Tour 23 - Step Theory

Tour Guide Richie normally jumps the stairs in the Chase Building three or four steps at a time. But he hurt his foot the other day, so he can only hop the stairs one or two steps at a time.

Let S_n be the number of different ways can he get up a flight of n steps, if he only hops the stairs one or two steps at a time. We shall investigate the amazing properties of S_n.

Determine S_1, S_2, S_3, S_4, S_5. What do you notice?

Prove the relationship between S_n and F_n, where F_n is the n^{th} term of the Fibonacci sequence. The Fibonacci sequence starts with $F_1 = 1$ and $F_2 = 1$.

Using *Step Theory*, prove each of the following Fibonacci identities. All of these can be generalized quite nicely.

1. $F_{11} = {F_6}^2 + {F_5}^2$
2. $F_{10} = {F_6}^2 - {F_4}^2$
3. $F_{10} = F_4F_7 + F_3F_6$
4. $F_6F_5 = {F_5}^2 + {F_4}^2 + {F_3}^2 + {F_2}^2 + {F_1}^2$
5. ${F_6}^2 = F_6F_5 + F_5F_4 + F_4F_3 + F_3F_2 + F_2F_1$
6. ${F_5}^2 = F_5F_4 + F_4F_3 + F_3F_2 + F_2F_1 + 1$
7. The Fibonacci From Pascal's Triangle Relation:

   $F_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \ldots.$