# Solutions To Assignment 1

1. Colour the squares black and white, like in a chessboard. Since James and John stand on diagonally opposite corner squares, they start on squares of the same colour. Say they both start on a black square. With every move, each of them moves to a square of the opposite colour. So James moves to a white square, then John moves to a white square, and the whole process repeats. Whenever it is James's turn to move, both James and John are on the same colour square, so James will always be moving to a square of the opposite colour to the square that John is on. Hence regardless of how long this chase continues, James will never be able to move to the square John is on, and hence will never be able to catch John.

2. Since there were 2001 people, everyone shook hands with at least 0 people, and at most 2000 people (you can't shake hands with yourself). Suppose that no two people shook the same number of hands. The only way this could possibly happen is if each of the 2001 people shook a *different* number of hands from 0 to 2000. However, if someone shook 2000 hands, then she shook hands with everybody else at the party. So everyone at the party shook at least one hand. That contradicts the fact that someone at the party shook 0 hands. Thus, there must have been two people who shook the same number of hands.

   *Alternative solution:* We'll prove this statement for all $n$. If everyone shook at least one hand, then the possible number of handshakes for each person must have been between 1 and $n-1$. Otherwise, at least one person did not shake a hand, and so nobody at the party could have shaken all $n-1$ hands. Thus, in this scenario, the possible number of handshakes for each person must have been between 0 and $n-2$. In each case, we have $n$ people and $n-1$ possible numbers of handshakes for each person, and so by the Pigeonhole Principle, at least two people must have shaken the same number of hands.

3. For each $i$ from 0 to $n-1$, we define the function $f(i)$ as follows: rotate the table $i$ positions to the right. Define $f(i)$ to be the number of people who match up with their entrees after rotating the table in this manner.

   So originally, we have $f(0) = 0$, since no one matches up with their entrees. Now, we must have $f(1) + f(2) + \ldots + f(n-1) = n$, since for each of the $n$ people, you can rotate the table exactly one way to match that person up with the correct entree. From here, we can proceed in two ways:

   *Method 1:* By definition, $f(i)$ represents the number of people that are matched up with their correct entrees when you rotate the table $i$ positions to the right. So each of the $n$ people must be matched up with the correct entree in exactly one of the $n-1$ possible rotations. We can describe this as follows: write down the numbers $1, 2, \ldots, n$, with one number for each of the $n$ people. We also have $n-1$ slots, labelled Slot 1, Slot 2, up to Slot $n-1$. Then we take each of our $n$ numbers, and throw them into one of $n-1$ slots, depending on how many table positions you have to rotate to match that number up with his correct entree. For example, if person 7 is matched up with his correct entree by moving the table 4 positions to the right, then we throw the number 7 into Slot 4. So we have $n$ numbers, and $n-1$ slots. By the Pigeonhole Principle, one of these slots will contain two numbers. And so $f(i) \geq 2$ for some $i$. This proves that if we rotate the table $i$ positions to the right, then at least two people will be matched up with their correct entrees.

*Method 2:* Suppose that the conclusion is false, that $f(i) \leq 1$ for each $i = 1, 2, \ldots, n-1$. Then $n = f(1) + f(2) + \ldots + f(n-1) \leq 1 + 1 + \ldots + 1 = n - 1$, a contradiction. Hence, we must have $f(i) \geq 2$ for some $i$, and we are done.

4. Let $S$ be the number of *odd* numbers on the board. So at the beginning, $S = 5$, because the numbers 1, 3, 5, 7, and 9 are on the board. We shall show that the parity of $S$ remains invariant throughout the problem. In every step, Riham removes two numbers $a$ and $b$ and replaces them by $|a - b|$. If $a$ and $b$ are both odd, $|a - b|$ will be even, so $S$ decreases by 2. If $a$ and $b$ are both even, $|a - b|$ will be even, so $S$ stays the same. Finally, if $a$ is odd and $b$ is even (or vice-versa), $|a - b|$ will be odd, and so $S$ stays the same. Thus, in every case, $S$ decreases by 2 or stays the same, and hence the parity of $S$ will remain *invariant*, i.e. the parity of $S$ never changes. Since $S = 5$ at the beginning, $S$ can never go down to 0. So the final number on the board cannot be even, and so it must be odd.

