

## Tour 4 - Proof By Contradiction

Rise and shine, my friend. It's 9:00 AM, and time to start the next leg of our journey. Yeah, yeah, I know it's early. But I know a whole bunch of university students who have classes at 8:30 in the morning, and *none* of them have ever complained about having to wake up that early in the morning.

"Have you ever heard of the International Date Line?" , I ask.

"Yeah, I've heard of it, but I can't remember what it is", you reply, yawning.

It's one of those things that many people have heard of but not many people know what it is. The International Date Line is an imaginary line somewhere in the Pacific Ocean, just east of New Zealand. The countries that lie west of that line are always one day ahead of the countries that lie east of that line. So today is Tuesday in Canada. But it's Wednesday in Japan.

"I see. But why do we need it? I mean, what's the point of having such a line? Why can't it be Tuesday in Japan too?" , you reply.

That's a great question. Let's investigate why we need the international date line. Suppose the International Date Line did not exist - what would happen?

Well, it's 9:00 AM in Waterloo, and today is Tuesday. Let's keep track of that. Now there is a three hour time difference between Waterloo and Vancouver. In other words, it is currently 6:00 AM in Vancouver. So if I hop on my private jet right now and fly to Vancouver in *exactly* three hours, what time would I arrive in Vancouver?

"It would be noon Waterloo time, but since you'd be in Vancouver, you would arrive in Vancouver at exactly 9:00 AM Vancouver time", you reply.

Exactly - and it would be 9:00 AM on Tuesday.

You are still skeptical. "I don't see where you are going with this" .

Trust me - this is leading to a very interesting place. Let's just continue this. Now we are in Vancouver at exactly 9:00 AM on Tuesday. The time difference between Vancouver and Tokyo is exactly eight hours, so I guess it would be 1:00 AM in Tokyo right now. Here's a key question: what day would it be in Tokyo?

"Since we're assuming that the International Date Line doesn't exist, it would be Tuesday in Tokyo."

Right. So let me hop on my private jet right now and fly to Tokyo in exactly eight hours. When I land, what time will it be in Tokyo?

"Well, it would be 9:00 AM on Tuesday as well, since you left when it was one AM in Tokyo on Tuesday and you arrived eight hours later."

Excellent. Now the time difference between Tokyo and London is eight hours. So if I hop on my private jet and fly from Tokyo to London in exactly eight hours, once again we arrive in London at exactly 9:00 AM on Tuesday.

"Yeah, I get all that", you say, "but what's the point of doing this?"

We're almost there. Here's where it gets exciting. Now, we're in London. The time difference between London and Waterloo is five hours. So because it's 9:00 AM on Tuesday

in London, it would be 4:00 AM on Tuesday in Waterloo. Now let me hop on my private jet again and fly to Waterloo in exactly five hours. What time will it be when I arrive in Waterloo?

“It would be 9:00 AM on Tuesday in Waterloo”, you say, unimpressed.

Okay, when and where did I *start* my round-the-world excursion?

“You started 9:00 AM on Tuesday in Waterloo... wait a minute, that can’t be right!”

Well, did you agree with everything I was talking about? I mean, I traveled around the world so that I would always arrive at my destination at 9:00 AM on Tuesday, right?

“Sure, but what went wrong? You can’t travel the whole world in zero time. I mean, Superman can, but you sure as heck ain’t Superman!”

Well, I’m no Superman, but everything I said from the beginning was perfectly correct. We started with the simple statement that no international date line exists, and we made several logical *deductions* and concluded that I can travel the entire world in zero time. And that doesn’t make any sense. So what went wrong?

You crack a big smile. “Your first statement was incorrect.”

Precisely. But what does that mean?

“Well, the statement that an International Date Line doesn’t exist must be false, and so that must mean that an International Date Line *must* exist!”

You got it. So in our little geography lesson, you just discovered one of the most powerful ideas in mathematics, and it is known as **Proof by Contradiction**. Here is the idea: to prove a statement is true, you assume the *negation* of that statement, and show that this leads to a contradictory result that cannot possibly be correct. By *negation* I am referring to the exact opposite of that statement. So if a statement is “prime numbers are always odd”, then the negation of that (false) statement is “prime numbers are not always odd”, which is a true statement. Note that the negation of this statement is *not* “prime numbers are always even”. Do you see the subtle difference? In essence, if you can show that the negation of a statement is false, then that is equivalent to showing that the statement is true.

