QUILLEN MODEL STRUCTURES, $*$-AUTONOMOUS CATEGORIES AND ADHERENCE SPACES

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A Thesis submitted to the
Faculty of Graduate and Postdoctoral Studies
University of Ottawa
in partial fulfillment of the requirements for the
Ph.D. degree in the

Ottawa-Carleton Institute for Graduate Studies and Research in Mathematics and Statistics

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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Abstract

Linear logic has been intensively studied since its introduction almost twenty years ago. Originally introduced as a proof theory, two distinct semantic traditions have evolved around linear logic: the *denotational semantics* of linear logic, and the *Geometry of Interaction*. In this thesis we explore how abstract homotopy theory may be used to reconcile these semantic traditions.

This approach is in some sense already suggested by the fact that, in denotational semantics, one is forced to take equivalence classes of proofs, and not proofs *per se*, as morphisms. Our approach amounts to taking a coarser equivalence relation than is needed to construct a denotational model, in order to create a category more closely resembling those which occur in Geometry of Interaction.

A new class of denotational models, called *adherence spaces* and in some sense tailor-suited to the problem at hand, are introduced. Then it is shown how a Quillen model structure may be imposed on a category of adherence spaces in such a way that the resulting homotopy category is compact closed.

Acknowledgements

I would like to gratefully acknowledge some of the many people and organisations that have helped me throughout this degree and in writing this thesis.

Firstly, the University of Ottawa (in general) and my sponsors Rick Blute and Phil Scott (in particular) for giving me a second chance at doctorhood; not everyone would take on such damaged goods so enthusiastically, and even if you may have occasionally wondered whether it was such a wise decision, I hope I have not soured you on the experience.

Rick Blute again, for being such a great supervisor; I know better than most the difference a supervisor can make, and your guidance, your encouragement and, yes, even your nagging (perhaps especially your nagging) made the difference to me.

The examiners Inna Bumagin, Susan Niefield, Paul-Eugène Parent and Phil Scott who gave many valuable suggestions and raised several interesting questions for future work.

The National Science and Engineering Research Council who provided generous financial support.

My fellow graduate students—especially Guy, Eric, Emile, Raman, Benoît, Gilles and Bob—who anchored me (i.e., kept me sane) to an even greater extent than I think they realise.

Finally, my parents—to whom I also dedicate this thesis—who did everything so that I could have the opportunities in life which I have almost thrown away (twice).

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Introduction

Proof Theory

The central paradigm of categorical proof theory is the creation of categories whose objects are the elements of some fixed formal language, and whose arrows are equivalence classes of formal proofs—by which we mean derivations with respect to some fixed sequent calculus defined on the formal language in question.

The fact that we take equivalence classes of proofs is key: otherwise we will not obtain a category at all. Indeed, we expect more than a category: we expect a category structured in a way which properly reflects the logic under consideration (—in the case of multiplicative-additive linear logic, which will be the primary focus of this thesis, the structure we expect is a $*$-*autonomous structure* with *finite products and coproducts*); to obtain such a category, we must impose further restrictions on the equivalence relation of proofs we use.

But what notions of equivalence of proofs exist?

Traditional emphasis, reasonably enough, has been to find the finest possible equivalence relation which results in a category with the desired structure. This yields the usual denotational semantics, first described by Lambek. There is, of course, also a coarsest possible equivalence relation: if we declare all proofs of the same sequent to be equivalent, then we recover the usual posetal semantics of Tarski et al.

But what lies in between? It is this question which our thesis aims to shed some light on.

Note that this question may be rephrased in terms of pure category theory; it is equivalent to asking what suitable congruences exist on the denotational model.

Now it turns out that in the case of *monoidal-closed* structures, there already exists a formalism for producing such congruences: that of *monoidal Quillen model structures*, which originally arose in algebraic topology.

Abstract Homotopy Theory

Homotopy theory seems to have always lain near the heart of categorical proof theorists, at least as a source of inspiration.

Indeed, as a student at McGill, I distinctly remember hearing Lambek speak (metaphorically) of the steps of the cut-elimination procedure as "continuous deformations" of proofs.

There have also been several attempts at directly modelling concurrent phenomena (closely related to linear logic) in terms of algebraic topology—see, for instance, [24, 19, 20].

But, as far as we know, we are the first to attempt a direct application of the formalisms of abstract homotopy theory to denotational semantics.

Our specific objective is to see whether it is possible to create a Quillen model structure on a $*$-autonomous category with finite products and coproducts whose homotopy category is compact closed.

This may seem a bizarre project, since compact closed categories are often regarded as degenerate denotational models. But our initial motivation lay in the attempt to better understand Girard's *Geometry of Interaction*.

Geometry of Interaction

Motivated in part by a *monist* philosophy, Girard suggested a radically new paradigm for proof theory, called *Geometry of Ineraction*; its mathematical worth lies in its analysis of the *cut-elimination procedure* for linear logic.

Recall that the *cut rule*, introduced by Gentzen, is a generalisation of *modus*

ponens:

$$\frac{\vec{a} \vdash \vec{x}, q \quad q, \vec{b} \vdash \vec{y}}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}} \text{ (X)}$$

where q denotes a single formula and $\vec{a}, \vec{b}, \vec{x}, \vec{y}$ denote arbitrarily long strings of formulae. A *cut-elimination theorem* states that

> If there exists a proof T of $\vec{a} \vdash \vec{x}$, then there exists a proof T' of $\vec{a} \vdash \vec{x}$ which contains no instance of the rule (X).

with respect to some sequent calculus. Proofs containing no instance of (X) are called *cut-free*. The cut-elimination theorems for classical and intuitionstic sequent calculus were proven by Gentzen [21], and that for linear sequent calculus by Girard [22]. A *cut-elimination procedure* is a (non-deterministic) algorithm which, when given an arbitrary proof, results in a cut-free proof; both Gentzen and Girard prove their respective cut-elimination theorems by demonstrating the existence of a cut-elimination procedure.

Cut-elimination, in both its guises (theorem and procedure), plays a key role in all approaches to proof theory, categorical and non-categorical. For instance, in denotational semantics, any proof is deemed equivalent to all the cut-free proofs that may result from applying the cut-elimination procedure to it. This is essential in demonstrating the associativity of composition. But Girard finds this identification of a proof with its cut-free normal form(s) objectionable; it suppresses the dynamics of the re-writing system itself.

In the original Geometry of Interaction paper [23], a proof T is modelled by a pair of partial isometries on an infinite-dimensional Hilbert space: $[\![T]\!]$ and σ_T. The latter is intended to represent where, and how, instances of the cut rule occur in T. In particular, if T is cut-free, then $\sigma_T = 0$. Moreover, if T' is a cut-free normal form of T, then their models are related by Girard's *execution formula*.

$$[\![T']\!] = Ex([\![T]\!], \sigma_T)$$

Over the course of the last sixteen years, however, it has been shown that Girard's original model can be captured by categorical constructions [25], and that other Geometry of Interaction models can be constructed using essentially the same [2], and

similar [3], categorical techniques. [Though not in that order!]

The theory of compact closed categories plays an important role in this process. In particular, the execution formula can be understood as an instance of *tracing*—a now common concept in monoidal category theory.

The relationship between compact closed categories and traced monoidal categories is a direct generalisation of that between (discrete or ordered) abelian groups and (discrete or ordered) commutative cancellation monoids. Namely, every compact closed category carries a canonical trace, and every traced monoidal category can be embedded into a compact closed category, via the so-called **Int** construction [28].

Combining all of the above

It is our contention that Quillen model structures can provide a bridge between denotational semantics (as embodied in the theory of $*$-autonomous category) and Geometry of Interaction (as embodied in the theory of compact closed categories). This is because many denotational models satisfy an extra inference, called the *mix rule*.

$$\frac{}{x, y \vdash x, y}$$

Categorically, this manifests itself as a natural transformation between two 'products' on the denotational model, which we will denote $\otimes$ and $\bowtie$. The definition of a compact closed category amounts to the requirement that this very natural transformation be invertible.

Now the technical advantange of Quillen model structures, as a means of pursuing abstract homotopy theory, is that one must specify a class of maps to be inverted; the members of this specified class are called *weak equivalences*.

Thus, including the components of the natural transformation $\otimes \to \bowtie$ among the weak equivalences of a Quillen model structure will result in them being invertible in the resultant homotopy category. If the Quillen model structure is also chosen in such a way that the $*$-autonomous structure is preserved, then the homotopy category will therefore be compact closed.

Overview

The layout of this thesis is as follows.

In Chapter 2, we introduce the various notions of monoidal category which form the foundation of the categorical interpretation of linear logic (and so much of the rest of mathematics).

In Chapter 3, we discuss both classical and linear logic, and categorical models of these.

In Chapter 4, we review Abstract Homotopy Theory. This chapter also contains some new results, where we have had to eliminate the assumption of arbitrary finite limits and colimits in favour of just discrete ones.

In Chapter 5, we give a first example of Abstract Homotopy Theory arising naturally in the general theory of monoidal categories. Namely, how Barr's little-chu category may be viewed as a homotopy category of his big-Chu category.

In Chapter 6, we introduce a new class of $*$-autonomous categories (denotational models for linear logic) which we loosely call categories of *adherence spaces*. These are closely related to Schalk and de Paiva's poset-valued sets, and the precise nature of this relationship is discussed.

In Chapter 7, we fix one category of adherence spaces and prove the existence of a QMS on it. This QMS restricts to a full subcategory of adherence spaces which is $*$-autonomous but not compact closed, and which has products and coproducts but not arbitrary limits and colimits, and whose resultant homotopy category is compact closed. Thus we kill two birds with one stone: we show that pullbacks and pushouts are optional, as well as answering our original question. We also discuss possibilities for future research.

Chapter 1

Category Theory

In this chapter we review the basic concepts of monoidal category theory.

We shall assume familiarity with categories, functors, natural and dinatural transformations, adjunctions, limits and colimits. For us, the word *equivalence* signifies what some authors call *adjoint equivalence*.

We usually do not specify the domain and codomain functors of a natural transformation explicitly, but rather implicitly, by displaying a generic component.

