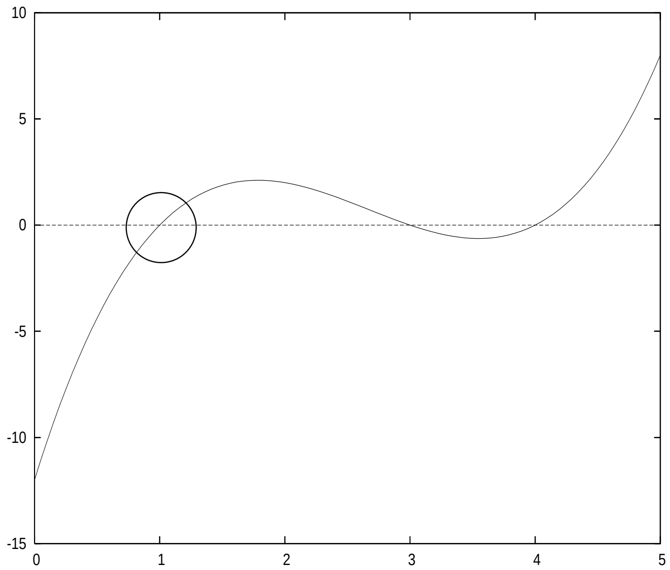
Math 2400 - Numerical Analysis
Mid-Term Test Solutions

1. Short Answers

(a) A sufficient and necessary condition for the bisection method to find a root of $f(x)$ on the interval $[a, b]$ is

$f(a)f(b) < 0$ or $f(a)$ and $f(b)$ are of opposite sign.

(b) The bisection method is applied to the function $y = (x-1)(x-3)(x-4)$ (graph given below). The initial interval used for the method is $[a, b] = [0, 5]$. Which root will the method converge to(circle it on the graph).



For the first iteration $c = 2.5$. At $x = 2.5$, the function is positive. Since the function is negative at $x = 0$, we move b over to 2.5 Then the only root left is the root at $x = 1$, so the method must converge to 1.

(c) The MATLAB command

```
format long, pi*cos(pi/2)
```

returns

```
ans = 1.923670693721790e-16
```

Complete each of the following statements regarding the accuracy of the approximation method used by MATLAB to compute $\pi \cos(\pi/2)$.

- The absolute error in MATLAB's approximation to $\pi \cos(\pi/2)$ is:
  1.923670693721790e-16

- The relative error in MATLAB's approximation to $\pi \cos(\pi/2)$ is ________________
  (fill in the most appropriate answer).
  Undefined or infinite.

(d) Give one advantage of using Newton's Divided Difference to construct an interpolating polynomial over using Langrange interpolation.

The main advantage is that it is easy to add more interpolation points

2. Consider the sum,

$$S_5 = \sum_{i=1}^{5} \frac{1}{i^3}$$

Use 3 digit rounding in each step of the following calculations.

(a) Find S_5 by adding up the terms in increasing order.
Here is the result from each calculation

$$\begin{aligned} 1 + 0.125 &= 1.13 \\ 1.13 + 0.0370 &= 1.17 \\ 1.17 + 0.0156 &= 1.19 \\ 1.19 + .008 &= 1.20\,. \end{aligned}$$

(b) Find S_5 by adding up the terms in decreasing order.

$$\begin{aligned} 0.008 + 0.0156 &= 0.0236 \\ 0.0236 + 0.0370 &= 0.0606 \\ 0.0606 + 0.125 &= 0.186 \\ 1 + 0.186 &= 1.19\,. \end{aligned}$$

(c) If the true answer is $S_5 \sim 1.185662037$ find the relative error in each of the above calculations.
Relative error for part a):

$$\left|\frac{1.20 - 1.185662037}{1.1856627}\right| \sim 0.012$$

Relative error for part b):

$$\left|\frac{1.19 - 1.185662037}{1.1856627}\right| \sim 0.0037$$

(d) When adding up a group of numbers with different magnitudes is it better to start with the smaller numbers or the larger. Why?
The relative error for part b) is roughly 1/3 that of the relative error for part a). The reason is that when you add a small number to a very larger number and then round off, you loose many of the digits in the small number. If this only happens once or twice it doesn't have any effect. However after many additions, the effect can grow and have cause the relative error of the resulting calculation to become significant.

