Math 2400 - Numerical Analysis
Homework #5 Solutions

Hand in printouts of all program listings as well as output with the homework assignment.

1. Let $f(x)$ be a given function which can be evaluated at any point. In the following question, h refers to the step size or the distance between the equally spaced points used in the approximation.
   (a) Find a 2nd order method (i.e., truncation error $O(h^2)$) approximating $f'''(x_0)$. Give the formula as well as an expression for the truncation error.
   We begin by expanding $f(x)$ about x_0 as follows:
$$f(x_0+h) = f(x_0) + hf'(x_) + \frac{h^2}{2}f''(x_) + \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f''''(x_0) + \frac{h^5}{120}f^{(5)}(\xi_1)\,, \quad (1)$$
$$f(x_0-h) = f(x_0) - hf'(x_) + \frac{h^2}{2}f''(x_) - \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f''''(x_0) - \frac{h^5}{120}f^{(5)}(\xi_2)\,, \quad (2)$$
   Now we look at (1)-(2),
$$f(x_0+h) - f(x_0-h) = 2hf'(x) + \frac{h^3}{3}f'''(x_0) - \frac{h^5}{60}f^{(5)}(\xi_3)\,. \quad (3)$$
   We want to get rid of the $f'(x_0)$ term. We will need more equations to do this, so we look at more points. We could repeat the same process we just did for $f(x_0 \pm 2h)$, or we could just sub in $2h$ in (3):
$$f(x_0+2h) - f(x_0-2h) = 4hf'(x) + \frac{8h^3}{3}f'''(x_0) - \frac{32h^5}{60}f^{(5)}(\xi_4)\,. \quad (4)$$
   To get rid of the $f'(x_0)$ term, we look at (4)-2(3):
$$f(x_0+2h) - f(x_0-2h) - 2f(x_0+h) + 2f(x_0-h) = 2h^3 f'''(x_0) - \frac{h^5}{2}f^{(5)}(\xi)\,. \quad (5)$$
   Now that is left is to solve for $f'''(x_0)$.
$$f'''(x_0) = \frac{1}{2h^3}(f(x_0+2h) - 2f(x_0+h) + 2f(x_0-h) - f(x_0-2h)) + \frac{h^2}{4}f^{(5)}(\xi)\,. \quad (6)$$
   (b) Use your formula to find approximations to $f'''(0)$ for the function $f(x) = e^x$ employing values $h = 10^{-1}, 10^{-2}, \ldots, 10^{-9}$. Verify for larger values of h, your formula is indeed 2nd order accurate[1]. Which value of h gives the closest approximation to $e^0 = 1$.
   We start with the table of values

h	approx. to $f'''(x_0)$	absolute error
1.0000e-01	1.0025e+00	2.5025e-03
1.0000e-02	1.0000e+00	2.5000e-05
1.0000e-03	1.0000e+00	2.4927e-07
1.0000e-04	9.9998e-01	2.2122e-05
1.0000e-05	1.0547e+00	5.4712e-02
1.0000e-06	5.5511e+01	5.4511e+01
1.0000e-07	1.6653e+05	1.6653e+05
1.0000e-08	5.5511e+07	5.5511e+07
1.0000e-09	-1.1102e+11	1.1102e+11

[1] Apply the formula with one value of h and then $\frac{h}{2}$. If your formula is 2nd order, the error should be reduced by a factor of 4

As can bee seen for the first 3 values of n, when we reduce h by a factor of 10, the error goes down by a factor of 100, so the method is second order.

(c) For the formula derived in (a), how does the roundoff error behave as a function of h as $h \to 0$.
As can be seen from the table, when h gets very small, the error blows up. As discussed in class this is due to round off error. If we repeat the calculation of roundoff error from class, we find that the roundoff error is inversely proportional to h^3.

2. Consider the integral

$$\int_0^4 e^{-x^2}\,dx$$

(a) Approximate the integral using the composite trapezoid method with $n = 2$. First I have an mfile to evaluate the function in Matlab

```
function y=fn(x)

  y=exp(-x.^2);
end
```

To evaluate the integral with the trapaziod method, $h = \frac{4-0}{2} = 2$. Here is the matlab session to evaluate the integral:

```
octave:2> t2=2/2*(fn(0)+2*fn(2)+fn(4))
t2 =  1.0366
```

(b) Approximate the integral using Gaussian Quadrature and $n = 3$. We can just sub into the formula. Here is the matlab session I used to evaluate the integral:

```
octave:3> x1=-sqrt(3/5);
octave:4> x2=0;
octave:5> x3=sqrt(3/5);
octave:6> c1=5/9;
octave:7> c2=8/9;
octave:8> c3=5/9;
octave:9> g3=4/2*(c1*fn((4*x1+4)/2)+c2*fn((4*x2+4)/2)+c3*fn((4*x3+4)/2))
g3 =  0.93934
```

(c) Given that the exact answer is 0.8862269120 find the relative error for both methods. Comment on the result. Here is the matlab session where I evaluate the relative errors of the methods.

```
octave:10> tr=quad(@fn,0,4)
tr =  0.88623
octave:11> regauss=(g3-tr)/tr
regauss =  0.059926
octave:12> retrap=(t2-tr)/tr
retrap =  0.16971
octave:13> diary off
```

Both methods just use three function evaluations, yet the relative error for Gaussian quadrature is about 1/3 that of the trapazoidal approximation.