Math 2790 Final Examination

A1. You throw a sequence of three darts at a dartboard, aiming for the centre. Assume each one of your throws is equally skillful, that is, you don't have a tendency to get better (or worse!) with each throw. Suppose that your second throw is better than your first. What is the probability that your third throw will be better than your second?

A2. Chantel, Chris, Elizabeth, Garrett, James, John, Karin, Rich, Riham, Sable, Vernon and Victoria are stranded on a deserted island. These twelve people are to be split into three teams, with four people on each team.

(a) Chantel, Chris, and Elizabeth are selected to be "team captains". So these three people get to select the remaining members of their team. How many ways can the teams be formed?

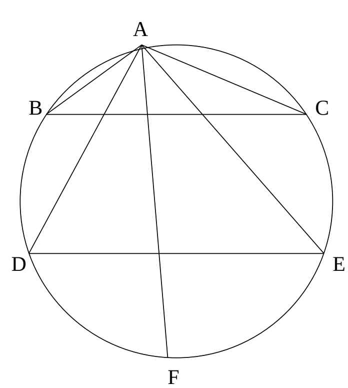
Carefully explain why the answer is $\binom{9}{3}\binom{6}{3}\binom{3}{3}$.

(b) Chantel, Chris, and Elizabeth feel that living on a deserted island is dumb, so they decide to swim back to Halifax. So there are only nine people left. Now the remaining people want to split off into three teams with three people on each team. How many ways can this be done? Carefully explain your answer.

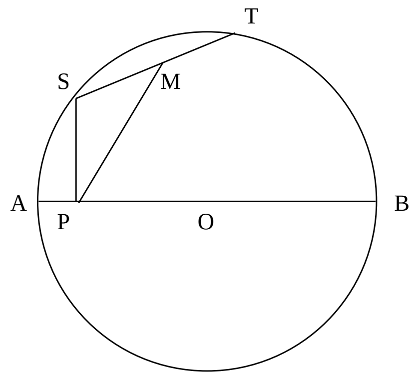
A3. (a) Determine the number of solutions $(a_1, a_2, a_3, a_4, a_5)$ that satisfy the equation $a_1 + a_2 + a_3 + a_4 + a_5 = 10$, where a_1, a_2, a_3, a_4, a_5 are all positive integers.

(b) Determine the number of solutions $(a_1, a_2, a_3, a_4, a_5)$ that satisfy the equation $a_1 + a_2 + a_3 + a_4 + a_5 = 20$, where each of a_1, a_2, a_3, a_4, a_5 is a positive integer that is *at least* 3. (For example, $(4, 5, 4, 3, 4)$ is a solution, but $(4, 7, 3, 2, 4)$ is not).

B1. Suppose that two parallel chords form the bases of two triangles that share a vertex on the circumference of the circle, as illustrated in the diagram. Prove that at this common vertex, both triangles share the same internal angle bisector.



B2. In the circle, ST is a chord of length 10. Let M be the midpoint of ST. P is the foot of the perpendicular from S to diameter AB. If $\angle SPM = 45°$, determine the area of the circle.

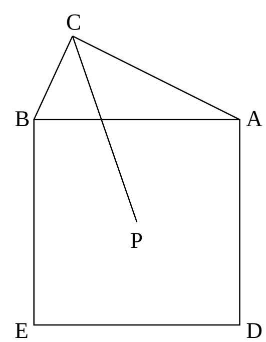


B3. $BADE$ is a square with centre P. We construct right-angled triangle ABC with AB as the hypotenuse, as shown. Suppose that $BC = x$ and $CA = y$.

(a) Explain why $\angle APB = 90°$.

(b) Prove that CP bisects $\angle ACB$.

(c) Prove that $CP = \dfrac{x+y}{\sqrt{2}}$.



C1. (a) Find *three* solutions in positive integers (x, y) to the equation $x^2 - 3y^2 = 1$.

(b) Prove that there are *no* solutions in positive integers (x, y) to the equation $x^2 - 3y^2 = 3$.

C2. I'm thinking of a quadratic polynomial $ax^2 - bx + c$. I'll give you three properties of this polynomial.

1. a, b, c are all *prime* numbers, with $b > a > c$.
2. The two roots of this polynomial are both *rational*.
3. The value of $a^2 + b^2 + c^2$ is a multiple of 5.

Find an ordered triplet (a, b, c) that satisfies all three properties above, and describe how you arrived at your answer. *BONUS: prove that your solution is unique.*

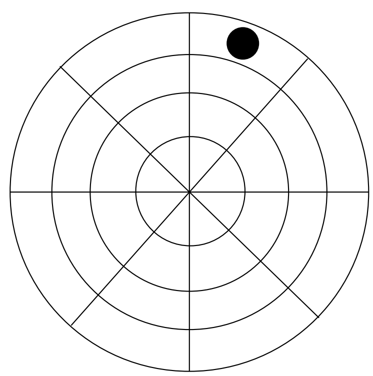
C3. Determine all solutions in positive integers (x, y) to the equation $|3^x - 2^y| = 1$.

