1. Short Answer
2. We have the following information about a funcition $f(x)$.
   - $f(1) = 2$
   - The first order divided difference $f[x_1, x_2] = 2$ for all values $x_1 \neq x_2$.

   What can you say about this function.

   There are a few ways to do this question. I think the most straightforward method is to consider,

   $$f[x_1, x_2] = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = 2\,.$$

   Now we let x_1 be any value and take the limit as $x_2 \to x_1$, we then have $f'(x_1) = 2$ for all values of x_1. This says that $f(x)$ is a straight line with a slope of 2 and going through $(1, 2)$. Thus,

   $$f(x) = 2(x - 1) + 2\,,$$

3. Determine the missing entries in the following divided difference table.

$x_0 = 0.0$	$f[x_0] = a$		
$x_1 = 0.4$	$f[x_1] = b$	$f[x_0, x_1] = c$	
$x_2 = 0.7$	$f[x_2] = 6$	$f[x_1, x_2] = 10$	$f[x_0, x_1, x_2] = \frac{50}{7}$

   We start at the rightmost table entries. First we can find c:

   $$\frac{10 - c}{0.7 - 0} = \frac{50}{7}$$

   So $c = 5$. Now we can find b,

   $$\frac{6 - b}{0.7 - 0.4} = 10$$

   So $b = 3$. Finally,

   $$\frac{3 - a}{0.4 - 0} = 5$$

   So $a = 1$.

4. We wish to find an interpolating cubic polynomial for a function on the interval $[0, 3]$. If we wish to minimize the global error bound for $x \in [0, 3]$, what points should we choose for interpolation.

   In order to minimize the interpolation error, we use the shifted zeros of the quartic Chebyshev polynomial. First the unshifted values.

   $$z_i = \cos\left(\frac{(2i - 1)\pi}{8}\right)\,,\ i = 1, \ldots, 4$$

   Now we need a linear transform that takes -1 to 0 and 1 to 3. So we let,

   $$x = \frac{3}{2}z + \frac{3}{2}$$

   So the optimal interpolation points are,

   $$t_i = \frac{3}{2}\cos\left(\frac{(2i - 1)\pi}{8}\right) + 1\,,\ i = 1, \ldots, 4$$

5. We are given the data set $\{(x_i, y_i)\}_{i=0}^{n}$ and wish to construct a **quadratic** spline to interpolate the data. Assume the spline is of the form,

   $$S(x) = \begin{cases} S_0(x) & x_0 \leq x \leq x_1 \\ S_1(x) & x_1 < x \leq x_2 \\ \vdots & \vdots \\ S_{n-1}(x) & x_{n-1} < x \leq x_n \end{cases} \tag{1}$$

where

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 \,. \tag{2}$$

(a) Write down, but do not solve, equations which will make $S(x)$

i. Interpolate the given data set.

ii. Continuous.

iii. Smooth (have a continuous derivative).

I just said write down, I didn't state any particular form. So as long as you provide all the required information, it doesn't matter how you provide it. It is sufficient to write the following:

i. $S(x_i) = y_i \,,\ i = 0, \ldots, n$

ii. $S_{i-1}(x_i) = S_i(x_i) \,,\ i = 1, \ldots, n-1$

iii. $S'_{i-1}(x_i) = S'_i(x_i) \,,\ i = 1, \ldots, n-1$

You can expand these equations if you wish, but this is sufficient for full marks.

(b) If all the above conditions are met, how many more conditions will be needed to completely specify $S(x)$ (There are a total of $3n$ unknowns in (1)).

We just need to count up how many equations we have given.

i. $n+1$ equations.

ii. $n-1$ equations.

iii. $n-1$ equations.

So far I have specified $3n-1$ conditions. I will need one more condition to completely specify S.

6. Assume we have the data set $\{(x_i, y_i)\}_{i=0}^n$. We wish to determine the constants C_1 and C_2 which minimize the sum of the square of the error for the model $y = \frac{C_1}{x+C_2}$.

