Probability of Winning at Craps

Recall that this is how the game of craps works. Roll two dice.

(1) If the total is 7 or 11, you win.

(2) If the total is 2, 3, or 12, you lose.

(3) In the other cases (when the sum is 4, 5, 6, 8, 9, or 10), your sum is referred to as your "point". You get the dice again. Now you keep rolling the dice until the sum is either 7, in which case you lose, or the sum is equal to your "point", in which case you win.

Let us calculate the probability of *winning* at the game of craps.

The probability of rolling a 7 with two dice is $\frac{6}{36}$, since there are 36 possible outcomes for the two dice, and there are six desired outcomes, i.e., those that add up to seven, namely $(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$. Similarly, the probability of rolling an 11 with two dice is $\frac{2}{36}$, since the only ways to roll 11 are $(5,6)$ and $(6,5)$. Therefore, the probability of winning on the first roll is $\frac{6}{36}+\frac{2}{36}=\frac{8}{36}=\frac{2}{9}$.

Now, let's examine what happens when we roll a $4,5,6,8,9$, or 10. Let's consider each case separately. First, consider the case when the point is 4.

We continue to roll the dice until the sum is either 4 (in which case we win), or roll a 7 (in which case we lose). We know that the game does not end until either of these two scenarios occur, so we want to determine the probability that the sum is 4 *given that* either the sum 4 or 7 has occurred. To elaborate, to find out the probability of winning the game when the point is 4, this is simply the probability that we roll 4 before we roll 7. So, this is the same as saying, what is the probability that we roll a 4 given that we roll either a 4 or a 7.

This brings us back to *conditional probability*. Recall that $P(A|B)$ represents the probability that event A occurred *given that* event B occurred.

There is a nice formula for conditional probability:

$$P(A|B)=\frac{P(A\cap B)}{P(B)}.$$

So in our example, let A be the event that the sum of the dice is 4, and let B be the event that the sum of the dice is either 4 or 7. We wish to find $P(A|B)$. Well, $A\cap B$ is simply A, namely the event that the sum is 4. Hence $P(A\cap B)=P(A)=\frac{3}{36}$, and $P(B)=\frac{3}{36}+\frac{6}{36}=\frac{1}{4}$, since there are three ways to roll a 3 with two dice, and six ways to roll a 7.

Hence, $P(\text{Roll 4} \mid \text{Roll 4 or 7})=\frac{3/36}{9/36}=\frac{1}{3}$.

Similarly, we find that

$P(\text{Roll 5} \mid \text{Roll 5 or 7})=\frac{4/36}{10/36}=\frac{2}{5}$.

$P(\text{Roll 6} \mid \text{Roll 6 or 7})=\frac{5/36}{11/36}=\frac{5}{11}$.

$P(\text{Roll 8} \mid \text{Roll 8 or 7}) = \frac{5/36}{11/36} = \frac{5}{11}.$

$P(\text{Roll 9} \mid \text{Roll 9 or 7}) = \frac{4/36}{10/36} = \frac{2}{5}.$

$P(\text{Roll 10} \mid \text{Roll 10 or 7}) = \frac{3/36}{9/36} = \frac{1}{3}.$

Hence, the probability of winning when our first roll is 4 is

$$P(\text{Initial Roll Is 4}) \times P(\text{Roll 4} \mid \text{Roll 4 or 7}) = \frac{3}{36} \cdot \frac{1}{3} = \frac{1}{36}.$$

Similarly, we have

$$P(\text{Initial Roll Is 5}) \times P(\text{Roll 5} \mid \text{Roll 5 or 7}) = \frac{4}{36} \cdot \frac{2}{5} = \frac{2}{45}.$$
$$P(\text{Initial Roll Is 6}) \times P(\text{Roll 6} \mid \text{Roll 6 or 7}) = \frac{5}{36} \cdot \frac{5}{11} = \frac{25}{396}.$$
$$P(\text{Initial Roll Is 8}) \times P(\text{Roll 8} \mid \text{Roll 8 or 7}) = \frac{5}{36} \cdot \frac{5}{11} = \frac{25}{396}.$$
$$P(\text{Initial Roll Is 9}) \times P(\text{Roll 9} \mid \text{Roll 9 or 7}) = \frac{4}{36} \cdot \frac{2}{5} = \frac{2}{45}.$$
$$P(\text{Initial Roll Is 10}) \times P(\text{Roll 10} \mid \text{Roll 10 or 7}) = \frac{3}{36} \cdot \frac{1}{3} = \frac{1}{36}.$$

So all of these fractions above represent the probabilities of winning when the first roll is 4, 5, 6, 8, 9, and 10. Notice the symmetry of the numbers. So this is saying that the probability of winning by rolling 10 on your first roll is only $\frac{1}{36}$, which is about three percent!

To find the probability of winning at the game, we just add all these probabilities:

$$\frac{2}{9} + \frac{1}{36} + \frac{2}{45} + \frac{25}{396} + \frac{25}{396} + \frac{2}{45} + \frac{1}{36} = \frac{244}{495}.$$

Since $\frac{244}{495}$ is approximately 49.3 percent, the probability of winning at this game is just under fifty percent.

In a casino, you can bet money on the "Pass Line", or on the "Don't Pass Line". If you bet 5 on the Pass Line, you win 5 dollars if the roller wins the game. If you bet 5 on the Don't Pass Line, you win 5 dollars if the roller loses the game. Since we've just shown that the probability of winning the game is less than fifty percent, it is statistically better to always bet on the "Don't Pass Line" - in other words, always bet that the roller will lose.

If you were to bet on the Don't Pass Line one thousand times, statistically you would expect to win 507 times and lose 493 times. If you were to bet ten dollars each time, your *expected profit* is $507 \cdot 10 - 493 \cdot 10 = 140$ dollars. And that's not bad for an evening's work. However, if you celebrate every time the roller "loses", you will probably get beaten up because everybody else bets hundreds of dollars on the roller to win.