Math 2400 – Practice Term Test Solutions

1. The two roots of the quadratic $ax^2 + bx + c$ are given by,

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1}$$

Here, x_+ corresponds to the root with the plus sign in front of the square root and x_- the other root.

(a) Explain in one or two sentences what problems could result from using this equation when $4ac$ is very small compared to b^2.

If $4ac$ is small compared to b^2 then $\sqrt{b^2 - 4ac} \approx |b|$. Then in one of the calculations, we will be subtracting nearly equal numbers. This will cause a loss of significant digits. If a is very small, the loss of precision may increased.

(b) These problems can be solved by calculating one of the roots using (1) and the other using,

$$x_- x_+ = \frac{c}{a}. \tag{2}$$

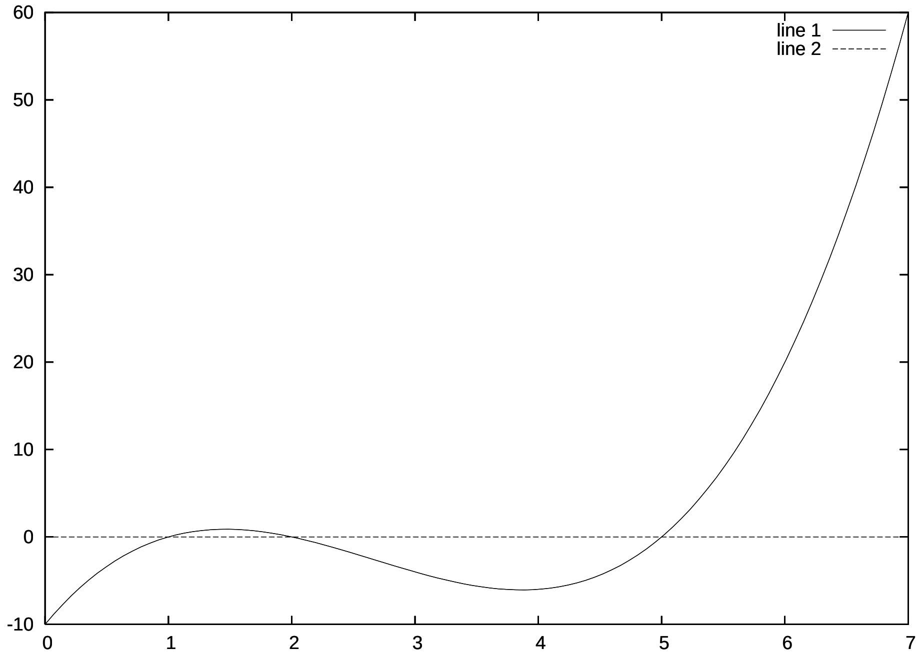
Which root should be calculated by equation (1) and why?

Which root we should solve using (1) depends on the sign of b. If $b > 0$ then we will have problems with x_+, so we use (1) to find x_- and (2) to find x_+. If $b < 0$ then we go the other way.

2. The graph of the function

$$f(x) = x^3 - 8\,x^2 + 17\,x - 10$$

is given by:



(a) If the points used for the bisection method are given by $a = 0$ and $b = 6$, what are the next two iterations of the bisection method?

The first value of $c = \frac{a+b}{2} = 3$. Since $f(3) < 0$ and $f(6) > 0$, we let $a = 3$. The next guess is then $\frac{6+3}{2} = 4.5$.

(b) How many iterations will be required to ensure an error of less then 10^{-4}.

The length of the n^{th} interval will be $\frac{6}{2^n}$. So we want $\frac{6}{2^n} < 10^{-4}$. Or $2^n > 6 \times 10^4$. To solve this we can takes the natural logarithm of both sides to get, $n > \frac{\ln(60000)}{\ln(2)}$. Or $n \geq 16$.

(c) If an initial guess of $x_0 = 6$ is used, what is the first iteration of Newton's method?

Here, $f'(x) = 3x^2 - 16x + 17$, so Newton's iteration formula is

$$x_{i+1} = x_i - \frac{x^3 - 8\,x^2 + 17\,x - 10}{3x^2 - 16x + 17}\,.$$

So if $x_0 = 6$, then

$$x_1 = 6 - \frac{20}{29}$$

3. To calculate any value of $\cos(x)$ we only need to be able to find $\cos(x)$ for $0 \leq x \leq \frac{\pi}{2}$ and use then use symmetry. We have the following exact values for the function $\cos(x)$ (and the decimal approximations) in this interval:

x	$\cos(x)$
0	1
$\frac{\pi}{6} \approx 0.52360$	$\frac{\sqrt{3}}{2} \approx 0.86603$
$\frac{\pi}{4} \approx 0.78540$	$\frac{1}{\sqrt{2}} \approx 0.70711$
$\frac{\pi}{3} \approx 1.04720$	$\frac{1}{2}$
$\frac{\pi}{2} \approx 1.57080$	0

(a) Use divided differences to find the unique polynomial of degree at most 4 which interpolates these points.