1.1 Monoidal and monoidal closed categories

Definition 1.1.1

A *monoidal category* consists of: a category $\mathcal{K}$, a distinguished object i of $\mathcal{K}$, a functor $\mathcal{K} \times \mathcal{K} \xrightarrow{\otimes} \mathcal{K}$, and natural isomorphisms

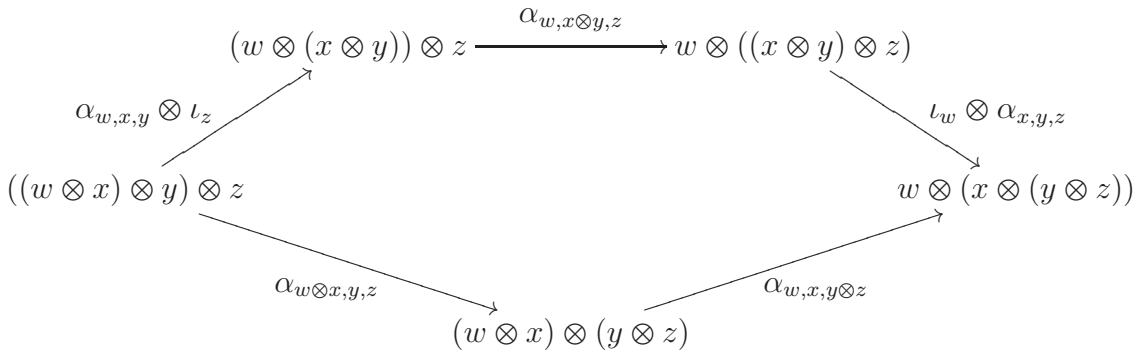
$$i \otimes x \xrightarrow{\vec{\upsilon}_x} x \xleftarrow{\overleftarrow{\upsilon}_x} x \otimes i$$

$$(x \otimes y) \otimes z \xrightarrow{\alpha_{x,y,z}} x \otimes (y \otimes z)$$

which satisfy the following *coherence axioms*:

$$\begin{array}{ccc} (x \otimes i) \otimes y & \xrightarrow{\alpha_{x,i,y}} & x \otimes (i \otimes y) \\ & {\scriptstyle \overleftarrow{\upsilon}_x \otimes \iota_y} \searrow \quad \swarrow {\scriptstyle \iota_x \otimes \vec{\upsilon}_y} & \\ & x \otimes y & \end{array}$$

commutes for every $x, y \in \mathsf{obj}[\mathcal{K}]$, and



commutes for every $w, x, y, z \in \mathsf{obj}[\mathcal{K}]$.

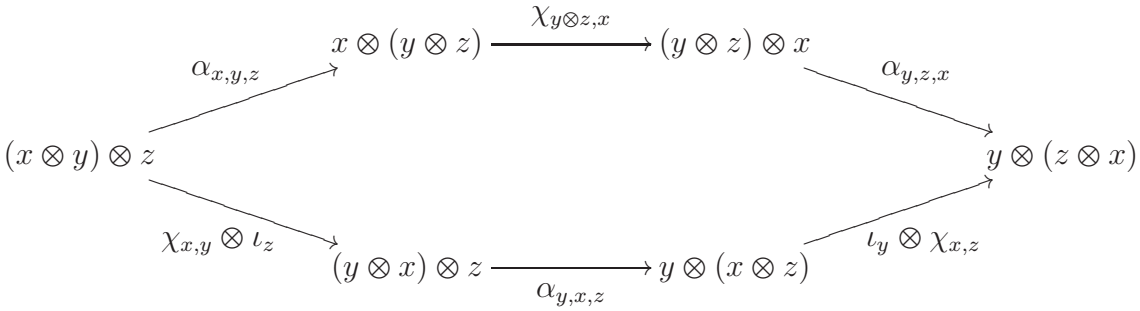
We usually suppress $\alpha, \vec{\upsilon}, \overleftarrow{\upsilon}$ in notation. So, for example, we say that $(\mathcal{K}, i, \otimes)$ is a monoidal category, rather than writing out $(\mathcal{K}, i, \otimes, \alpha, \vec{\upsilon}, \overleftarrow{\upsilon})$.

Definition 1.1.2

A *symmetry* on a monoidal category $(\mathcal{K}, i, \otimes)$ is a natural isomorphism

$$x \otimes y \xrightarrow{\chi_{x,y}} y \otimes x$$

satisfying the coherence condition that



should commute for all $x, y, z \in \mathsf{obj}[\mathcal{K}]$, and that the equation $\chi_{x,y} = \chi_{y,x}^{-1}$ should hold for all $x, y \in \mathsf{obj}[\mathcal{K}]$.

A *symmetric monoidal category* is a monoidal category $(\mathcal{K}, i, \otimes)$ together with a chosen symmetry χ.

As with α, $\vec{\upsilon}$ and $\overleftarrow{\upsilon}$, we shall frequently suppress χ in notation. We shall also use $\sim$ to denote any reasonably evident combination of natural isomorphisms.

Examples 1.1.3

1. The category of sets and functions, **Set**, can be made into a symmetric monoidal category in two obvious ways. Let 0 denote the empty set, 1 any singleton—say $1 = \{*\}$, $+$ the disjoint union of sets, and $\times$ cartesian product. Then both $(\mathbf{Set}, 0, +)$ and $(\mathbf{Set}, 1, \times)$ are symmetric monoidal categories [34].

2. Similarly, both $(k\text{-}\mathbf{Lin}, o, \oplus)$ and $(k\text{-}\mathbf{Lin}, k, \otimes)$ are symmetric monoidal categories, where k is a field, k-**Lin** denotes the category of k-vector spaces and k-linear transformations, and $o = \{0_k\}$.

3. Let $\mathcal{K}$ be any category, and let $\mathcal{V} = \mathsf{End}[\mathcal{K}]$ be the category of endofunctors of $\mathcal{K}$ with natural transformations between them. Then $(\mathcal{V}, i, \cdot)$ is a monoidal category, where i denotes the identity functor on $\mathcal{K}$, and $\cdot$ represents composition of functors and horizontal composition of natural transformations. [We reserve $\circ$ for the composition of arrows in $\mathcal{V}$—i.e., vertical composition of natural transformations.]

   As a special case, note that if $\mathcal{O}$ is the empty category, then $\mathcal{T} := \mathsf{End}[\mathcal{O}]$ is the terminal category—it has only one object, i, and only one arrow ι_i.

The last example above is somewhat atypical in that $f \cdot i = f = i \cdot f$ and $(f \cdot g) \cdot h = f \cdot (g \cdot h)$. This means that we can—and do—choose $\vec{\upsilon}_f$, $\overleftarrow{\upsilon}_f$ and $\alpha_{f,g,h}$ to be the appropriate identity natural transformations. [Note that the coherence conditions are trivially satified!] Such monoidal categories are called *strict*; there is a famous theorem [34] that every monoidal category is equivalent to a strict one.

In the next definition, we introduce a new symbol, $\boxtimes$, for certain types of monoidal structures; the reason for this should become clear by the end of section 2.3.

Definition 1.1.4

A *monoidal closed* category is a monoidal category $(\mathcal{K}, e, \boxtimes)$ together with a pair of functors

$$\mathcal{K}^{\mathsf{op}} \times \mathcal{K} \xrightarrow{\quad \multimap \quad} \mathcal{K} \xleftarrow{\quad \circ\!\!- \quad} \mathcal{K} \times \mathcal{K}^{\mathsf{op}}$$

and, for every $x, y \in \mathsf{obj}[\mathcal{K}]$, natural transformations

$$x \boxtimes (x \multimap (-)) \xrightarrow{\vec{\varepsilon}^{(x)}} (-) \qquad (-) \xrightarrow{\vec{\eta}^{(x)}} x \multimap (x \boxtimes (-))$$
$$((-) \circ\!\!- y) \boxtimes y \xrightarrow{\overleftarrow{\varepsilon}^{(y)}} (-) \qquad (-) \xrightarrow{\overleftarrow{\eta}^{(y)}} ((-) \boxtimes y) \circ\!\!- y$$

satisfying the usual 'triangle' identities—i.e., making

$$\mathcal{K} \underset{x \boxtimes (-)}{\overset{x \multimap (-)}{\rightleftarrows}} \mathcal{K} \qquad \text{and} \qquad \mathcal{K} \underset{(-) \boxtimes y}{\overset{(-) \circ\!\!- y}{\rightleftarrows}} \mathcal{K}$$

into adjunctions.

If $(\mathcal{K}, e, \boxtimes)$ is a symmetric monoidal category, then the functors $a \boxtimes (-)$ and $(-) \boxtimes a$ are isomorphic. Therefore, by the uniqueness of adjoints up to natural isomorphism, we have $a \multimap (-) \cong (-) \circ\!\!- a$ for every $a \in \mathsf{obj}[\mathcal{K}]$. Indeed, to show that a symmetric monoidal category is closed, it suffices to demonstrate the existence of a suitable $\multimap$ and then to define $\circ\!\!-$ via $b \circ\!\!- a = a \multimap b$.

This argument does not apply to non-symmetric monoidal categories. If a monoidal category admits the existence of a suitable $\multimap$, then we call it *left*-closed; if it admits a suitable $\circ\!\!-$, then we call it *right*-closed. Most authors call non-symmetric monoidal closed categories *bi*-closed in order to emphasise the existence of both $\multimap$ and $\circ\!\!-$.

Examples 1.1.5

1. $(\mathbf{Set}, 1, \times)$ can be made into a monoidal closed category by setting $x \multimap y$ (and $y \circ\!\!- x$) to equal the set of functions $x \to y$.

2. Similarly, $(k\text{-}\mathbf{Lin}, k, \otimes)$ can be made into a monoidal closed category by setting $x \multimap y$ (and $y \circ\!\!- x$) to equal the space of k-linear transformations $x \to y$.

While the preceding examples are familiar, the following are less so. Indeed, I do not believe that they appear in the literature, although related concepts can be found, for example, in [43].

Example 1.1.6

Let $\mathcal{K}$ be any category, and let $\mathcal{V} = \mathsf{End}^{+\infty}[\mathcal{K}]$ be the category of "endofunctors with infinitely many right adjoints". More precisely, the objects of $\mathcal{V}$ are infinite sequences of adjunctions

$$f = (f_0 \dashv f_1 \dashv f_2 \dashv \cdots)$$

and an arrow $f \longrightarrow g$ is simply a natural transformation $f_0 \longrightarrow g_0$. [This does in fact induce natural transformations $f_{2n} \longrightarrow g_{2n}$ and $g_{2n+1} \longrightarrow f_{2n+1}$, for all $n \in \mathbb{N}$.]

Then $(\mathcal{V}, i, \cdot)$ is a strict monoidal category where i denotes the constant sequence of identity functors on $\mathcal{K}$, and $\cdot$ is defined by

$$(f \cdot g)_{2n} = f_{2n} \cdot g_{2n} \quad \text{and} \quad (f \cdot g)_{2n+1} = g_{2n+1} \cdot f_{2n+1}$$

Moreover, we can make $(\mathcal{V}, i, \cdot)$ into a monoidal left-closed category as follows.

First, we define f^* to be the sequence given by $f^*_n = f_{n+1}$; then we set $f \multimap h = f^* \cdot h$.

Intuitively, we have that a natural transformation

$$f_0 \cdot g_0 = (f \cdot g)_0 \longrightarrow h_0$$

corresponds to an arrow

$$g_0 \longrightarrow f_1 \cdot h_0 = (f \multimap h)_0$$

because we can just apply the adjunction $f_0 \dashv f_1$. More technically, we should say that we can define $\vec{\varepsilon}^{(f)}$ and $\vec{\eta}^{(f)}$ as

$$(f \cdot (f \multimap h))_0 = (f_0 \cdot f_1) \cdot h_0 \xrightarrow{\varepsilon^{(01)} \cdot h_0} h_0$$

and

$$h_0 \xrightarrow{\eta^{01} \cdot h_0} (f_1 \cdot f_0) \cdot h_0 = (f \multimap (f \cdot h))_0$$

where $\eta^{01}, \varepsilon^{01}$ are the unit and counit of $f_0 \dashv f_1$ respectively.

