Osculating Interpolation

We are given the set of points \( \{x_i\}_{i=1}^{q} \). Assume that at each point \( x_i \), we are given the values of the function and up to the \( m_i^{th} \) derivative of the function. So we have \( q \) function values and \( \sum_{i=1}^{q} m_i \) derivative values. Thus, we have \( q + \sum_{i=1}^{q} m_i \) different values. The maximum degree of the resulting polynomial is then \( r \) where,

\[
r = q - 1 + \sum_{i=0}^{q-1} m_i ,
\]

(recall that \( n \) pieces of information results in a polynomial of degree at most \( n - 1 \)).

To find the osculating polynomial which goes through each of the data points and matches the derivative values, we write out the data in the following way:

\[
(z_1, z_2, \ldots, z_r) = (x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_q, \ldots, x_q),
\]

\[
m_1 + 1 \text{ times } m_2 + 1 \text{ times } m_q + 1 \text{ times}
\]

Or in words, if have the function value at \( x_1 \) and \( m_1 \) derivative values, \( z_1 \) through \( z_{m_0} \) will all equal \( x_1 \) and so on till we get to \( z_r \). Now when we wish to find the osculating polynomial we use divided differences with a slight change. We write out the \( z_i \) values where we wrote up the \( x_i \)’s before. We are now going to find the coefficients \( f[z_1] \) up to \( f[z_1, z_2, \ldots, z_q] \). We define the \( 0^{th} \) order divided difference in the same manner as before, \( f[z_i] = f(z_i) \). For the higher order divided differences we use,

\[
f[z_k, \ldots, z_l] = \begin{cases} 
  \frac{f[z_{k+1}, \ldots, z_l] - f[z_{k}, \ldots, z_{l-1}]}{z_i - z_k} & z_i \neq z_k \\
  \frac{f[z_{k}, \ldots, z_l]}{1} & z_i = z_k
\end{cases}
\]

The recursive relationship is the same as in the previous case if \( z_i \neq z_k \). If \( z_i = z_k \) then we just use the derivative information. We illustrate with the following example.

We have the following table of data,

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( f(x_i) )</th>
<th>( f'(x_i) )</th>
<th>( f''(x_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
<td>.5</td>
<td>.25</td>
</tr>
<tr>
<td>1.0</td>
<td>1.5</td>
<td>.7</td>
<td></td>
</tr>
</tbody>
</table>

Now we have 2 points, so \( q = 2 \). At \( x_1 = 0 \) we know \( f' \) and \( f'' \), so \( m_2 = 2 \). At \( x_2 = 1 \) we just know \( f' \) so \( m_1 = 1 \). Here \( r = 4 \) and \( (z_1, z_2, z_3, z_4, z_5) = (0, 0, 0, 1, 1) \). We now fill in the table.

0 \quad f[z_1] = 1
0 \quad f[z_2] = 1 \quad f[z_1, z_2] = .5
0 \quad f[z_3] = 1 \quad f[z_2, z_3] = .5
1 \quad f[z_4] = 1.5 \quad f[z_3, z_4] = \frac{f[z_2, z_3] - f[z_1, z_2]}{z_4 - z_3} = .5
1 \quad f[z_5] = 1.5 \quad f[z_4, z_5] = 0.7

The last entry which doesn’t fit is,

\[
f[z_1, z_2, z_3, z_4] = \frac{f[z_2, z_3, z_4] - f[z_1, z_2, z_3]}{z_5 - z_1} = .325.
\]

The osculating polynomial is then given by,

\[
P_4(x) = 1 + .5(x - 0) + .125(x - 0)^2 - .125(x - 0)^3 + .325(x - 0)^3(x - 1).
\]

I checked this polynomial out with Maple and it has all the require properties.