The Pigeonhole Principle

The **Pigeonhole Principle** states that if $n + 1$ pigeons are placed into n pigeonholes, then one of the pigeonholes must contain at least 2 pigeons.

For example, if there are eight people in a room, then at least two people must be born on the same day of the week. In this example, the seven days of the week correspond to our seven pigeonholes (i.e., $n = 7$). Of course, if we had more than eight people, then we can also claim that at least two people must be born on the same day of the week. However, the conclusion does not hold with only seven people, for the seven people could be born on seven different days.

Here are a couple more examples of the Pigeonhole Principle. Convince yourself that they are legitimate statements.

Among any three people, you can find two that are of the same gender.

Among any thirty-two people, you can find two that are born on the same day of the month.

Note that the Pigeonhole Principle does not say that among any 32 people, "exactly" two were born on the same day of the month. For we could pick 32 people all born on September 11^{th}. The Pigeonhole Principle states that you will be able to find "at least" two people who were born on the same day of the month. Just like if we had three people in a room, *at least* two must be of the same gender. If we said *exactly* two must be of the same gender, that would be false because we could have three girls, or we could have three boys.

Let's get a bit more complicated. Convince yourself that the following statements are correct.

Among any seven people, you can find four that are of the same gender.
Among any fifty people, you can find eight that were born on the same day of the week.
Among any twenty-five people, you can find three that were born in the same month.

This is the **Generalized Pigeonhole Principle**: if $kn + 1$ pigeons are placed into n pigeonholes, then one of the pigeonholes must contain at least $k + 1$ pigeons. For example, we have $k = 2$ and $n = 12$ in the last example above.

Note: if we just let $k = 1$, the Generalized Pigeonhole Principle just becomes the regular Pigeonhole Principle.

The Pigeonhole Principle (and the Generalized Pigeonhole Principle) can be used to solve some very difficult problems in an elegant manner.