The analogously-defined category $\mathsf{End}^{-\infty}[\mathcal{K}]$ of "endofunctors with infinitely many left adjoints"—i.e., of sequences of adjunctions

$$f = (\cdots \dashv f_{-2} \dashv f_{-1} \dashv f_0)$$

—is right-closed: one sets $h \circ\!\!- \ g = h \cdot {}^*g$, where ${}^*g$ is the sequence defined by ${}^*g_n = g_{n-1}$.

Theorem 1.1.7

For any object z in any monoidal closed category $\mathcal{K}$, we have an adjunction

$$\mathcal{K}^{\mathsf{op}} \underset{z \circ\!\!- \ (-)}{\overset{(-) \multimap z}{\rightleftarrows}} \mathcal{K}$$

Proof

An arrow from $z \circ\!\!- \ y \longrightarrow x$ in $\mathcal{K}^{\mathsf{op}}$ is precisely an arrow $x \longrightarrow z \circ\!\!- \ y$ in $\mathcal{K}$. By the second adjunction in Definition 1.1.4, such arrows are in natural bijective correspondance with arrows of the form $x \boxtimes y \longrightarrow z$ in $\mathcal{K}$. But these arrows, using the first adjunction in Definition 1.1.4, are in natural bijective correspondance with those of the form $y \longrightarrow x \multimap z$ in $\mathcal{K}$.

More precisely, the unit of the adjunction $(-) \longrightarrow (z \circ\!\!- \ (-)) \multimap z$ is the transpose of $(z \circ\!\!- \ (-)) \boxtimes (-) \xrightarrow{\overleftarrow{\varepsilon}_z^{(-)}} z$; the co-unit is defined similarly. Q.E.D.

Remark 1.1.8

If we think of an arrow $x \boxtimes y \xrightarrow{\psi} z$ as a "function of two variables"

$$\begin{array}{ccc} x \boxtimes y & \xrightarrow{\psi} & z \\ \alpha, \beta & \mapsto & \psi(\alpha, \beta) \end{array}$$

then we should think of the corresponding arrow $y \longrightarrow x \multimap z$, which will henceforth be denoted $'\psi$, as the "second-order function"

$$\begin{array}{ccc} y & \xrightarrow{'\psi} & x \multimap z \\ \beta & \mapsto & [\alpha \Rightarrow \psi(\alpha, \beta)] \end{array}$$

Similarly, the other corresponding arrow $x \longrightarrow z \circ\!\!- y$, denoted ψ', should be thought of as

$$\begin{array}{ccc} x & \xrightarrow{\psi'} & z \circ\!\!- y \\ \alpha & \mapsto & [\psi(\alpha, \beta) \leftrightarrows \beta] \end{array}$$

Using this notation we can express the idea that if, say, $w \xrightarrow{\omega} x$ and $y \xrightarrow{\psi} z$, then $x \multimap y \xrightarrow{\omega \multimap \psi} w \multimap z$ represents "internal composition" by ψ and ω as follows:

$$\begin{array}{ccc} x \multimap y & \xrightarrow{\omega \multimap \psi} & w \multimap z \\ \phi & \mapsto & [\alpha \Rightarrow \psi(\phi(\omega(\alpha)))] \end{array}$$

Lemma 1.1.9

Given arrows $a \xrightarrow{\lambda} b$, $x \xrightarrow{\rho} y$, $a \boxtimes c \xrightarrow{\psi} x$, and $b \boxtimes c \xrightarrow{\omega} y$, the following diagrams are equivalent. [In the sense that if one commutes then so do all the others]

$$\begin{array}{ccc} a \boxtimes c & \xrightarrow{\psi} & x \\ {\scriptstyle \lambda \boxtimes \iota_c}\downarrow & & \downarrow{\scriptstyle \rho} \\ b \boxtimes c & \xrightarrow[\omega]{} & y \end{array} \qquad \begin{array}{ccc} a & \xrightarrow{\psi'} & x \circ\!\!- c \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \rho \circ\!\!- \iota_y} \\ b & \xrightarrow[\omega']{} & y \circ\!\!- c \end{array} \qquad \begin{array}{ccc} c & \xrightarrow{{}'\omega} & b \multimap y \\ {\scriptstyle {}'\psi}\downarrow & & \downarrow{\scriptstyle \lambda \multimap \iota_y} \\ a \multimap x & \xrightarrow[\iota \multimap \rho]{} & a \multimap y \end{array}$$

Proof

The equivalence of the first two diagrams is a routine exercise in adjunctions.

But the equivalence of the first and third relies on the final observation of Remark 1.1.8:

$$(\lambda \multimap \iota_y) \circ {}'\omega = {}'(\omega \circ (\lambda \boxtimes \iota_c))$$

since the left-hand side maps

$$\zeta \mapsto {}'\omega(\zeta) \mapsto [\alpha \Rightarrow \iota_y({}'\omega(\zeta)(\lambda(\alpha)))]$$

while the expression inside evaluates as

$$\iota_y({}'\omega(\zeta)(\lambda(\alpha))) = \omega(\lambda(\alpha), \zeta) = \omega((\lambda \boxtimes \iota_c)(\alpha, \zeta))$$

Similarly,

$$(\iota_y \multimap \rho) \circ {}'\psi = {}'(\rho \circ \psi)$$

Q.E.D.

1.2 Monoids and monoidal functors

Definition 1.2.1

Let $(\mathcal{H}, i, \odot)$ and $(\mathcal{K}, j, \otimes)$ be monoidal categories.

A *monoidal functor* $(\mathcal{H}, i, \odot) \longrightarrow (\mathcal{K}, j, \otimes)$ consists of a functor $\mathcal{H} \xrightarrow{M} \mathcal{K}$ together with an arrow $j \xrightarrow{\eta} M(i)$ and a natural transformation

$$M(x) \otimes M(y) \xrightarrow{\mu_{x,y}} M(x \odot y)$$

satisfying the diagrams

$$\begin{array}{ccc} j \otimes M(x) & \xrightarrow{\eta \otimes \iota_{M(x)}} & M(i) \otimes M(x) \\ {\scriptstyle \vec{\upsilon}_{M(x)}}\downarrow & & \downarrow{\scriptstyle \mu_{i,x}} \\ M(x) & \xleftarrow[M(\vec{\upsilon}_x)]{} & M(i \odot x) \end{array}$$

$$\begin{array}{ccc} M(x) \otimes j & \xrightarrow{\iota_{M(x)} \otimes \eta} & M(x) \otimes M(i) \\ {\scriptstyle \overleftarrow{\upsilon}_{M(x)}}\downarrow & & \downarrow{\scriptstyle \mu_{x,i}} \\ M(x) & \xleftarrow[M(\overleftarrow{\upsilon}_x)]{} & M(x \odot i) \end{array}$$

$$\begin{array}{ccc} (M(x) \otimes M(y)) \otimes M(z) & \xrightarrow{\alpha_{M(x),M(y),M(z)}} & M(x) \otimes (M(y) \otimes M(z)) \\ {\scriptstyle \mu_{x,y} \otimes \iota_{M(z)}}\downarrow & & \downarrow{\scriptstyle \iota_{M(x)} \otimes \mu_{y,z}} \\ M(x \odot y) \otimes M(z) & & M(x) \otimes M(y \odot z) \\ {\scriptstyle \mu_{x \odot y, z}}\downarrow & & \downarrow{\scriptstyle \mu_{x, y \odot z}} \\ M((x \odot y) \odot z) & \xrightarrow[M(\alpha_{x,y,z})]{} & M(x \odot (y \odot z)) \end{array}$$

If both η and μ are invertible, then (M, η, μ) is called a *strong* monoidal functor.

Example 1.2.2

The forgetful functor k-**Lin** $\xrightarrow{|\ |}$ **Set** can be made into a monoidal functor $(k\text{-}\mathbf{Lin}, k, \otimes) \longrightarrow (\mathbf{Set}, 1, \times)$ by equipping it with the natural transformation whose components are the universal bi-linear transformations

$$|v| \times |w| \longrightarrow |v \otimes w|$$

and the map $1 \xrightarrow{\eta} |k|$ given by $\eta(*) = 1_k$.

The same functor can also be viewed—or made into—a monoidal functor $(k\text{-}\mathbf{Lin}, o, \oplus) \longrightarrow (\mathbf{Set}, 1, \times)$ by equipping it with the identity natural transformation $|v| \times |w| \xrightarrow{=} |v \oplus w|$ and the unique function $1 \longrightarrow |o|$.

Note that the latter example is strong, while the former is not.

Definition 1.2.3

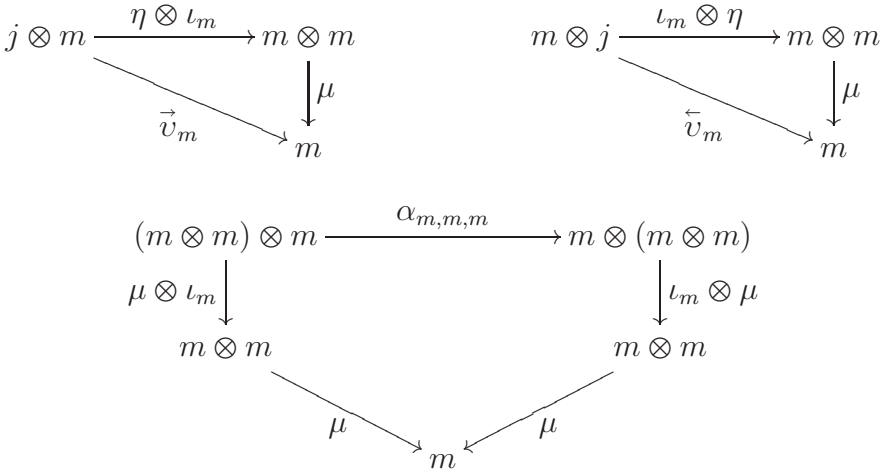
Let $(\mathcal{K}, j, \otimes)$ be a monoidal category, and $(\mathcal{T}, i, \cdot)$ be the terminal category with its unique monoidal structure (see Example 1.1.3.3).

Then, a *monoid* in $(\mathcal{K}, j, \otimes)$—or, more colloquially, a $\otimes$-*monoid* in $\mathcal{K}$—is a monoidal functor $(\mathcal{T}, i, \cdot) \xrightarrow{(M,\eta,\mu)} (\mathcal{K}, j, \otimes)$.

Remark 1.2.4

Note that, since $\mathcal{T}$ has only one object, μ has only one component—namely, $M(i) \otimes M(i) \xrightarrow{\mu_{i,i}} M(i)$.