D1. Consider the set $\{\pi, 2\pi, 3\pi, 4\pi, \ldots, 99\pi\}$. Prove that in this set there must be at least one term that is within $\frac{1}{100}$ of an integer.

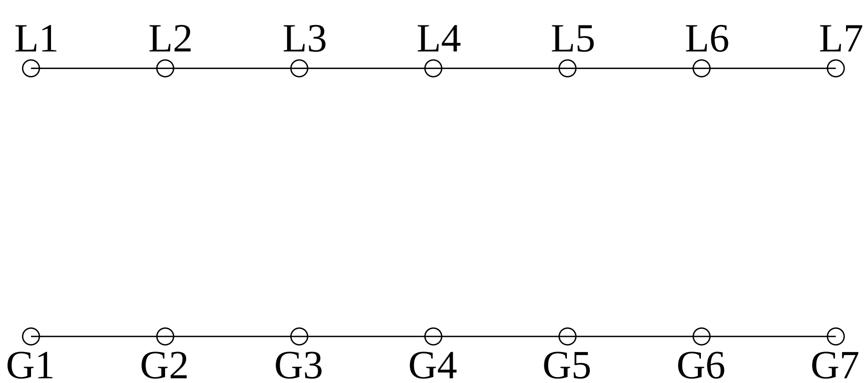
D2. Alison and Ian play a game by alternately moving a disk on a circular board. The game starts with the disk already on the board as shown. A player may move either clockwise one position or one position toward the centre, but cannot move to a position that has been previously occupied. The last person who is able to move wins the game. Alison moves first.

(a) Find a winning strategy for Alison, and prove that it is indeed a winning strategy (i.e., no matter how well Ian plays, Alison will always be able to win).

(b) Generalize the problem: suppose the board has n concentric circles with r regions in each ring. (Note: in our diagram below, we have $n = 4$ and $r = 8$). For what values of n and r does Ian have a winning strategy? Justify your answer.

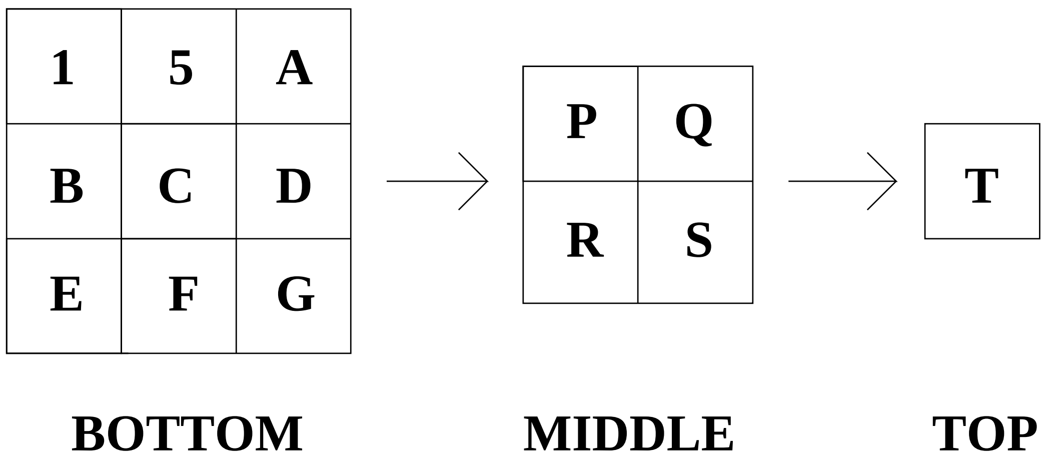


D3. In a ballroom seven gentlemen are sitting equally spaced in a straight line, each directly opposite each of seven ladies, who are also in a straight line and equally spaced. The gentlemen all go to and invite a different lady to dance. Is it possible for each of the gentlemen to walk a different distance to his partner?



BONUS.

There are 14 unit cubes, arranged in a pyramid. There are nine cubes on the bottom layer, four cubes on the middle layer, and one cube at the top. The nine cubes on the bottom layer are each assigned a *unique* digit from 1 to 9. Each cube on the middle and top layers is assigned the number that is the *average* of the four cubes directly below it. Suppose that our pyramid is numbered as follows:



(For example, P is the average of 1, 5*, B, and C.)*

The bottom layer has been numbered in such a way that when you determine P, Q, R, S, and T, you will find that these five numbers are all *distinct integers*. This is enough information to *uniquely* solve for the twelve unknown variables!

Determine A, B, C, D, E, F, G, P, Q, R, S, and T.