(a) Give the nonlinear least squares equations which must be satisfied for optimal values C_1 and C_2. **Note:** Just give the equations which C_1 and C_2 must satisfy. You do not need to consider any iterative method for finding the values.

We wish to minimize

$$\sum_{i=0}^{n} \left(y_i - \frac{C_1}{x_i + C_2} \right)^2$$

So we take the derivatives with respect to C_1 and C_2 and set the result to 0. First with respect to C_1

$$2 \sum_{i=0}^{n} \left(y_i - \frac{C_1}{x_i + C_2} \right) \frac{-1}{x_i + C_2} = 0$$

Then with respect to C_2

$$2 \sum_{i=0}^{n} \left(y_i - \frac{C_1}{x_i + C_2} \right) \left(\frac{C_1}{(x_i + C_2)^2} \right) = 0$$

(b) Use $\frac{1}{y} = \frac{x}{C_1} + \frac{C_2}{C_1}$ to find a transform of the data which will allows the use of linear least squares.

If we define $z_i = \frac{1}{y_i}$, then the transformed data should follow the linear relation

$$z_i = \frac{1}{C_1} x_i + \frac{C_2}{C_1}$$

(c) Find the normal equations need to solve for C_1 and C_2 which minimizes the error for the transformed data using linear least squares.

First we define $b_0 = \frac{C_2}{C_1}$ and $b_1 = \frac{1}{C_1}$. Then the normal equations for standard linear regression give us,

$$\begin{pmatrix} (n+1) & \sum_{i=0}^n x_i \\ \sum_{i=0}^n x_i & \sum_{i=0}^n x_i^2 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^n \frac{1}{y_i} \\ \sum_{i=0}^n \frac{x_i}{y_i} \end{pmatrix}$$

Once b_0 and b_1 are found, we just solve

$$\frac{1}{C_1} = b_1$$

and then

$$\frac{C_2}{C_1} = b_0$$

.

(d) If the error of the i^{th} data is has the form $y = \frac{C_1}{x+C_2} + \epsilon_i$, use an order one Taylor series about $\epsilon_i = 0$ to show how the transform modifies the error. What effect will this have on the resulting approximation.
First we write the expression with error as

$$y = \frac{C_1 + \epsilon_i(x + C_2)}{x + C_2}$$

So for the transformed variables, we have

$$z = \frac{x + C_2}{C_1 + \epsilon_i(x + C_2)}$$

Now we expand the above in a Taylor series about $\epsilon_i = 0$

$$z = \frac{x + C_2}{C_1} - \frac{(x + C_2)^2}{C_1^2}\epsilon_i + O(\epsilon_i^2)$$

So the errors for x large will be weighted more heavily then the errors for which x is small.

7. Numerical differentiation Find an $O(h^2)$ approximation to $y'(x_0)$ using the values $y(x_0)$, $y(x_0+h)$ and $y(x_0+2h)$. Find the error term as well.

We expand $y(x_0 + h)$ and $y(x_0 + 2h)$ in a Taylor series.

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2}y''(x_0) + \frac{h^3}{6}y'''(z_1)\,, \tag{3}$$

$$y(x_0 + 2h) = y(x_0) + 2hy'(x_0) + \frac{4h^2}{2}y''(x_0) + \frac{8h^3}{6}y'''(z_2)\,. \tag{4}$$

Now we want to get rid of the y'' term, so we take $4\times(3)-(4)$. We note that although $z_1 \neq z_2$ in general, we can always evaluate both terms at some other point z and get the same value for the error. Thus we ahve

$$4y(x_0 + h) - y(x_0 + 2h) = 3y(x_0) + 2hy'(x_0) - \frac{4}{6}h^3 y'''(z)$$

Now we just solve for $y'(x_0)$:

$$y'(x_0) = \underbrace{\frac{4y(x_0 + 2h) - y(x_0 + 3) - 3y(x_0)}{2h}}_{\text{approximation}} + \underbrace{\frac{1}{6}h^2 y'''(z)}_{\text{error term}}$$

Here $x_0 \leq z \leq x_0 + 2h$.