In our International Date Line example, we pretended that the line did not exist and with that information, we proved that we could travel the entire world in zero time, which is a contradictory statement. Therefore, we can conclude that the International Date Line *must* exist. Now this proof does not tell us *where* the International Date Line is, but it does tell us that there must be such an imaginary line somewhere.

Let’s use this idea to solve a collection of really hard mathematics problems.

**Example Problem:** *Prove that there are infinitely many prime numbers*

Let’s ask ourselves, what is the negation of that statement? Well, it would be that there are *not* infinitely many prime numbers, i.e., that the list of prime numbers is finite. This means that we can list the prime numbers: 2, 3, 5, 7, 11, and we would eventually stop at some prime  $P$ , which is the *largest* prime number. Our task is to show that this setup leads to a contradiction.

Consider the number  $N = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdots P + 1$ . So we multiply all of the prime numbers,

and we add 1. This is a very large number! Let's look at the prime factors of  $N$ .

When we divide  $N$  by 2, we get a remainder of 1. That is because  $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdots P$  is divisible by 2, so when we add 1 to this number, we will get a remainder of 1 when we divide  $N$  by 2. So  $N$  is not divisible by 2. Likewise, when we divide  $N$  by 3, we get a remainder of 1, so  $N$  is not divisible by 3. By the same argument,  $N$  is not divisible by 5, or 7, or 11, and so on. We get to the end, and we conclude that  $N$  is also not divisible by  $P$ . Therefore, this number  $N$  is not divisible by any of the primes in our list. But our list contained all of the prime numbers, and so by definition,  $N$  must be prime.

However,  $N$  is larger than any of the numbers in the list  $\{2, 3, 5, 7, 11, \dots, P\}$ . But our list contains *all* the prime numbers, and we have found a prime number  $N$  that is *not* on our list. This is a contradiction. Therefore, our assumption that there are only finitely many primes is false. Hence, we conclude that there are infinitely many prime numbers.

**Problem 4.1:** Prove that  $\sqrt{2}$  is an irrational number.

Suppose that  $\sqrt{2}$  is rational. In other words, suppose that we can write this number as a fraction (reduced to lowest terms). So there exist positive integers  $a$  and  $b$ , with  $\gcd(a, b) = 1$  so that  $\sqrt{2} = \frac{a}{b}$ . We will establish a contradiction.

Let's square both sides. Thus,  $2 = \frac{a^2}{b^2}$ , and we can rewrite this as  $a^2 = 2b^2$ . Since  $2b^2$  is even, that implies that  $a^2$  is even. If  $a^2$  is even, that must mean that  $a$  is even. (For if  $a$  were odd,  $a^2$  would be odd and therefore not even). Hence,  $a$  is an even integer, and so  $a = 2m$ , for some positive integer  $m$ .

Substituting into our equation, we have  $2b^2 = a^2 = (2m)^2 = 4m^2$ , which reduces to  $b^2 = 2m^2$ . Now we have the same argument as before: since  $2m^2$  is even, that means  $b^2$  is even. And this implies that  $b$  is even.

So we have proven that  $a$  and  $b$  must both be even integers. Then these two numbers have a common factor of 2, but this contradicts the given stipulation that  $\gcd(a, b) = 1$ . Therefore, we have established a contradiction, and so  $\sqrt{2}$  cannot be expressed in the form  $\frac{a}{b}$ , where  $a$  and  $b$  are positive integers. Hence, we have proven that  $\sqrt{2}$  is irrational.

**Problem 4.2** Does there exist an infinite sequence of real numbers  $a_1, a_2, a_3, \dots$  so that the sum of any two consecutive terms is positive and the sum of any three consecutive terms is negative?

There are several different ways this can be done. Here is probably the quickest way. Suppose that such a sequence exists. Just look at the first six terms of the sequence:  $a_1, a_2, a_3, a_4, a_5, a_6$ . Let  $S$  be the sum of these six terms.