Writing $m = M(i)$ and taking advantage of the fact that $\mathcal{T}$ is a strict monoidal category, we see that a monoid in $(\mathcal{K}, j, \otimes)$ amounts to a diagram $j \xrightarrow{\eta} m \xleftarrow{\mu} m \otimes m$ in $\mathcal{K}$ satisfying the diagrams



Examples 1.2.5

A $\times$-monoid in **Set** is, in effect, the same as a monoid in the usual sense of the word. The last of three diagrams above state that μ is an associative binary operation on m, the previous two that $\eta(*)$ is a two-sided unit for μ.

By contrast, each set carries a unique +-monoid structure, since there is only possible choice for $0 \xrightarrow{\eta} m$, and there is only map $m + m \xrightarrow{\mu} m$ compatible with η, namely the *co-diagonal* (or *squish*) map.

Similarly, a monoid in $(k\text{-}\mathbf{Lin}, k, \otimes)$ is, essentially, a unital k-algebra, whereas the only monoids in $(k\text{-}\mathbf{Lin}, o, \oplus)$ are given by pre-existing additive structure on each vector space v: $v \oplus v \xrightarrow{+_v} v \xleftarrow{0_v} o$.

Definition 1.2.6

A *co-monoid* in $(\mathcal{K}, j, \otimes)$ is a monoid in $(\mathcal{K}^{\mathsf{op}}, j, \otimes)$.

1.3 Linear distributive and $*$-autonomous categories

Definition 1.3.1

Let $(\mathcal{K}, e, \invamp, \multimap, \circ\!\!-)$ be a monoidal closed category. Then a *dualising object* for $\mathcal{K}$ is a d such that the adjunction

$$\mathcal{K}^{\mathsf{op}} \underset{d \circ\!\!- (-)}{\overset{(-) \multimap d}{\rightleftarrows}} \mathcal{K}$$

(see 1.1.7) is actually an equivalence of categories.

This means that the canonical arrows

$$x \longrightarrow (d \circ\!\!- x) \multimap d \quad \text{and} \quad x \longrightarrow d \circ\!\!- (x \multimap d)$$

should be invertible for every x in $\mathsf{obj}[\mathcal{K}]$.

A (symmetric) $*$-*autonomous category* is a (symmetric) monoidal closed category together with a choice of dualising object.

We write x^* for $x \multimap d$ and ${}^*x$ for $d \circ\!\!- x$.

Example 1.3.2

Let $k\text{-}\mathbf{Lin}_{fd}$ denote the category of finite-dimensional k-vector spaces and k-linear transformations. One obtains a monoidal closed structure on $k\text{-}\mathbf{Lin}_{fd}$ by restricting that of $k\text{-}\mathbf{Lin}$.

Moreover k is a dualising object for $(k\text{-}\mathbf{Lin}_{fd}, k, \otimes, \multimap, \circ\!\!-)$.

Example 1.3.3

Let $\mathcal{K}$ be any category, and let $\mathcal{V} = \mathsf{End}^{\pm\infty}[\mathcal{K}]$ be the category of "endofunctors with infinitely many adjoints on both sides"—i.e., of doubly-infinite sequences of adjunctions

$$f = (\cdots \dashv f_{-1} \dashv f_0 \dashv f_1 \dashv \cdots).$$

Then, the left-closed structure of $\mathsf{End}^{+\infty}[\mathcal{K}]$ (Example 1.1.6) and the right-closed structure of $\mathsf{End}^{-\infty}[\mathcal{K}]$ both restrict to $\mathcal{V}$. Moreover i is a dualising object for $\mathcal{V}$.

To see this, first note that our notation is consistent, since $f \multimap i$ and $i \circ\!\!- g$ are, by the given definition of $\multimap$ and $\circ\!\!-$, equal to $f^* \cdot i = f^*$ and $i \cdot {}^*g = {}^*g$, respectively. Then, note that the canonical maps $f \longrightarrow ({}^*f)^*$ are actually identities.

Thus $(\mathcal{V}, i, \cdot, \multimap, \circ\!\!-, i)$ is a $*$-autonomous category.

These examples are all atypical in that they are *(non-symmetrically) compact closed*—definition to follow in the next section. In particular, we have $(v \otimes w)^* \cong w^* \otimes v^*$ and $(f \cdot g)^* = g^* \cdot f^*$, for all v and w, and all f and g, respectively. Equivalently,

$$v \otimes w \cong {}^*(w^* \otimes v^*)$$

$$f \cdot g = {}^*(g^* \cdot f^*)$$

for all v, w, f and g.

But for a general $*$-autonomous category $(\mathcal{K}, e, \invamp, \multimap, \circ\!\!-, d)$, the object ${}^*(y^* \invamp x^*)$ is neither equal nor isomorphic to $x \invamp y$. We devote the rest of this section to its study.

Theorem 1.3.4 (M. Barr, [9])

$(\mathcal{K}, d, \bowtie)$ is a monoidal category whenever $(\mathcal{K}, e, \invamp, \multimap, \circ\!\!-, d)$ is $*$-autonomous and $\mathcal{K} \times \mathcal{K} \xrightarrow{\bowtie} \mathcal{K}$ is defined by the formula

$$x \bowtie y := {}^*(y^* \invamp x^*)$$

Moreover, $x \bowtie y \cong ({}^*y \invamp {}^*x)^*$.

Thus, a $*$-autonomous category is doubly monoidal. Moreover, while $\invamp$ and $\bowtie$ do not generally coincide, they do interact in a highly non-trivial fashion.

Lemma 1.3.5

The duality of a $*$-autonomous category $\mathcal{K}$ can be "internalised" in the sense that we have natural isomorphisms

$$x \multimap z \cong x^* \circ\!\!- z^* \qquad \text{and} \qquad z \circ\!\!- y \cong {}^*z \multimap {}^*y$$

Moreover, the *boolean implication* identities hold; i.e., we have natural isomorphisms

$$x \multimap z \cong x^* \invamp z \qquad \text{and} \qquad z \circ\!\!- y \cong z \invamp {}^*y$$

Proof

For any object d in any monoidal closed category $\mathcal{K}$, we have internal composition maps

$$(x \multimap z) \otimes (z \multimap d) \longrightarrow x \multimap d$$
$$(d \circ\!\!- z) \otimes (z \circ\!\!- y) \longrightarrow d \circ\!\!- y$$

which can be transposed into maps of the form

$$x \multimap z \xrightarrow{\rho_{x,z}} (x \multimap d) \circ\!\!- (z \multimap d) = x^* \circ\!\!- z^*$$
$$z \circ\!\!- y \xrightarrow{\lambda_{z,y}} (d \circ\!\!- z) \multimap (d \circ\!\!- y) = {}^*z \multimap {}^*y$$

Moreover, if d is a dualising object, then $\rho_{x,z}$ is inverted by

$$x^* \circ\!\!- z^* \xrightarrow{\lambda_{x^*,z^*}} {}^*(x^*) \multimap {}^*(z^*) \cong x \multimap {}^*(z^*) \cong x \multimap z$$

and similarly $\lambda_{z,y}$ is inverted by

$${}^*z \multimap {}^*y \xrightarrow{\rho_{{}^*z,{}^*y}} ({}^*z)^* \circ\!\!- ({}^*y)^* \cong z \circ\!\!- ({}^*y)^* \cong z \multimap y$$

Now it follows that we have natural bijections between:

- maps of the form $x \otimes y \longrightarrow z$,
- maps of the form $y \longrightarrow x \multimap z \cong x^* \circ\!\!- z^*$,
- maps of the form $y \otimes z^* \longrightarrow x^*$, and

- maps of the form $x \cong {}^*(x^*) \longrightarrow {}^*(y \parr z^*)$.

So, by the uniqueness of adjoints up to isomorphism, we have $z \circ\!\!- y \cong {}^*(y \parr z^*)$. But, $z \otimes {}^*y \cong {}^*((^*y)^* \parr z^*) \cong {}^*(y \parr z^*)$ also.

Similarly, $x \multimap z \cong ({}^*z \parr x)^* \cong x^* \otimes z$. Q.E.D.

Theorem 1.3.6

In a $*$-autonomous category $\mathcal{K}$ we have natural transformations

$$w \parr (x \otimes y) \xrightarrow{\vec{\kappa}_{w,x,y}} (w \parr x) \otimes y$$
$$(x \otimes y) \parr z \xrightarrow{\overleftarrow{\kappa}_{x,y,z}} x \otimes (y \parr z)$$

Proof

The easiest proof uses the previous Lemma. By transposing

$$x \parr (x \multimap y) \parr z \xrightarrow{\vec{\varepsilon}_y^{(x)} \parr \iota_z} y \parr z$$

we obtain a canonical arrow

$$(x^* \otimes y) \parr z \cong (x \multimap y) \parr z \longrightarrow x \multimap (y \parr z) \cong x^* \otimes (y \parr z)$$

Replacing x by ${}^*x$ and using the isomorphisms $x \longrightarrow ({}^*x)^*$, we obtain the second of the desired maps. Q.E.D.

The theorem above motivates the following definition. Note that $\vec{\kappa}$ and $\overleftarrow{\kappa}$ are not, in general, invertible—see [14, section 5.2].

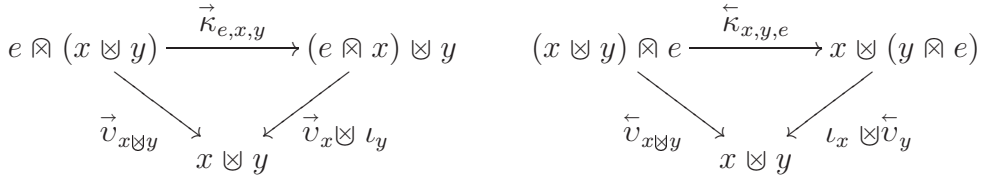
Definition 1.3.7

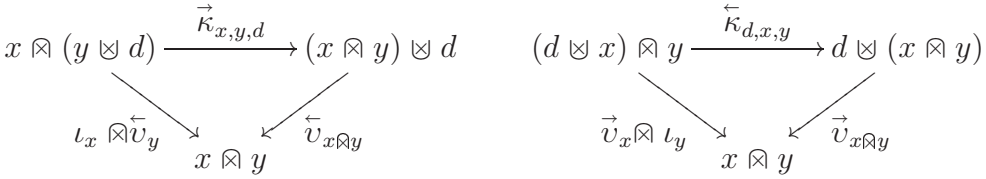
A *linear-distributive category* consists of a category $\mathcal{K}$, two distinguished objects d and e of $\mathcal{K}$, two functors $\mathcal{K} \times \mathcal{K} \xrightarrow{\parr} \mathcal{K}$ and $\mathcal{K} \times \mathcal{K} \xrightarrow{\otimes} \mathcal{K}$, natural isomorphisms which make both $(\mathcal{K}, e, \parr)$ and $(\mathcal{K}, d, \otimes)$ into monoidal categories, and two coherent natural transformations

$$w \parr (x \otimes y) \xrightarrow{\vec{\kappa}_{w,x,y}} (w \parr x) \otimes y$$
$$(x \otimes y) \parr z \xrightarrow{\overleftarrow{\kappa}_{x,y,z}} x \otimes (y \parr z)$$

—where in this case, coherent means that $\vec{\kappa}$ and $\overleftarrow{\kappa}$ should satisfy: one diagram each with respect to each unit (d and e); plus, six variants of the pentagon in Definition 1.1.1.

