Problems For Tour 8

A *Diophantine equation* is an equation in two or more variables that we wish to find *integer* solutions for. One of the most famous problems in mathematics is a Diophantine equation:

Fermat's Last Theorem states that there are no solutions in integers to the equation $x^n + y^n = z^n$, where n is a positive integer greater than or equal to 3. This was finally proved by Andrew Wiles of Princeton University several years ago, after it had baffled countless mathematicians for over three centuries!

Let's solve some Diophantine equations!

1. Determine all solutions (x, y) in integers to the equation $xy = y^2 + 3$.
2. Determine all integers x for which $(x + 5)(x - 5)$ is a perfect square.
3. Show that a number of the form p^n, where p is prime, can never be perfect.

   Hint: the formula for the sum of a geometric series is:

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}.$$

4. Determine all solutions (x, y) in integers to the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{6}$.
5. Determine all positive integers n for which $n^2 - 19n + 99$ is a perfect square.
6. Find four solutions in positive integers to the equation $x^2 - 3y^2 = 1$.