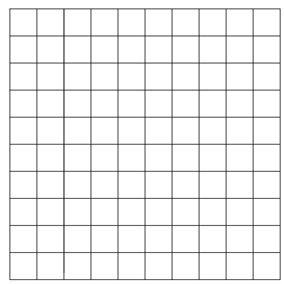
Proof for Week 1

Due on Thursday, September 13^{th}

Prove that a 10 by 10 board cannot be covered with 25 L-shaped quadrominoes.

This proof uses a method known as Proof By Contradiction, which we will discuss in detail later. Here's the idea: we assume that the board can be covered with 25 L-shaped quadrominoes, and from that, we derive a contradictory statement (such as 0 = 1). Therefore, our assumption must be false, and thus we may conclude that it is impossible to cover the board with 25 L-shaped quadrominoes.

Number the board with 0's and 1's, as shown in the diagram.



Assume that we *can* place 25 L-shaped quadrominoes so that every square on the board is covered. Since $25 \times 4 = 100$, notice that every square is covered exactly once, and there is no overlap. Consider such a covering of the board. We will prove that the sum of the numbers of the one hundred squares is *odd*.

Pick a L-shaped quadromino on the board, and add up the numbers of the four squares that the piece covers. This sum must be odd (*). Call this sum S_1. Repeat this with all the other quadrominoes, and we get an odd sum each time. So $S_2, S_3, \ldots, S_{25}$ are all odd. Now, the sum of the numbers of the one hundred squares in the board is just $S_1+S_2+S_3+\ldots+S_{25}$, since the twenty-five quadrominoes cover the entire board without any overlap.

But $S_1 + S_2 + S_3 + \ldots + S_{25}$ is odd (*). So the sum of the numbers of the one hundred squares is odd, and that is a contradiction (*). Therefore, it is impossible to cover a 10 by 10 board with 25 L-shaped quadrominoes.

This proof is correct, but several key steps require justification, and they are marked with a (*). Your task is to provide that justification.

Establish the following:

a) When you place an L-shaped quadromino on the numbered board, it covers four squares. Prove that when you add up the numbers of these four squares, you must get an odd number.

b) We have twenty-five numbers, $S_1, S_2, \ldots, S_{25}$. All of these numbers are odd. Prove that their sum, $S_1 + S_2 + \ldots + S_{25}$, must be odd.

c) Explain why we have a contradiction. (Hint: look at the way we numbered the board at the beginning of our proof).