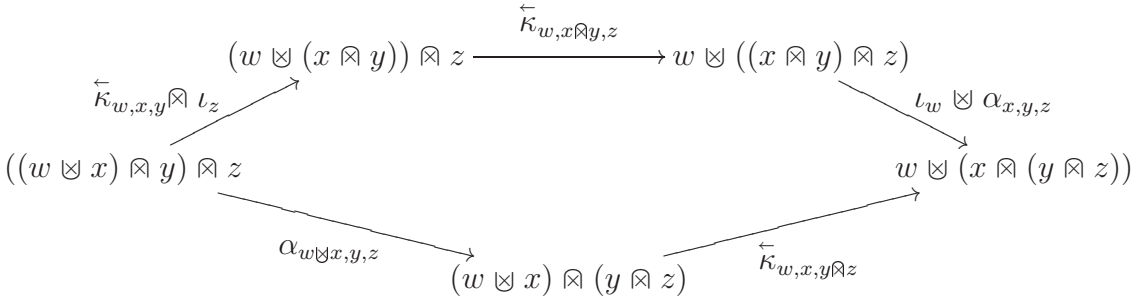
The four unit diagrams are:



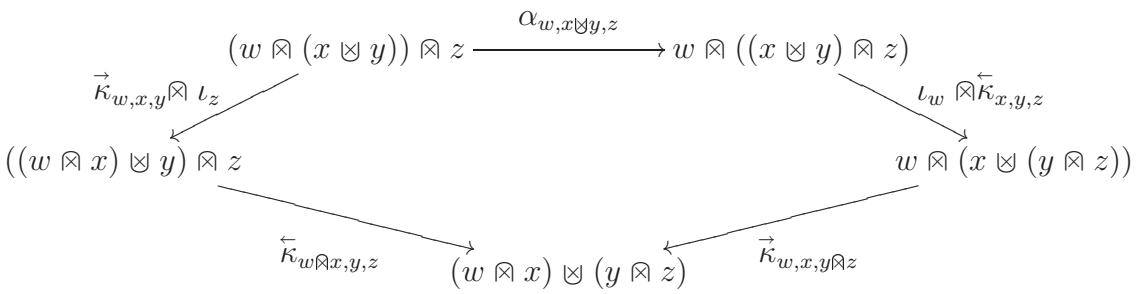


The six variants of the pentagon come from considering all possible combinations of $\boxtimes$ and $\bowtie$ replacing $\otimes$ in the expression $((w \otimes x) \otimes y) \otimes z$. [Of course, there are really eight such combinations—but the two constant cases, $\boxtimes\boxtimes\boxtimes$ and $\bowtie\bowtie\bowtie$, have already been handled by the requirement that $(\mathcal{K}, e, \boxtimes)$ and $(\mathcal{K}, d, \bowtie)$ both be monoidal categories.]

For example, the case $\bowtie\boxtimes\boxtimes$ yields



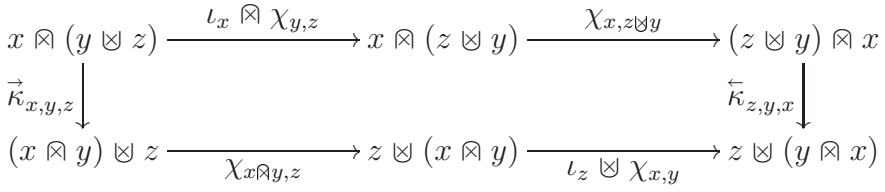
Similarly, the case $\invamp\otimes\invamp$ yields



Note how arrows must sometimes be reversed to accomodate the fact that $\vec{\kappa}$ are not generally invertible. In effect, $\vec{\kappa}$ takes the place of α^{-1} rather than that of α.

Definition 1.3.8

A *symmetric linear-distributive category* is a linear-distributive category together with symmetries (both denoted χ) for $\invamp$ and $\otimes$ which satisfy the further coherence condition:



Note that for a symmetric linear-distributive category, we obtain extra natural transformations

$$w \invamp (x \otimes y) \xrightarrow{\vec{\vec{\kappa}}_{w,x,y}} x \otimes (w \invamp y)$$
$$(x \otimes y) \invamp z \xrightarrow{\overleftarrow{\kappa}_{x,y,z}} (x \invamp z) \otimes y$$

This notation shall be useful even in the "trivial" case, when $\invamp$=$\otimes$.

As with Examples 1.1.6 and 1.3.3, I believe that the following is new—although it is easily motivated by the considerations of section 2.2.

Example 1.3.9

Let $(\mathcal{K}, t, \invamp)$ be a monoidal category whose unit, t, is a terminal object for $\mathcal{K}$. Suppose further that $\mathcal{K}$ has finite colimits and that the functors $x \invamp (-)$, $(-) \invamp y$

preserve these. If we then define $x \bowtie y$ as the pushout of :

$$\begin{array}{ccccc} x \boxtimes y & \longrightarrow & t \boxtimes y & \xrightarrow{\sim} & y \\ \downarrow & & & & \\ x \boxtimes t & & & & \\ \sim\downarrow & & & & \\ x & & & & \end{array}$$

then $\bowtie$ is associative, unital (with any initial object o as unit), and permits linear distributions given by universal property. For example, $\vec{\kappa}$ is defined as follows:

$$\begin{array}{ccccc} x \boxtimes (y \boxtimes z) & \longrightarrow & x \boxtimes z & \longrightarrow & z \\ \downarrow & & \downarrow & & \downarrow \\ x \boxtimes y & \longrightarrow & x \boxtimes (y \bowtie z) & \xrightarrow{\vec{\kappa}_{x,y,z}} & (x \boxtimes y) \bowtie z \end{array}$$

All the coherence conditions are easily verified using universality.

1.4 Compact closed and traced monoidal categories

The last example shows that, while $x \boxtimes y$ may not be isomorphic to $x \bowtie y$ in general, there may be a canonical map (in this case, a coprojection) $x \boxtimes y \longrightarrow x \bowtie y$.

This also occurs whenever the unit, e, of a monoidal-closed category $(\mathcal{K}, e, \boxtimes, \multimap, \circ\!\!-)$ is also a dualising object; that is to say, when $\mathcal{K}$ can be made into a $*$-autonomous category by choosing $d = e$.

As it turns out, these maps form a natural transformation $\boxtimes \longrightarrow \bowtie$, which can also be constructed under much weaker hypotheses.

Definitions 1.4.1

1. A *reboot* is a map $d \xrightarrow{\xi} e$ such that the following diagram commutes

$$\begin{array}{ccccc}
x \invamp (d \otimes y) & \xrightarrow{\vec{\kappa}_{x,d,y}} & (x \invamp d) \otimes y & \xrightarrow{(\iota_x \invamp \xi) \otimes \iota_y} & (x \invamp e) \otimes y \\
\sim\uparrow & & & & \downarrow\sim \\
x \invamp y & & & & x \otimes y \\
\sim\downarrow & & & & \uparrow\sim \\
(x \otimes d) \invamp y & \xrightarrow[\vec{\kappa}_{x,d,y}]{} & x \otimes (d \invamp y) & \xrightarrow[\iota_x \invamp (\xi \otimes \iota_y)]{} & x \otimes (e \invamp y)
\end{array}$$

2. A *(∗-)mix category* is a linearly-distributive category $(\mathcal{K}, e, \invamp, d, \otimes)$ (respectively, a $*$-autonomous category $(\mathcal{K}, e, \invamp, \multimap, \circ\!\!-, d)$) together with a choice of reboot.

3. The common composite

$$x \invamp y \longrightarrow x \otimes y$$

   is called a *mix map*. Collectively, the mix maps form a natural transformation $\invamp \longrightarrow \otimes$.

4. A *compact closed category* is a $*$-mix category for which the natural transformation $\invamp \longrightarrow \otimes$ is invertible.

5. The *core* of a $(*$-$)$mix category is the full subcategory consisting of those x such that for every y, $x \invamp y \longrightarrow x \otimes y$ is invertible.

6. A *(∗-)isomix category* is a $(*$-$)$mix category whose reboot is invertible.

In any $*$-autonomous category $(\mathcal{K}, e, \invamp, \multimap, d)$ we have canonical maps

$$e \xrightarrow{\tau_x} x^* \otimes x \qquad \text{and} \qquad y \invamp y^* \xrightarrow{\gamma_y} d$$

—the former being the transpose of $\overleftarrow{\upsilon}_x$ followed by the isomorphism $x \multimap x \cong x^* \otimes x$, and the latter being a component of $\vec{\varepsilon}^{(d)}$.

Therefore, every symmetric compact closed category is "cancellative" in the following sense: given an arrow $x \otimes t \xrightarrow{\psi} y \otimes t$, we can construct an arrow

$$\begin{array}{ccccc} x & \xrightarrow{\iota_x \otimes \tau_t} & x \otimes (t^* \otimes t) & & y \otimes (t \otimes t^*) \xrightarrow{\iota_y \otimes \gamma_t} y \\ & & \downarrow \sim & & \sim \uparrow \\ & & (x \otimes t) \otimes t^* & \xrightarrow{\psi \otimes \iota_{t^*}} & (y \otimes t) \otimes t^* \end{array}$$

—this process is called *tracing* in honour of the following example.

Example 1.4.2

If x, y and t are, respectively, m, n and p-dimensional k-vector spaces, then a k-linear transformation $x \otimes t \xrightarrow{\psi} y \otimes t$ may be thought of as an blocked $(mp \times np)$-matrix—with each block of size $(p \times p)$.

Taking the ordinary trace of each block results in an $(m \times n)$-matrix, which corresponds to the same k-linear transformation $x \longrightarrow y$ as the process described above.

Since every commutative cancellation monoid embeds into an abelian group, it makes sense to ask whether every "cancellative" symmetric monoidal category can be embedded into a symmetric compact closed category.

This is in fact the case, but first we must give a more precise definition!