   *Alternative solution:* Let $T$ be the sum of the numbers on the board. So at the beginning, $T = 1 + 2 + 3 + \ldots + 10 = 55$. We shall show that the parity of $T$ remains invariant throughout the problem. In every step, Riham removes two numbers $a$ and $b$ (assume $a \geq b$) and replaces them by $|a - b| = a - b$. Hence we reduce $T$ by $a + b$ and increase it by $a - b$. Thus, $T$ decreases by $2b$ in each step, and so $T$ always decreases by an *even* number, so the parity of $T$ remains invariant. So at the end, the value of $T$ will be the final number left on the board, since $T$ is the sum of the numbers left on the board. Since $T$ is odd, we conclude that this final number must be odd.

5. Look at the first row. There are seven squares, coloured with one of two colours. By the Pigeonhole Principle, four of these squares must have the same colour. Suppose that this colour is indigo. So we have (at least) four columns whose top square is coloured indigo. Just consider these columns, and ignore the other columns of the board. So we are now looking at a 3 by 4 chessboard.

   In this smaller chessboard, every square in the first row is coloured indigo. If there are at least two squares on the second row that are coloured indigo, then we will have a rectangle whose four corner squares are all indigo, then we will be done. So at most one square on the second row is coloured indigo, i.e., at least three squares on the second row are coloured teal. So there must be three columns, all of which have the top square being indigo and the middle square being teal. Just consider these columns. So now we are looking at a 3 by 3 chessboard.

   So in this reduced chessboard, the first row is all indigo, and the second row is all teal. The third row contains three squares, and so by the Pigeonhole Principle, at least two of these squares must have the same colour. If two of the squares are indigo, we have a rectangle (with the first row), and if two of the squares are teal, we have a rectangle (with the second row). Regardless of the situation, we must have a rectangle whose four corner squares are all the same colour.

If the chessboard is 3 by 6, the result does not hold. There are several counterexamples. For example, colour the first row IIITTT where I is indigo and T is teal. Colour the second row ITTIIT and the third row TITITI. Then this board will not contain a rectangle whose four corner squares are all the same colour.

*Alternative Solution:* Some of you came up with this clever solution. Rather than look at three rows of seven squares, flip the diagram clockwise and look at the seven rows, each row having three squares. Each row can be one of the following eight types: TTT, TTI, TIT, ITT, IIT, ITI, TII, III. Now, if any two rows are the same, then we immediately have a rectangle where all four corner squares are the same colour. (For example, if we have rows 3 and 6 being ITI, then we have a rectangle where all four corner squares are indigo). So in our seven rows, we cannot have the same row-type appear twice.

Suppose TTT appears in one of our seven rows. Then, we cannot have TTI, TIT, or ITT appear anywhere, or else we will have a rectangle where all four corner squares are teal. That leaves six rows that we need to fill with four row-types (IIT, ITI, TII, and III). But by the Pigeonhole Principle, at least one row-type will be duplicated twice, and so we will have a rectangle where all four corner squares are the same colour. Hence, TTT cannot appear in any of the rows. By symmetry, III cannot appear in any of the rows either.

So now we have only six possible row-types: TTI, TIT, ITT, IIT, ITI, and TII. But we have seven rows. By the Pigeonhole Principle, at least one row-type will be duplicated twice, and so we will have a rectangle where all four corner squares are the same colour. That clears all the cases, so we are done.

This argument does not hold for a rectangle with six rows and three columns. Just have the six rows being TTI, TIT, ITT, IIT, ITI, and TII. Then there is no rectangle where all four corner squares are the same colour.

6. Here is a winning strategy for Eve: she should start by placing a black checker in the middle. Then for whatever move Oddie makes, Eve should counter by placing a checker of the opposite colour in the diametrically opposite square. For example, if Oddie puts a black checker in the top right corner, then Eve's next move should be a white checker in the bottom left corner.

   Now we justify why this is a winning strategy for Eve. First of all, by Eve's strategy, she guarantees herself four points because every row, column, or diagonal that goes through the centre will have exactly two black checkers. So now we have four points accounted for. All we need to do is prove that Eve must win one more point somewhere among the outside two rows or outside two columns. We'll show that she actually wins exactly two more points.

   Look at the top row and the bottom row. If the top row has $x$ black checkers, then the bottom row has $x$ white checkers (i.e., $3 - x$ black checkers), due to Eve's reverse-copying strategy. Exactly one of $x$ and $3 - x$ will be even, and hence Eve will get one point from either the top row or bottom row. By the same argument, Eve will get exactly one point from the two columns, so this strategy guarantees that she will win by a score of 6 to 2.