Useful Table

i	1	2	3	4	5
$\frac{1}{i^3}$	1	0.125	0.03703703704	0.01562500000	0.008

3. Show that Newton's method applied to a linear function

$$f(x) = mx + b, m \neq 0$$

will converge to a root on the first iteration.

$f'(x) = m$, so the iteration formula is

$$\begin{aligned} x_{i+1} &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ &= x_i - \frac{mx_i + b}{m}, \\ &= \frac{mx_i - mx_i - b}{m}, \\ &= -\frac{b}{m} \end{aligned}$$

So, given any x_0, $x_1 = -\frac{b}{m}$ the root.

4. Consider the following data set

i	1	2	3	4
x_i	0	1	2	5
y_i	2	7	2	-1

(a) Construct a divided difference table for the data. This is where most of the work is done, I will just give the table

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	2			
1	7	5		
2	3	-5	-5	
5	1	-1	1	$\frac{6}{5}$

(b) Using the table find a quadratic polynomial going through the first 3 points$\{(x_i, y_i)\}_{i=1}^3$. The quadratic is given by

$$P_2(x) = 2 + 5x - 5x(x - 1)$$

(c) Find a polynomial going through all the points.

$$P_3(x) = 2 + 5x - 5x(x - 1) + \frac{6}{5}x(x - 1)(x - 2)$$

Given the following data set:

i	0	1	2	3
x_i	1	2	3	4
y_i	1.1	1.8	3.3	4.6

5. We wish to find the line $P(x) = c_0 + c_1 x$ which minimizes $\sum_{i=0}^3 (P(x_i) - y_i)^2$. Give the linear system which we must solve to find c_0 and c_1. (You do not need to solve the system)

This is just a linear least squares approximation. To find the elements in the matrix, we will need $\sum_{i=0}^3 1$, $\sum_{i=0}^3 x_i$, $\sum_{i=0}^3 x_i^2$, $\sum_{i=0}^3 y_i$ and $\sum_{i=0}^3 x_i y_i$. To make these calculations I will write out the table

x_i	1	2	3	4
x_i^2	1	4	9	16
y_i	1.1	1.8	3.3	4.6
$x_i y_i$	1.1	3.6	9.9	18.4

Adding up the rows, we have,

$$\sum_{i=0}^{3} 1 = 4\,,$$

$$\sum_{i=0}^{3} x_i = 10\,,$$

$$\sum_{i=0}^{3} x_i^2 = 30\,,$$

$$\sum_{i=0}^{3} y_i = 10.8\,,$$

$$\sum_{i=0}^{3} x_i y_i = 33\,.$$

So the system we need to solve is given by,

$$\begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 10.8 \\ 33 \end{pmatrix}.$$

6. A free cubic spline s for a function f is defined on $[1, 3]$ by

$$s(x) = \begin{cases} s_0(x) = 2(x-1) - (x-1)^3 & 1 \le x < 2 \\ s_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3 & 2 \le x \le 3 \end{cases}$$

Find a, b, c, d.

First we apply $s_0(2) = s_1(2)$ this gives us $a = 1$. We next apply $s_0'(2) = s_1'(2)$. This gives up, $b = -1$. Now we apply $s_0''(2) = s_1''(2)$. This gives us $c = -3$. There are two more conditions to apply, the free spline boundary conditions. $s_0''(0) = 0$ is already satisfied. Now we apply $s_2''(3) = 0$. This gives us $d = 1$ and we are done.

7. Construct a parametric interpolating polynomial which passes through the following points:

i	1	2	3
x_i	-1	0	1
y_i	0	1	0.5

We start by setting $\Delta t = \frac{1}{n-1} = \frac{1}{2}$, so $t_1 = 0$, $t_2 = \frac{1}{2}$ and $t_3 = 1$. We then have the following divided difference tables:

t_i	$f[t_i]$	$f[t_i, t_{i+1}]$	$f[t_i, t_{i+1}, t_{i+2}]$
0	-1		
$\frac{1}{2}$	0	2	
1	1	2	0

t_i	$f[t_i]$	$f[t_i, t_{i+1}]$	$f[t_i, t_{i+1}, t_{i+2}]$
0	0		
$\frac{1}{2}$	1	2	
1	$\frac{1}{2}$	-1	-3

So the parametric representation of the polynomial is given by,

$$x(t) = -1 + 2(t - 0)\,,$$
$$y(t) = 0 + 2(t - 0) - 3(t - 0)(t - \frac{1}{2})\,.$$