Definition 1.4.3

A *trace* on a symmetric monoidal category $(\mathcal{K}, i, \otimes)$ is a *sesqui-natural* transformation

$$[x \otimes t, y \otimes t]_{\mathcal{K}} \xrightarrow{\mathsf{Tr}_t} [x, y]_{\mathcal{K}}$$

—where (in this case) sesqui-natural means natural in x and y and dinatural in t—satisfying the following axioms:

1. if $x \otimes i \xrightarrow{\psi} y \otimes i$, and $\tilde{\psi}$ denotes the composite

$$x \xrightarrow{\overleftarrow{\upsilon}_x^{-1}} x \otimes i \xrightarrow{\psi} y \otimes i \xrightarrow{\overleftarrow{\upsilon}_y} y$$

then $\mathsf{Tr}_i(\psi) = \tilde{\psi}$;

2. if $x \otimes (s \otimes t) \xrightarrow{\psi} y \otimes (s \otimes t)$, and $\tilde{\psi}$ denotes the composite

$$(x \otimes s) \otimes t \xrightarrow{\alpha_{x,s,t}} x \otimes (s \otimes t) \xrightarrow{\psi} y \otimes (s \otimes t) \xrightarrow{\alpha^{-1}_{y,s,t}} (y \otimes s) \otimes t$$

then $\mathsf{Tr}_{s\otimes t}(\psi) = \mathsf{Tr}_s(\mathsf{Tr}_t(\tilde{\psi}))$;

3. if $a \otimes t \xrightarrow{\psi} x \otimes t$, $b \xrightarrow{\omega} y$ and χ denotes the composite

$$(a \otimes b) \otimes t \xrightarrow{\overleftarrow{\kappa}_{a,b,t}} (a \otimes t) \otimes b \xrightarrow{\psi \otimes \omega} (x \otimes t) \otimes y \xrightarrow{\overleftarrow{\kappa}_{x,t,y}} (x \otimes y) \otimes t$$

then $\mathsf{Tr}_t(\chi) = \mathsf{Tr}_t(\psi) \otimes \omega$; and,

4. $\mathsf{Tr}_x(\chi_{x,x}) = \iota_x$.

A *traced monoidal category* is a symmetric monoidal category equipped with a chosen trace.

Definition 1.4.4

Let $(\mathcal{K}, i, \otimes)$ be a traced monoidal category. Then we define a new category $\mathcal{G}[\mathcal{K}]$ as follows:

- the objects of $\mathcal{G}[\mathcal{K}]$ are 'fractions' ${}^{j}\!/\!_{k}$, where j and k are arbitrary objects of $\mathcal{K}$;
- an arrow ${}^{j}\!/\!_{k} \xrightarrow{\omega} {}^{m}\!/\!_{n}$ in $\mathcal{G}[\mathcal{K}]$ is defined to be a map in $\mathcal{K}$ of the form

$$j \otimes n \xrightarrow{\omega} m \otimes k$$

- the identity arrow ${}^{j}\!/\!_{k} \xrightarrow{\iota} {}^{j}\!/\!_{k}$ is defined to be the identity map $j \otimes k \xrightarrow{\iota} j \otimes k$.
- the composite of a pair of arrows ${}^{j}\!/\!_{k} \xrightarrow{\omega} {}^{m}\!/\!_{n} \xrightarrow{\psi} {}^{p}\!/\!_{q}$ is defined by tracing: let χ denote the composite

$$\begin{array}{ccccccc} (j \otimes q) \otimes n & & & & & & (p \otimes k) \otimes n \\ \downarrow{\scriptstyle \overleftarrow{\kappa}_{j,q,n}} & & & & & & \uparrow{\scriptstyle \overleftarrow{\kappa}_{p,n,k}} \\ (j \otimes n) \otimes q & \xrightarrow{\omega\otimes\iota} & (m \otimes k) \otimes q & \xrightarrow{\overleftarrow{\kappa}_{m,k,q}} & (m \otimes q) \otimes k & \xrightarrow{\psi\otimes\iota} & (p \otimes n) \otimes k \end{array}$$

then $\psi \circ \omega$ is defined to be (the arrow ${}^{j}\!/\!_{k} \to {}^{p}\!/\!_{q}$ corresponding to) $j \otimes q \xrightarrow{\mathsf{Tr}_n(\chi)} p \otimes k$.

Theorem 1.4.5

$\mathcal{G}[\mathcal{K}]$ is a compact closed category containing an isomorphic copy of $\mathcal{K}$; moreover the induced trace on $\mathcal{K}$ is the same as the original.

Proof

See [28] for the associativity of composition and the unit laws.

There is an obvious bijective correspondence between maps of the form

$$(j \otimes m) \otimes q \to p \otimes (n \otimes k)$$

—i.e, arrows ${}^{j\otimes m}\!/\!_{n\otimes k} \to {}^{p}\!/\!_{q}$—and maps of the form

$$j \otimes (m \otimes q) \to (p \otimes n) \otimes k$$

—i.e., arrows ${}^{j}\!/\!_{k} \to {}^{p\otimes n}\!/\!_{m\otimes q}$.

This naturally leads one to define:

$${}^{j}\!/\!_{k} \otimes {}^{m}\!/\!_{n} = {}^{j\otimes m}\!/\!_{n\otimes k} \text{ and } {}^{m}\!/\!_{n} \multimap {}^{p}\!/\!_{q} = {}^{p\otimes n}\!/\!_{m\otimes q}$$

It is easy to see that ${}^{i}\!/\!_{i}$ is both a tensor unit and a dualising object since

$${}^{i}\!/\!_{i} \otimes {}^{m}\!/\!_{n} \cong {}^{m}\!/\!_{n} \text{ and } {}^{m}\!/\!_{n} \multimap {}^{i}\!/\!_{i} \cong {}^{n}\!/\!_{m}$$

$\mathcal{K}$ embeds into $\mathcal{G}[\mathcal{K}]$ via the mapping $x \mapsto {}^{x}\!/\!_{i}$.

Q.E.D.

Chapter 2

Linear Logic

2.1 Introduction

Mathematical logic has its roots in the observation that certain steps of an argument can be made without reference to the meaning of the phrases involved—i.e., that there exists an underlying arithmetic of sentences.

For this reason, one introduces the notion of *formal language*. A formal language is a set of artificial statements; i.e., strings of symbols which are structured to look like statements, but which lack any inherent meaning.

Instead of attempting to assign a notion of absolute truth to elements of a formal language, $\mathbb{L}$, it is common to assign a notion of relative truth. That is, we equip $\mathbb{L}$ with a(n at first) binary relation $\vdash$, such that intended interpretation of $a \vdash x$ is that (the truth of) x *follows* from (the truth of) a. This relation should evidently be subject to axioms such as reflexivity and transitivity. Axioms concerning $\vdash$ are called *inference rules*.

For visual clarity, inference rules are usually displayed vertically, with hypotheses written above their conclusion. For example, reflexivity and transitivity are written as

$$\frac{}{a \vdash a} \qquad \text{and} \qquad \frac{a \vdash b \quad b \vdash c}{a \vdash c}$$

respectively.

Inference rules come in two sorts: *structural rules*, which express the intended

nature of sequents *per se*; and *logical rules*, which relate to the peculiarities of $\mathbb{L}$. [Both of the rules above are structural.]

For technical reasons, Gentzen was led to generalise $\vdash$ so that it became a binary relation on the free monoid generated by $\mathbb{L}$. Hence, we now consider expressions of the form

$$a_1, \ldots, a_n \vdash x_1, \ldots, x_m$$

where $a_1, \ldots, a_n$,$x_1, \ldots, x_m$ are all elements of $\mathbb{L}$. These expressions are called $\mathbb{L}$-*sequents*, or just *sequents* if $\mathbb{L}$ is understood.

The elements to the left of $\vdash$ are called *antecedents* and those to the right *succedents*. Note that the cases $n = 0$ and $m = 0$, are allowed. [They correspond to having the neutral element of the free monoid in the antecedent and succedent respectively.] We shall use vector notation $\vec{a}, \vec{b}, \vec{c}, \ldots$ to denote arbitrary elements of the free monoid on $\mathbb{L}$.

The classical interpretation of a sequent is that (the truth of) *at least one of* the succedents follows from (the truth of) *all of* antecedents. Or, in Gentzen's own words, [21]:

> $a_1, \ldots, a_n \vdash x_1, \ldots, x_m$ *bedeutet inhaltweise genau so wie* (signifies the same content as) $a_1 \cap \cdots \cap a_n \Rightarrow x_1 \cup \cdots \cup x_m$.

In particular, $a \vdash$ means that a contradiction follows from a, whereas $\vdash x$ means that x follows from nothing—i.e., it is (absolutely) true.

The asymmetry between the interpretation of the antecedent $a_1, \ldots, a_n$ and the succedent $x_1, \ldots, x_m$ may seem paradoxical at first. But it better serves Gentzen's purpose, and in any case there would seem little point (classically) in interpreting $a_1, \ldots, a_n \vdash x_1, \ldots, x_m$ as meaning that all of $x_1, \ldots, x_m$ follow from all of $a_1, \ldots, a_n$ since that can be expressed as m separate assertions: $a_1, \ldots, a_n \vdash x_j$ for $1 \leq j \leq m$.

One consequence of this interpretation is that $\vdash$ should no longer be transitive. I.e., we do not have

$$\frac{\vec{a} \vdash \vec{b} \quad \vec{b} \vdash \vec{c}}{\vec{a} \vdash \vec{c}}$$

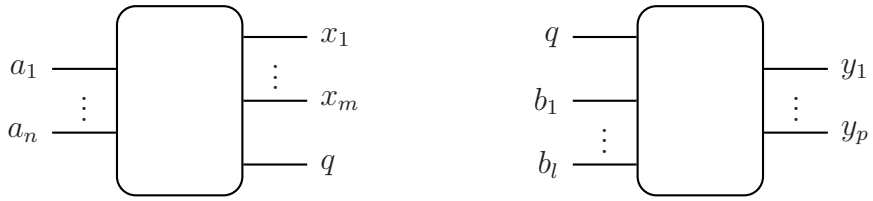
unless $\vec{b}$ is a string of length one.

Instead of transitivity, $\vdash$ is required to satisfy the rule

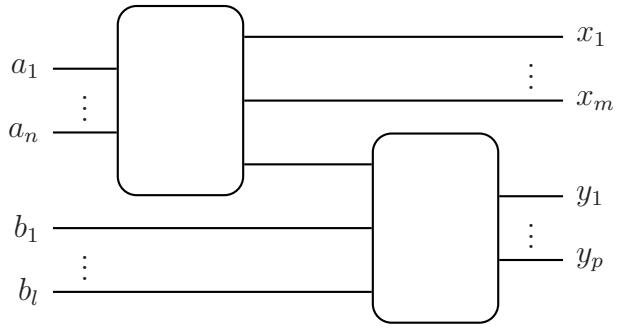
$$\frac{\vec{a} \vdash \vec{x}, q \quad q, \vec{b} \vdash \vec{y}}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}}$$

—which is called the *cut rule*.

The intuition behind the cut rule is best expressed in terms of logical circuits. The hypotheses, $\vec{a} \vdash \vec{x}, q$ and $q, \vec{b} \vdash \vec{y}$ are represented by boxes:



The conclusion $\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}$ is then the result of "plugging" the q output socket of the first box into the the q input socket of the second.



