Tour 10 - More Magnificent Modular Arithmetic

Recall that if a and b are both integers, a is *congruent* to b *modulo* m if a and b both give the same remainder when divided by m. We write this as $a \equiv b \pmod{m}$.

Note that saying $a \equiv b \pmod{m}$ is equivalent to saying that $a - b$ is a multiple of m. So if $a \equiv b \pmod{m}$, then we can write a in the form $km + b$ for some integer k. For example, $27 \equiv 6 \pmod{7}$, and we have $27 = 3 \cdot 7 + 6$.

Let's solve some problems. The last few problems were from the previous tour.

1. Determine whether the number $1234 \cdot 56 + 789 \cdot 100$ is divisible by 3.

2. i) If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, prove that $a + c \equiv b + d \pmod{m}$.

   ii) If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, prove that $ac \equiv bd \pmod{m}$.

   iii) If $a \equiv b \pmod{m}$, then prove that $a^n \equiv b^n \pmod{m}$, for all integers $n \geq 1$.

3. Prove that $2^{70} + 3^{70}$ is divisible by 13.

4. In the magical land of Camelot, there are 45 chameleons. Thirteen are lavender, fifteen are beige, and seventeen are aquamarine. Whenever two chameleons of different colours meet, they both turn into a chameleon of the third colour. Prove that no matter how many times these chameleons "meet", it is impossible for all of the chameleons to be of the same colour at one time. (Hint: analyze the problem in mod 3).

5. Show that $4^{3n+1} + 2^{3n+1} + 1$ is divisible by 7 for all positive integers n.

6. Pick any 55 numbers from the set $\{1, 2, 3, \ldots, 100\}$. Prove that among those 55 numbers, you can find two of them that differ by 9.

7. Prove Fermat's Little Theorem: if p is prime and a is not divisible by p, show that $a^{p-1} \equiv 1 \pmod{p}$.

   (Hint: look at the set $\{a, 2a, 3a, \ldots, (p-1)a\}$ *and reduce each element modulo p. For example, if* $a = 3$ *and* $p = 7$*, the set becomes* $\{3, 6, 9, 12, 15, 18\}$*, which reduces to* $\{3, 6, 2, 5, 1, 4\}$ *in modulo* 7*. What do you notice?)*