jacob follows this strategy.) Ryan just wants to beat Jacob. So he will spin the wheel a second time if his spin is less than Jacob's total, and will naturally stay if his spin is higher than Jacob's total. In the case of a tie, say that Ryan will spin the wheel a second time if his spin is 50 or less, and will elect to stay otherwise. In the latter situation, there will be a tie and the two will restart the game from the beginning. Once again, Jacob will go first, and Ryan will follow. The rules for this "second round" remain the same as before.

B2. For each positive integer n, define $f(n)$ to be the number of 1's used in writing down all the integers from 1 through n. For example, $f(12) = 5$ because in the sequence $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$, there are exactly five occurrences of the digit 1. Determine all positive integers n for which $f(n) = n$.