2.2 Structural Rules

There are eight classical structural rules:

$$\frac{}{a \vdash a}\ (\text{T}) \qquad \frac{\vec{a} \vdash \vec{x}, q \quad q, \vec{b} \vdash \vec{y}}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}}\ (\text{X})$$

$$\frac{\vec{a} \vdash \vec{x}}{\vec{a}, b \vdash \vec{x}}\ (\text{DA}) \qquad \frac{\vec{a}, b, b \vdash \vec{x}}{\vec{a}, b \vdash \vec{x}}\ (\text{CA}) \qquad \frac{\vec{a} \vdash \vec{x}}{\vec{a}_\sigma \vdash \vec{x}}\ (\text{PA})$$

$$\frac{\vec{a} \vdash \vec{x}}{\vec{a} \vdash \vec{x}, y}\ (\text{DS}) \qquad \frac{\vec{a} \vdash \vec{x}, y, y}{\vec{a} \vdash \vec{x}, y}\ (\text{CS}) \qquad \frac{\vec{a} \vdash \vec{x}}{\vec{a} \vdash \vec{x}_\tau}\ (\text{PS})$$

[Here $\vec{a}_\sigma$ and $\vec{x}_\tau$ are abbreviations for $\{a_{\sigma(1)}, \ldots, a_{\sigma(n)}\}$ and $\{x_{\tau(1)}, \ldots, x_{\tau(m)}\}$—where σ, τ are permutations on $\{1, \ldots, n\}$,$\{1, \ldots, m\}$ respectively.]

Two of these have already been discussed in the introduction: tautology (T) and cut (X).

The *dilution* rules, (DA) and (DS), simply express that one is allowed to have superfluous hypotheses, and dually, superfluous conclusions—given that we only expect one of those conclusions to hold. The dilution rules are also called *weakening* rules.

A distaste for conventional, or *material*, implication has motivated the definition of *relevance logic* which excludes dilution but retains the remaining structural rules[6, 7]. In this case, a sequent of the form $a_1, \ldots, a_n \vdash x$ should be interpreted as meaning that the premisses $a_1, \ldots, a_n$ suffice to conclude x, and that each premiss is relevant, in the sense that no proper subset of $\{a_1, \ldots, a_n\}$ suffices to conclude x. Similarly, a sequent of the form $a \vdash x_1, \ldots, x_m$ should also imply a minimality condition on the set $\{x_1, \ldots, x_m\}$—in model-theoretic terms, one might imagine this as meaning: for each k between 1 and m, there should exist a model of a which satisfies no element of $\{x_1, \ldots, x_m\}$ other than x_k.

The *contraction* rules, (CA) and (CS), and the *permutation* rules, (PA) and (PS), together serve to remove the "vectorness" from the strings of premisses and conclusions; i.e., they express the fact that $\{a_1, \ldots, a_n, b, b\} = \{a_1, \ldots, a_n, b\}$, and $\{a_1, \ldots, a_n\} = \{a_{\sigma(1)}, \ldots, a_{\sigma(n)}\}$, whenever σ is a permutation of the set $\{1, \ldots, n\}$.

If we omit the contraction rules, (CA) and (CS), we enter the realm of *linear* logic: here the elements of our formal language are no longer assumed to represent sentences but resources. In *affine* logic, which retains the dilution rules, a sequent $\vec{a} \vdash \vec{x}$ should be read as saying that the resources contained in the multi-set $\{a_1, \ldots, a_n\}$ suffice to produce at least one of the resources in $\{x_1, \ldots, x_m\}$. In linear logic proper, which dispenses with both the contraction and dilution rules, $\vec{a} \vdash \vec{x}$ must be interpreted with care: it implies that all the resources in $\{a_1, \ldots, a_n\}$ must be used, or *consumed*, but also that each non-empty sub-multi-set of $\{x_1, \ldots, x_m\}$ is a possible outcome. From a computation-theoretic point of view this may be explained in terms of non-determinism.

Further generalisations occur if we omit or restrict the permutation rules. For

example, [44] restricts the permutation rules to cyclic permutations. Fully non-commutative logic, which admits no permutation rules is sometimes known as *observational logic*—at least in the intuitionistic setting [4, 37]. Here the elements of our formal language are regarded as events, and a sequent $\vec{a} \vdash \vec{x}$ means that the occurrence of one or more of the events $x_1, \ldots, x_m$—in that order—follows from the occurrence of the events $a_1, \ldots, a_n$—in that order.

2.3 Logical Rules for Classical Logic

Definition 2.3.1

Let X denote a set of propositions. Then $\mathbb{L}_c[X]$ is defined to be the smallest set containing X and satisfying the following closure properties:

1. $\top, \bot \in \mathbb{L}_c[X]$,
2. if $x \in \mathbb{L}_c[X]$ then $\neg x \in \mathbb{L}_c[X]$, and
3. if $x, y \in \mathbb{L}_c$ then $(x \cap y), (x \cup y) \in \mathbb{L}_c$.

As common in classical logic we define $(x \Rightarrow y)$ as an abbreviation for $(\neg x \cup y)$.

A classical (propositional) theory consists of a set of propositions, X, together with a set of axioms, A, each of which is a $\mathbb{L}_c[X]$-sequent.

Gentzen defined the following sequent calculus for $\mathbb{L}_c[X]$.

Definition 2.3.2

[LK] consists of all eight structural rules listed in section 2.2 together with the following logical rules.

$$\frac{\vec{a}, c \vdash \vec{x}}{\vec{a}, (b \cap c) \vdash \vec{x}}\ (\text{MAL}) \qquad \frac{\vec{a} \vdash \vec{x}, y \quad \vec{a} \vdash \vec{x}, z}{\vec{a} \vdash \vec{x}, (y \cap z)}\ (\text{MS}) \qquad \frac{\vec{a}, b \vdash \vec{x}}{\vec{a}, (b \cap c) \vdash \vec{x}}\ (\text{MAR})$$

$$\frac{\vec{a} \vdash \vec{x}, z}{\vec{a} \vdash \vec{x}, (y \cup z)}\ (\text{JSL}) \qquad \frac{\vec{a}, b \vdash \vec{x} \quad \vec{a}, c \vdash \vec{x}}{\vec{a}, (b \cup c) \vdash \vec{x}}\ (\text{JA}) \qquad \frac{\vec{a} \vdash \vec{x}, y}{\vec{a} \vdash \vec{x}, (y \cup z)}\ (\text{JSR})$$

$$\frac{}{\bot \vdash \vec{x}}\ (\text{JUA}) \qquad \frac{\vec{a} \vdash \vec{x}}{\vec{a} \vdash \vec{x}, \bot}\ (\text{JUS}) \qquad \frac{\vec{a} \vdash \vec{x}}{\vec{a}, \top \vdash \vec{x}}\ (\text{MUA}) \qquad \frac{}{\vec{a} \vdash \top}\ (\text{MUS})$$

$$\frac{\vec{a} \vdash x, \vec{y}}{\vec{a}, \neg x \vdash \vec{y}}\ (\text{NA}) \qquad \frac{\vec{a}, x \vdash \vec{y}}{\vec{a} \vdash \neg x, \vec{y}}\ (\text{NS})$$

Definition 2.3.3

An [LK]-*derivation*, or [LK]-*proof*, is a finite tree whose nodes are instances of the above rules.

One says that a sequent *can be derived*, or *proven*, in the system [LK], if it appears at the root of an [LK]-derivation.

Note that it follows from this definition that the leaves of said tree must be instances of the tautology rule (T), (MUS), or (JUA).

Here are a pair of derivations showing that the usual distributive law

$$(p \cap (q \cup r)) \equiv ((p \cap q) \cup (p \cap r))$$

is provable in the system [LK]. [$x \equiv y$ means that both $x \vdash y$ and $y \vdash x$.]

Examples 2.3.4

1.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,(\mathrm{T})}{p, q \vdash p}\,(\mathrm{DA}) \quad \dfrac{\dfrac{}{q \vdash q}\,(\mathrm{T})}{q, p \vdash q}\,(\mathrm{DA})}{p, q \vdash p}\,(\mathrm{PA})}{p, q \vdash (p \cap q)}\,(\mathrm{MS})}{p, q \vdash ((p \cap q) \cup (p \cap r))}\,(\mathrm{JSR}) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,(\mathrm{T})}{p, r \vdash p}\,(\mathrm{DA}) \quad \dfrac{\dfrac{}{r \vdash r}\,(\mathrm{T})}{r, p \vdash r}\,(\mathrm{DA})}{p, r \vdash r}\,(\mathrm{PA})}{p, r \vdash (p \cap r)}\,(\mathrm{MS})}{p, r \vdash ((p \cap q) \cup (p \cap r))}\,(\mathrm{JSL})$$
$$\dfrac{\dfrac{\dfrac{\dfrac{p, (q \cup r) \vdash ((p \cap q) \cup (p \cap r))}{p, (p \cap (q \cup r)) \vdash ((p \cap q) \cup (p \cap r))}\,(\mathrm{MAR})}{(p \cap (q \cup r)), p \vdash ((p \cap q) \cup (p \cap r))}\,(\mathrm{PA})}{(p \cap (q \cup r)), (p \cap (q \cup r)) \vdash ((p \cap q) \cup (p \cap r))}\,(\mathrm{MAL})}{(p \cap (q \cup r)) \vdash ((p \cap q) \cup (p \cap r))}\,(\mathrm{CA})$$

2.
$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,(\mathrm{T})}{(p \cap q) \vdash p}\,(\mathrm{MAR}) \quad \dfrac{\dfrac{\dfrac{}{q \vdash q}\,(\mathrm{T})}{(p \cap q) \vdash q}\,(\mathrm{MAL}) \quad \dfrac{\dfrac{}{q \vdash q}\,(\mathrm{T})}{q \vdash (q \cup r)}\,(\mathrm{JSR})}{(p \cap q) \vdash (q \cup r)}\,(\mathrm{X})}{(p \cap q) \vdash (p \cap (q \cup r))}\,(\mathrm{MS}) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,(\mathrm{T})}{(p \cap r) \vdash p}\,(\mathrm{MAR}) \quad \dfrac{\dfrac{\dfrac{}{r \vdash r}\,(\mathrm{T})}{(p \cap r) \vdash r}\,(\mathrm{MAL}) \quad \dfrac{\dfrac{}{q \vdash q}\,(\mathrm{T})}{r \vdash (q \cup r)}\,(\mathrm{JSL})}{(p \cap r) \vdash (q \cup r)}\,(\mathrm{X})}{(p \cap r) \vdash (p \cap (q \cup r))}\,(\mathrm{MS})}{((p \cap q) \cup (p \cap r)) \vdash (p \cap (q \cup r))}\,(\mathrm{JA})$$

Remark 2.3.5

The first half of Example 2.3.4 contains special instances of two extremely important arguments, which we now record for future reference.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vec{a}, b, c \vdash x}{\vec{a}, b, (b \cap c) \vdash x}\,(\text{MAR})}{\vec{a}, (b \cap c), b \vdash x}\,(\text{PA})}{\vec{a}, (b \cap c), (b \cap c) \vdash x}\,(\text{MAL})}{\vec{a}, (b \cap c) \vdash x}\,(\text{CA}) \qquad \dfrac{\dfrac{\vec{a} \vdash x}{\vdots}\,(\text{DA}) \quad \dfrac{\dfrac{\vec{b} \vdash y}{\vdots}\,(\text{DA})}{\vec{b}, \vec{a} \vdash y}\,(\text{DA})}{\vec{a}, \vec{b} \vdash x \qquad \vec{a}, \vec{b} \vdash y}\,(\text{PA})}{\vec{a}, \vec{b} \vdash (x \cap y)}\,(\text{MS})$$