7. a) We shall prove that Alison can always win the game by moving clockwise on each of her moves. Notice that whoever enters the centre ring first will *lose*, because there are eight regions and the second player will occupy the eighth and final region in the centre ring, causing the first player to lose. So neither player wants to enter the centre ring first, and so each player wants to force the other to enter the centre ring to force a victory. If a player (say Ian) enters the third ring first, the other player will just move clockwise until either Ian has moven into the centre ring or is forced into the centre ring because Alison moves into the final unoccupied region in that ring. Similarly, neither player wants to enter the second ring first. So the person who can win the game is the person who can force the opponent into the second ring. Since Alison moves first, if she moves clockwise on every move, either Ian will go into the second ring or be forced into the second ring (since there are eight regions in the ring). Then Alison can wait until Ian moves into the third and centre rings, and she will win the game. So her strategy is to never move towards the centre, and always move clockwise in each move, and she is guaranteed to win.

   b) We shall prove that Ian can always win. To analyze this game, let's work backwards. Whoever enters the centre ring first will win, because there are nine regions, and the person who moves into the first region in the centre ring will also move into the ninth and final region, causing that player to win. So each player wants to enter the centre ring first. Thus both players want to avoid moving into the fourth ring, because then the opponent will jump immediately into the centre ring, guaranteeing victory. Hence the winner will be able to force the opponent to move into the fourth ring. Whoever enters the third ring first will be able to occupy the final region in that ring (by continually moving clockwise), and so the opponent will be forced to move into the fourth ring. So the strategy is to get to the third ring first. Thus, both players want to avoid entering the second ring, because the opponent will immediately move into the third ring. So the winning strategy is to force your opponent to move to the second ring. Since there are nine regions, if both Alison and Ian move clockwise (since they both want to avoid moving to the second ring), Alison will be forced into the second ring (or she will move there voluntarily). Then Ian immediately moves to the third ring, and Alison will be forced into the fourth ring (or move there voluntarily). Finally, Ian jumps into the centre ring, and will occupy the ninth and final region of the ring and will win the game. Therefore, Ian has the winning strategy. Another way to say this is that Ian's winning strategy is to copy exactly what Alison does (see how this is the same strategy as the one outlined above?), and by doing that, Ian is guaranteed to win.

8. Notice that the situation for any table is the same as any other table whose sides are in the same ratio. For example, a 6 by 10 table will give the same pocket and the same number of reflections as a 3 by 5 table. Hence instead of looking at an $m$ by $n$ table, we can look at a table whose dimensions are $x$ by $y$, where $x = \frac{m}{\gcd(m,n)}$ and $y = \frac{n}{\gcd(m,n)}$. Since $x < y$, the width of the table is $x$ and the length of the table is $y$.

   So we have our $x$ by $y$ pool table, and let us produce infinitely many copies of this pool table by reflecting the sides about the lines $BC$ and $CD$. Draw a straight line from $A$ at a 45 degree angle until it hits a vertex of one of our "reflected" pool tables. If we let $A$ have coordinates $(0, 0)$, the other endpoint (i.e. pocket) of this straight line must be $(xy, xy)$, because $x$ and $y$ are relatively prime (i.e., $\gcd(x, y) = 1$). In other words, to get to the pocket that the ball goes into, the ball must go across $x$ tables and up $y$ tables.

By the symmetry of the problem, the number of reflections of the ball is identical to the number of *intersections* of the straight line with our set of pool tables. Notice that we go across $x$ tables and up $y$ tables. So our line makes $x - 1$ vertical intersections and $y - 1$ horizontal intersections (we don't include the endpoint $(xy, xy)$ in our count because that is a pocket). Notice that no vertical intersection point coincides with a horizontal intersection point because the straight line ends as soon as it hits a pocket of our pool table, and the straight line ends at $(xy, xy)$.

So there are a total of $(x-1)+(y-1) = x+y-2$ reflections. To determine the pocket the ball goes into, there are four cases to consider:

(1) if $x$ and $y$ are both odd, the ball goes into pocket $C$.

(2) if $x$ is odd and $y$ is even, the ball goes into pocket $D$.

(3) if $x$ is even and $y$ is odd, the ball goes into pocket $B$.

(4) if $x$ and $y$ are both even, the ball goes into pocket $A$.

However, we must reject the final case because $\gcd(x, y) = 1$, and so $x$ and $y$ cannot possibly both be even. So the ball never goes into pocket $A$.

Finally, let's check the case $m = 210$ and $n = 357$. Since $\gcd(210, 357) = 21$, we have $x = 10$ and $y = 17$. So the ball makes $10 + 17 - 2 = 25$ reflections. Since $x$ is even and $y$ is odd, we conclude that the ball will go into pocket $B$.