Random variables

In a random process, a *random variable* is a function from the sample space to the real numbers. So the random variables takes a numerical value which depends on the outcome. In this course, we will only consider random variables that take non-negative, integer values. Examples:

- A deck of 52 playing cards is dealt out randomly to 4 players, 13 cards each. The random variable X is the maximum number of cards of one suit that any player receives. Note that X must be between 4 and 13.
- Suppose 10 people attend a party on a rainy day. All bring an umbrella. Upon leaving, they all grab an umbrella at random from the collection. The random variable X is the number of people that leave with their own umbrella. So X can take values between 0 and 10.
- Consider random binary words of length 20. Let X be the number of ones in the word. Let Y be the number of runs in the word. What is the maximum value of Y?
- Consider randomly chosen grid paths from $(0,0)$ to $(8,8)$. Let X be the number of times the path touches the main diagonal.
- Suppose 11 people enter the elevator of a building with 11 stories. Each presses the button of their destination floor. Suppose the destination floors are chosen randomly from the 11 possibilities. Let X be the number of floors the elevator stops at. X can take values from 1 to 11.

Expectation

The *expectation* of a random variable X is defined as follows:

$$E(X) = \sum_{k=0}^{\infty} kP(X = k).$$

So the expectation is a weighted average of the different values the variable can take. An alternative definition is

$$E(X) = \sum_{s \in S} X(s)P(s),$$

where S is the sample space, which is assumed to be finite.
If X and Y are random variables on the same probability space, then we have that

$$E(X + Y) = E(X) + E(Y),$$

and for all real numbers α,

$$E(\alpha X) = \alpha E(X).$$

This is referred to as the *linearity of expectation*.

Indicator variables

An important tool in computing the expectation are *indicator variables*. An indicator variable is a Bernouilli variable (a variable only taking values 1 or 0) which indicates whether an event happened or not. Let A be an event. Then the indicator variable X_A for the event is given by: $X_A = 1$ if the outcome is in A, and 0 otherwise. The expected value of X_A equals:

$$E(X_A) = 1 \cdot P(X_A = 1) + 0 \cdot P(X_A = 0) = P(A).$$

Suppose we toss a fair coin 100 times in a row. Then the sample space consists of all possible outcomes, which can be represented as binary words of length 100 on the alphabet $\{H, T\}$. To be precise, H is for heads, T is for tails, and the i-th symbol of the word represents the outcome of the i-th coin toss. Since the coin is fair, the probability measure is uniform, so each outcome is equally likely. Since the sample space has size 2^{100}, each outcome has probability 2^{-100}.
Let X be the number of heads. Let us first compute the expectation using the definition. For this, we need to know the probability $P(X = k)$. The event "$X = k$ correspond to the set of outcomes where there are exactly k heads, and $100 - k$ tails. This corresponds to all binary words of length 100 with k heads and $100 - k$ tails. There are $\binom{100}{k}$ such words. Thus,

$$P(X = k) = \binom{100}{k} 2^{-100}.$$

Using the definition for expectation, we obtain that

$$E(X) = \sum_{k=0}^{100} k \binom{100}{k} 2^{-100} = \sum_{k=1}^{100} 100 \binom{99}{k-1} 2^{-100} = 100 \cdot 2^{99} \cdot 2^{-100} = 50.$$

Here, the second equality uses the fact that, for all $1 \leq k \leq n$, $k\binom{n}{k} = n\binom{n-1}{k-1}$. The third inequality uses the binomial theorem.
Now let us compute the expectation using indicator variables. For $1 \leq i \leq 100$, let X_i be the random variable representing the i-th coint toss. So $X_i = 1$ if the i-th toss is heads, and $X_i = 0$ if it is tails. Thus, $X = \sum_{i=1}^{100} X_i$. Now $E(X_i) = P(X_i = 1) = \frac{1}{2}$. By linearity of expectation,

$$E(X) = \sum_{i=1}^{100} E(X_i) = 100 \cdot \frac{1}{2} = 50$$

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Examples

1. Suppose 10 people attend a party on a rainy day. All bring an umbrella. Upon leaving, they all grab an umbrella at random from the collection. What is the expected number of people that go home with the same umbrella they came with?

   Let X be the number of people that go home with their own umbrella. Here we use indicator variables. For $1 \leq i \leq 10$, let X_i be defined so that $X_i = 1$ if person i goes home with his own umbrella, and $X_i = 0$ otherwise. So $X = \sum_{i=1}^{10} X_i$.

   Since every person picks an umbrella (uniformly) at random from the 10 umbrellas, the probability that a person goes home with his own umbrella is $\frac{1}{10}$. So $E(X_i) = P(X_i = 1) = \frac{1}{10}$. By linearity of expectation,

   $$E(X) = E(\sum_{i=1}^{10} X_i) = \sum_{i=1}^{10} E(X_i) = \sum_{i=1}^{10} \frac{1}{10} = 1.$$

2. Suppose 11 people enter the elevator of a building with 11 stories. Each presses the button of their destination floor. Suppose the destination floors are chosen randomly from the 11 possibilities. What is the expected number of stops the elevator makes?

   Let X be the number of stops of the elevator. For $1 \leq i \leq 11$, let X_i be the indicator variable defined so that $X_i = 1$ if the elevator stops at floor i, and $X_i = 0$ otherwise. Then $X = \sum_{i=1}^{11} X_i$.

   There are 10^{11} outcomes where none of the eleven people chooses floor i (so there are 10 choices for each of the 11 persons.) Thus the probability that the elevator does not stop on floor i equals $\frac{10^{11}}{11^{11}}$. So the probability that the elevator does stop on floor i, and thus $X_i = 1$, equals $1 - \frac{10^{11}}{11^{11}}$. Therefore,

   $$E(X) = \sum_{i=1}^{11} E(X_i) = \sum_{i=1}^{11} (1 - \frac{10^{11}}{11^{11}}) = 11 - \frac{10^{11}}{11^{10}}.$$

Work on these examples in class on Friday. We will discuss them on Monday, Sept. 28. Your responses to my questions will count in your 'class participation' mark.

3. Consider random bit strings of length 20. We can consider such as string as consisting of n independent Bernouilli variables $X_i \sim B(\frac{1}{2})$. What is the expected number of ones? A *run* is a string of contiguous ones, bordered by zeros, or a string of contiguous zeros, bordered by ones. For example, **0011101000** has five runs. What is the expected number of runs?

4. What is the expected number of times that a chosen grid path from $(0,0)$ to (n,n) meets the diagonal from $(0,0)$ to (n,n) ? (The beginning and end point do not count).

5. Consider the random graph $G(n,p)$. What is the expected number of triangles in the graph?

6. Consider a k-uniform hypergraph with vertex set S and set of hyper edges $\mathcal{F}$. Suppose the vertices are coloured randomly (independently, uniformly) with r colours. What is the expected number of monochromatic hyperedges?

7. Suppose that an urn contains n white balls and n black balls. A ball is selected at random from the urn and removed. This process is repeated until the urn contains only balls of one colour. What is the expected number of balls left at the end?