Fist we will construct a divided difference table.

x_i	0 DD	1DD	2DD	3DD	4DD
0.00000	1.00000				
0.52360	0.86603	-0.25587			
0.78540	0.70711	-0.60702	-0.44710		
1.04720	0.50000	-0.79109	-0.35154	0.09125	
1.57080	0.00000	-0.95493	-0.20861	0.13649	0.02880

So the polynomial is given by,

$$P(x) = 1 - 0.25587x - 0.44710x(x - 0.52360)+$$
$$0.09125x(x - 0.52360)(x - 0.788540) + 0.02880x(x - 0.52360)(x - 0.788540)(x - 1.04720)$$

(b) Find a bound on the maximum error of this approximation. Note for this bound, you may use $|x - x_i| \leq \frac{\pi}{2}$ even though this gives a very rough estimate.

The formula for the error bound is given by,

$$|e| \leq \frac{f^{(5)}(\xi)}{5!} x(x - 0.52360)(x - 0.78540)(x - 1.04720)(x - 1.57080)\,.$$

Since $f(x) = \cos(x)$, we have $f^{(5)}(\xi) \leq 1$ and $x(x - 0.52360)(x - 0.78540)(x - 1.04720)(x - 0.13649) \leq \left(\frac{\pi}{2}\right)^5$. So the the bound on the error is,

$$|e| \leq \frac{1}{120}\left(\frac{\pi}{2}\right)^5 \approx 0.08\,.$$

(c) What interpolation points would we use to minimize the global error?

To minimize the global error, we should use the translated roots of a Chebyshev polynomial as interpolation points. We just need to plug into the formula

$$\begin{aligned}
x_i &= \frac{b+a}{2} + \frac{b-a}{2}\cos\left(\frac{(2i-1)\pi}{2n}\right), \\
&= \frac{\pi}{4} + \frac{\pi}{4}\cos\left(\frac{(2i-1)\pi}{8}\right), \\
x_1 &= 1.5110\,, \\
x_2 &= 1.0860\,, \\
x_3 &= 0.48484\,, \\
x_4 &= 0.059785\,,
\end{aligned}$$

4. Circle the apprpriate descriptions below: The function defined by:

(a)

$$S(x) = \begin{cases} 10 - 5x + 2x^2 + x^3 & 0 \le x < 1 \\ 8 + 2(x-1) + 5(x-1)^2 + 10(x-1)^3 & 1 \le x \le 2 \end{cases}$$

Free Cubic Spline $\boxed{\text{Clamped Cubic Spline}}$ Not a cubic spline

$S(x)$ is continuous at 1 and $S'(x)$ is continuous at 1 as is $S''(x)$, so S is a cubic spline. However, $S''(0) \neq 0$, so we know it is not a free spline, so it must be a clamped spline.

(b)

$$S(x) = \begin{cases} 10 - 5x + x^3 & 0 \le x < 1 \\ 6 - 2(x-1) + 3(x-1)^2 - (x-1)^3 & 1 \le x \le 2 \end{cases}$$

$\boxed{\text{Free Cubic Spline}}$ Clamped Cubic Spline Not a cubic spline

In this case all the conditions for a free cubic spline are met.

(c)

$$S(x) = \begin{cases} 10 - 5x + x^3 & 0 \le x < 1 \\ 7 + 2(x-1) + 3(x-1)^2 - (x-1)^3 & 1 \le x \le 2 \end{cases}$$

Free Cubic Spline Clamped Cubic Spline $\boxed{\text{Not a cubic spline}}$

In this case $S'(x)$ is not continuous at 1.

5. Construct a parametric interpolating polynomial which passes through the following points:

i	1	2	3	4
x_i	-1	0	1	0
y_i	0	1	0.5	0

There are 4 points to interpolate, so we set $t_1 = 0$, $t_2 = \frac{1}{3}$, $t_3 = \frac{2}{3}$ and $t_4 = 1$. We then seek polynomials $x(t)$ and $y(t)$ which interpolate the following values:

t_i	0	.33333	.66667	1	t_i	0	.33333	.66667	1
x_i	-1	0	1	0	y_i	0	1	0.5	0

We can find the two polynomials using divided differences. First for $x(t)$:

t_i	0 DD	1DD	2DD	3DD
0.00000	-1.00000			
0.33333	0.00000	3.00000		
0.66667	1.00000	3.00000	0.00000	
1.00000	0.00000	-3.00000	-9.00000	-9.00000

Now $y(t)$:

t_i	0 DD	1DD	2DD	3DD
0.00000	0.00000			
0.33333	1.00000	3.00000		
0.66667	0.50000	-1.50000	-6.75000	
1.00000	0.00000	-1.50000	0.00000	6.75000

So the parametric curve is given by:

$$x(t) = -1 + 3t - 9t(t - 0.33333)(t - 0.66667)\,,$$
$$y(t) = 3t - 6.75t(t - 0.33333) + 6.75t(t - 0.33333)(t - 0.66667)\,.$$