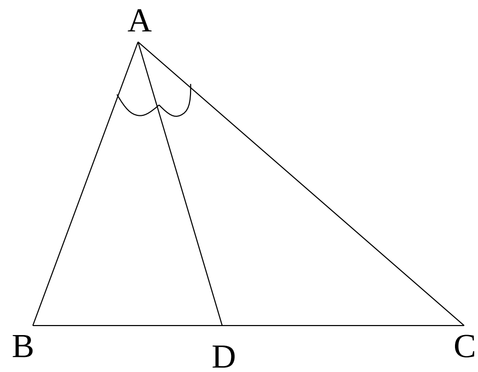
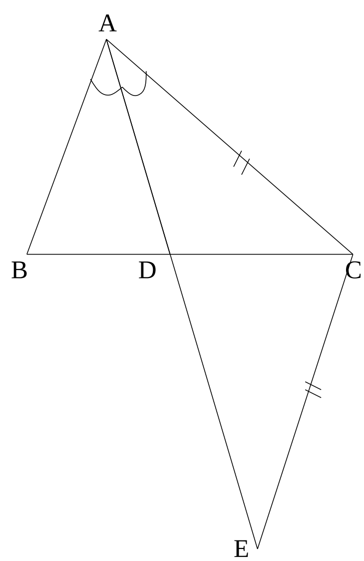
Tour 13 - All Triangles Are Equilateral!

Yes, you've read the title of this tour correctly. We shall prove that all triangles are equilateral. But before we do that, we first state and prove the Internal Angle Bisector Theorem.



The Internal Angle Bisector Theorem states that if AD is an internal angle bisector of $\angle BAC$, then $\dfrac{AB}{AC} = \dfrac{BD}{DC}$.

There are at least six different ways to prove this. Here is Euclid's proof, which was purely geometric, i.e., no trigonometry, no coordinates, nothing messy. This is quite nice.

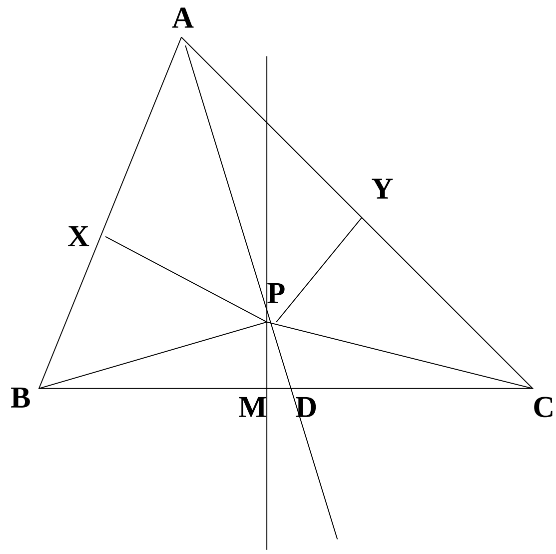


Construct point E on the line AD extended so that $AC = CE$. Since $\angle BAD = \angle CAD$ (defintion of angle bisector) and $\angle CAD = \angle CED$ (because $\triangle ACE$ is isosceles), we have $\angle BAD = \angle CED$. Furthermore, by the opposite angles theorem, we have $\angle BDA = \angle CDE$.

So by the Angle-Angle-Angle similarity test, $\triangle ABD \sim \triangle ECD$.

Hence, we have $\dfrac{AB}{BD} = \dfrac{CE}{CD} = \dfrac{AC}{CD}$, since $CE = AC$. Rearranging the terms, we get $\dfrac{AB}{AC} = \dfrac{BD}{DC}$, as required.

Now, let me show you an incredible proof that all triangles are equilateral.



Pick any arbitrary triangle ABC.

Let the internal angle bisector of $\angle A$ and the perpendicular bisector of side BC meet at point P, as shown. Construct PB and PC, and find points X and Y on AB and AC respectively so that $PX \perp AB$ and $PY \perp AC$.

Now, $\triangle APX \cong \triangle APY$, by Angle-Side-Angle (ASA) congruency test. Hence, $PX = PY$, and $AX = AY$. Since P is the perpendicular bisector of side BC, we have $BM = MC$, so by the Side-Angle-Side congruency test, $\triangle PMB \cong \triangle PMC$, and so $PB = PC$.

Finally, $\triangle PXB \cong \triangle PYC$ by the Hypotenuse-Side congruency test, since $PX = PY$, $PB = PC$, and $\angle PXB = \angle PYC = 90°$. Thus, $XB = YC$.

We have proven that $AX = AY$ and $XB = YC$. Therefore, $AX + XB = AY + YC$, which simplifies to $AB = AC$. Since we started with an arbitrary triangle ABC, and we proved that $AB = AC$, we conclude that for any triangle ABC, two of its sides must be equal. Rotating the diagram and applying the same argument on sides AC and BC, we get the conclusion that $AC = BC$. Therefore, all *three* sides of the triangle must be equal! Therefore, we have proven that all triangle are equilateral, Q.E.D.

Do you see the flaw in this proof? If so, can you prove it? Hint: use the Internal Angle Bisector Theorem.