8. Use three point Gaussian quadrature to approximate,

$$\int_0^1 e^{-x^2}\,dx$$

You may use the table below. You do not need to simplify your answer.

Points	Weighting Factors	Function Arguments
2	c1 = 1.000000000	x1 = -0.577350269
	c2 = 1.000000000	x2 = 0.577350269
3	c1 = 0.555555556	x1 = -0.774596669
	c2 = 0.888888889	x2 = 0.000000000
	c3 = 0.555555556	x3 = 0.774596669
4	c1 = 0.347854845	x1 = -0.861136312
	c2 = 0.652145155	x2 = -0.339981044
	c3 = 0.652145155	x3 = 0.339981044
	c4 = 0.347854845	x4 = 0.861136312

What is the highest order polynomial that this 3 point formula will provide an exact answer for?

To use Gaussian quadrature, we must first make a change of variables to bring the interval to $[-1, 1]$. So we let $t = 2x - 1$ then $dt = 2\,dx$. So we now approximate

$$\int_{-1}^{1} \frac{1}{2} e^{\left(\frac{t+1}{2}\right)^2} \, dt$$

Now we can just use the Gaussian quadrature formula to get

$$\int_{-1}^{1} \frac{1}{2} e^{\left(\frac{t+1}{2}\right)^2} \, dt \sim \frac{1}{2} \left(0.555555556 e^{\left(\frac{1-0.7746}{2}\right)^2} + 0.888888889 e^{\left(\frac{1}{2}\right)^2} + 0.555555556 e^{\left(\frac{.7746+1}{2}\right)^2} \right)$$

Finally an n point formula will integrate a degree $2n - 1$ polynomial exactly. So in this case, the formula will integrate any degree 5 polynomial exactly.

9. Find c_1, c_2 and c_3 such that the rule

$$\int_0^1 f(x)\, dx = c_1 f(0) + c_2 f(0.5) + c_3 f(1)$$

is exact for all polynomials for degree 2 or less.

In order to solve this problem, we just need to make sure the rule gives exact answers for the cases $f(x) = 1$, $f(x) = x$ and $f(x) = x^2$, We thus have the following equations:

$$\begin{aligned} \int_0^1 1\, dx &= 1 = c_1 + c_2 + c_3 \,, \\ \int_0^1 x\, dx &= \frac{1}{2} = c_2 \frac{1}{2} + c_3 \\ \int_0^1 x^2\, dx &= \frac{1}{3} = c_2 \frac{1}{4} + c_3 \,. \end{aligned}$$

Solving these equations gives:

$$\begin{aligned} c_1 &= \frac{1}{6} \,, \\ c_2 &= \frac{2}{3} \,, \\ c_3 &= \frac{1}{6} \,. \end{aligned}$$

10. Develop a first order method for approximating $f''(x)$ that uses the data $f(x - h)$, $f(x)$ and $f(x + 3h)$ only. Find the error term.

Since we are solving for f'', we will have to divide by h^2 to get the final formula. If the error term is to be $O(h)$, we will need the remainder term in the Taylor polynomial to be $O(h^3)$. We thus consider

$$f(x - h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f^{(3)}(\zeta_1) \,, \tag{5}$$

$$f(x + 3h) = f(x) + 3h f'(x) + \frac{9h^2}{2} f''(x) + \frac{27h^3}{6} f^{(3)}(\zeta_2) \,, \,. \tag{6}$$

To solve for $f''(x)$, we look at $3 \times (5) + (6)$.

$$3f(x + h) + f(x + 3h) = 4f(x) + 6h^2 f''(x) + 5h^3 f^{(3)}(\zeta) \,,$$

where we have used the intermediate value property to add up the two error terms. We can know solve for our approximation and error term.

$$f''(x) = \frac{3f(x + h) - 4f(x) + f(x + 3h)}{6h^2} - \frac{5}{6} h f^{(3)}(\zeta) \,.$$