Since the sum of any two consecutive terms is positive, we have  $a_1 + a_2 > 0$ ,  $a_3 + a_4 > 0$ , and  $a_5 + a_6 > 0$ . So if we add these three inequalities, we get  $S > 0 + 0 + 0 = 0$ . Furthermore, the sum of any three consecutive terms is negative, and so  $a_1 + a_2 + a_3 < 0$  and  $a_4 + a_5 + a_6 < 0$ . So if we add these two inequalities, we get  $S < 0 + 0 = 0$ . Hence, we have  $S > 0$  and  $S < 0$ , which is a contradiction. Thus, no such sequence exists.

**Problem 4.3** A permutation  $(a_1, a_2, \dots, a_n)$  of the integers  $1, 2, \dots, n$  is said to be *fantastic* if  $a_1 + a_2 + \dots + a_k$  is divisible by  $k$  for *each*  $k$  from 1 to  $n$ . For example,  $(3, 1, 2)$  is a fantastic permutation of  $1, 2, 3$  because 3 is divisible by 1,  $3 + 1$  is divisible by 2, and  $3 + 1 + 2$  is divisible by 3. However,  $(2, 1, 3)$  is not a fantastic permutation because  $2 + 1$  is not divisible by 2.

- a) Show that no fantastic permutation exists for  $n = 4$ .
- b) Does a fantastic permutation exist for  $n = 5$ ? Explain.
- c) (Just for fun) Find all fantastic permutations.

Suppose that there is a fantastic permutation of  $1, 2, 3, 4$ . Let this permutation be  $t_1, t_2, t_3, t_4$ . Since this permutation is fantastic,  $t_1 + t_2 + t_3 + t_4$  must be divisible by 4. However, regardless of how these four numbers are permuted,  $t_1 + t_2 + t_3 + t_4$  must equal 10. Since 10 is not divisible by 4, we have a contradiction, and so no fantastic permutation exists for  $n = 4$ .

Let's now do the case  $n = 5$ . Suppose that there is a fantastic permutation of  $1, 2, 3, 4, 5$ . Let this permutation be  $t_1, t_2, t_3, t_4, t_5$ . Note that  $t_1 + t_2 + t_3 + t_4 + t_5 = 15$ , which is indeed divisible by 5. So there is no contradiction to be found there. However, we know that  $t_1 + t_2 + t_3 + t_4$  is divisible by 4, since this permutation is fantastic. Well,  $t_1 + t_2 + t_3 + t_4 = 15 - t_5$ , and since this is divisible by 4, we must have  $t_5 = 3$ . Note that any other choice for  $t_5$  will make  $15 - t_5$  not divisible by 4, since  $t_5$  must be between 1 and 5 inclusive.

So we have now proven that if a fantastic permutation exists, the last term must be 3. Thus we have  $t_1 + t_2 + t_3 + t_4 = 15 - t_5 = 12$ . Since this permutation is fantastic, we know that  $t_1 + t_2 + t_3 = 12 - t_4$  is divisible by 3. The only value for  $t_4$  that makes  $12 - t_4$  divisible by 3 is  $t_4 = 3$ . Hence, we have proven that if a fantastic permutation exists, then  $t_5$  and  $t_4$  must both be 3. But then this is no longer a permutation, since the same term is listed twice! So we have a contradiction, and therefore we conclude that no fantastic permutation exists for  $n = 5$ .

By the way, here is another solution to part b), courtesy of Victoria Best. From above, we have  $t_5 = 3$ . Thus,  $(t_1, t_2, t_3, t_4)$  must be a permutation of  $\{1, 2, 4, 5\}$ . Since  $t_1 + t_2$  is divisible by 2, i.e., it is even, both  $t_1$  and  $t_2$  must have the same parity. The only possibilities therefore are  $(t_1, t_2)$  being  $(1, 5)$ ,  $(5, 1)$ ,  $(2, 4)$ , or  $(4, 2)$ . In each situation, we have  $t_1 + t_2 = 6$ . Note that  $t_1 + t_2 + t_3 = 6 + t_3$  must be a multiple of 3. Therefore,  $t_3$  must be a multiple of 3. This forces  $t_3 = 3$ , and once again we have a contradiction because we already have  $t_5 = 3$ .

Part c) can be done in exactly the same way part a) and b) are done. For  $n = 2000$ , follow the argument for  $n = 4$ . For  $n = 2001$ , follow the argument for  $n = 5$ .