[We express the first of these by saying that $\cap$ *represents comma on the left.*]

The second half of Example 2.3.5 may also be proven using special instances of comma representation:

$$\dfrac{\dfrac{\dfrac{}{p \vdash p}\,(\text{T})}{(p \cap q) \vdash p}\,(\text{MAR}) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{q \vdash q}\,(\text{T})}{q \vdash (q \cup r)}\,(\text{JSR})}{p, q \vdash (q \cup r)}\,(\text{DA})}{\vdots}}{(p \cap q) \vdash (q \cup r)}\,(\text{MS})}{(p \cap q) \vdash (p \cap (q \cup r))} \qquad \dfrac{\dfrac{\dfrac{}{p \vdash p}\,(\text{T})}{(p \cap r) \vdash p}\,(\text{MAR}) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{r \vdash r}\,(\text{T})}{r \vdash (q \cup r)}\,(\text{JSL})}{p, r \vdash (q \cup r)}\,(\text{DA})}{\vdots}}{(p \cap r) \vdash (q \cup r)}\,(\text{MS})}{(p \cap r) \vdash (p \cap (q \cup r))}$$

$$\dfrac{(p \cap q) \vdash (p \cap (q \cup r)) \qquad (p \cap r) \vdash (p \cap (q \cup r))}{((p \cap q) \cup (p \cap r)) \vdash (p \cap (q \cup r))}\,(\text{JA})$$

Although longer, this proof has the technical advantage of avoiding the cut rule; the avoidance of (X) shall become a theme later on.

2.4 Logical Rules for Linear Logic

One corollary of abandoning the contraction and dilution rules is that the arguments of Remark 2.3.5 are no longer available. One solution, first developed by Girard, is to refashion them as logical rules with respect to a new binary connective called *multiplicative conjunction*, or simply *tensor*. Dually, one also introduces a *multiplicative disjunction*, or *par*, satisfying the dual logical rules.

The connectives satisfying the old logical rules are now called *additive* conjunction and disjunction.

Several different notational conventions for these four connectives—but, mercifully, only one standard terminology—exist. We summarise a few of them here:

Girard	Conj.	Disj.
Multiplicative	$\otimes$	$\parr$
Additive	$\&$	$\oplus$

Cockett/Seely	Conj.	Disj.
Multiplicative	$\otimes$	$\oplus$
Additive	$\times$	$+$

Me	Conj.	Disj.
Multiplicative	$\boxtimes$	$\bowtie$
Additive	$\pitchfork$	$\uplus$

In Words	Conj.	Disj.
Multiplicative	*tensor*	*par*
Additive	*with*	*plus*

Each of these connectives is required to have a unit. The notational situation for units is even more chaotic: it is said that every linear logician has their own convention for units, and we are no exception.

Units	Conj.	Disj.
Multiplicative	$\overset{\frown}{1}$	$\underset{\smile}{1}$
Additive	$\overset{\frown}{0}$	$\underset{\smile}{0}$

In Words	Conj.	Disj.
Multiplicative	*boot*	*crash*
Additive	*terminal*	*initial*

We follow Girard's convention that negation is not a connective.

Definition 2.4.1

Let P denote a set of propositions. Then $\mathbb{L}_l[P]$ is defined to be the smallest set containing P and $\{\mathsf{p}^\perp \mid \mathsf{p} \in P\}$, and and satisfying the following closure properties:

1. $\overset{\frown}{1}, \underset{\smile}{1}, \overset{\frown}{0}, \underset{\smile}{0} \in \mathbb{L}_l[P]$
2. if $x, y \in \mathbb{L}_l[P]$ then $(x \boxtimes y), (x \bowtie y), (x \pitchfork y), (x \uplus y) \in \mathbb{L}_l[P]$.

$$(x \boxtimes y), (x \bowtie y), (x \pitchfork y), (x \uplus y) \in \mathbb{L}_l$$

$(-)^\perp$ is extended to a function $\mathbb{L}_l[P] \longrightarrow \mathbb{L}_l[P]$ by structural induction:

$$(x \boxtimes y)^\perp = x^\perp \bowtie y^\perp \qquad (x \bowtie y)^\perp = x^\perp \boxtimes y^\perp$$

$$(x \pitchfork y)^\perp = x^\perp \uplus y^\perp \qquad (x \uplus y)^\perp = x^\perp \pitchfork y^\perp$$

$$\overset{\frown}{1}^\perp = \underset{\smile}{1} \qquad \underset{\smile}{1}^\perp = \overset{\frown}{1} \qquad \overset{\frown}{0}^\perp = \underset{\smile}{0} \qquad \underset{\smile}{0}^\perp = \overset{\frown}{0} \qquad (\mathsf{p}_n^\perp)^\perp = \mathsf{p}_n$$

and $(x \multimap y)$ is treated as an abbreviation for $(x^\perp \bowtie y)$.

Definition 2.4.2

[LL] consists of the structural rules (T), (X), (PA), and (PS), together with the following logical rules:

$$\frac{\vec{a}, b, c \vdash \vec{x}}{\vec{a}, (b \parr c) \vdash \vec{x}} \qquad \frac{\vec{a} \vdash \vec{x}, y \quad \vec{b} \vdash \vec{x}, z}{\vec{a}, \vec{b} \vdash \vec{x}, (y \otimes z)}$$

$$\frac{\vec{a} \vdash \vec{x}, y, z}{\vec{a} \vdash \vec{x}, (y \parr z)} \qquad \frac{\vec{a}, b \vdash \vec{x} \quad \vec{a}, c \vdash \vec{y}}{\vec{a}, (b \otimes c) \vdash \vec{x}, \vec{y}}$$

$$\frac{\vec{a}, c \vdash \vec{x}}{\vec{a}, (b \with c) \vdash \vec{x}} \qquad \frac{\vec{a} \vdash \vec{x}, y \quad \vec{a} \vdash \vec{x}, z}{\vec{a} \vdash \vec{x}, (y \with z)} \qquad \frac{\vec{a}, b \vdash \vec{x}}{\vec{a}, (b \with c) \vdash \vec{x}}$$

$$\frac{\vec{a} \vdash \vec{x}, z}{\vec{a} \vdash \vec{x}, (y \oplus z)} \qquad \frac{\vec{a}, b \vdash \vec{x} \quad \vec{a}, c \vdash \vec{x}}{\vec{a}, (b \oplus c) \vdash \vec{x}} \qquad \frac{\vec{a} \vdash \vec{x}, y}{\vec{a} \vdash \vec{x}, (y \oplus z)}$$

$$\frac{\vec{a} \vdash x, \vec{y}}{\vec{a}, x^{\perp} \vdash \vec{y}}\ (\text{NA}) \qquad \frac{\vec{a}, x \vdash \vec{y}}{\vec{a} \vdash x^{\perp}, \vec{y}}\ (\text{NS})$$

$$\frac{}{\vec{a}, \bigcirc \vdash \vec{x}} \qquad \frac{}{\bigcirc \vdash} \qquad \frac{\vec{a} \vdash \vec{x}}{\vec{a} \vdash \vec{x}, \bigcirc} \qquad \frac{\vec{a} \vdash \vec{x}}{\vec{a}, \top \vdash \vec{x}} \qquad \frac{}{\vdash \top} \qquad \frac{}{\vec{a} \vdash \bot, \vec{x}}$$

Definition 2.4.3

An [LL]-*derivation*, or [LL]-*proof*, is a finite tree whose nodes are instances of the above rules.

One says that a sequent *can be derived*, or *proven*, in the system [LL], if it appears at the root of an [LL]-derivation.

It is natural to ask which connectives distribute with each other. But a cursory examination of Example 2.3.4 reveals that structural rules were used only to establish the multiplicative properties of $\cap$. Thus, with very little effort, they may be turned into proofs that $\otimes$ distributes $\oplus$:

Examples 2.4.4

1.
$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\ (\text{T}) \quad \dfrac{}{q \vdash q}\ (\text{T})}{p, q \vdash (p \otimes q)}}{p, q \vdash ((p \otimes q) \oplus (p \otimes r))} \quad \dfrac{\dfrac{\dfrac{}{p \vdash p}\ (\text{T}) \quad \dfrac{}{r \vdash r}\ (\text{T})}{p, r \vdash (p \otimes r)}}{p, r \vdash ((p \otimes q) \oplus (p \otimes r))}\ (\text{JSL})}{p, (q \oplus r) \vdash ((p \otimes q) \oplus (p \otimes r))}}{(p \otimes (q \oplus r)) \vdash ((p \otimes q) \oplus (p \otimes r))}$$

2.
$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,(\mathrm{T}) \quad \dfrac{\dfrac{}{q \vdash q}\,(\mathrm{T})}{q \vdash (q \uplus r)}\,(\mathrm{JSR})}{p, q \vdash (p \boxtimes (q \uplus r))}}{(p \boxtimes q) \vdash (p \boxtimes (q \uplus r))} \quad \dfrac{\dfrac{\dfrac{}{p \vdash p}\,(\mathrm{T}) \quad \dfrac{\dfrac{}{r \vdash r}\,(\mathrm{T})}{r \vdash (q \uplus r)}\,(\mathrm{JSL})}{p, r \vdash (p \boxtimes (q \uplus r))}}{(p \boxtimes r) \vdash (p \boxtimes (q \uplus r))}}{((p \boxtimes q) \uplus (p \boxtimes r)) \vdash (p \boxtimes (q \uplus r))}$$

Dually, it is possible to show that $\bowtie$ distributes $\oplus$—i.e.,

$$(p \bowtie (q \oplus r)) \equiv ((p \bowtie q) \oplus (p \bowtie r))$$

2.5 Cut-Elimination

The cut rule (X) is the only rule in which terms appearing above the line are not necessarily sub-terms of those appearing below. But this strength is unfortunate from a computational point of view: how is a machine, asked to find a derivation of a given sequent, to "guess" how to apply (X)?

As Girard writes, there is a great irony in Gentzen's *Hauptsatz*. It proves that the only useful (*utile*) structural rule is actually superfluous (*inutile*).

Theorem 2.5.1

Any sequent which can be derived in [LK] (respectively, [LL]) can be derived without using the rule (X).

It suffices to prove the theorem for proofs which have only one instance of (X), and that appearing at the root.

$$\dfrac{\dfrac{T_1}{\vec{a} \vdash \vec{x}, q} \quad \dfrac{T_2}{q, \vec{b} \vdash \vec{y}}}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}}\,(\mathrm{X})$$

We assign two numbers to each such proof: the *degree*, which is defined to be the number of connectives in q, including units; and the *rank*, which is defined to be the number of appearances of q as a succedent in T_1 plus the number of appearances of q as a antecedent in T_2. [Briefly, the degree represents the logical complexity of q, and the rank is the number of relevant appearances of q in the rest of the proof.]

The argument proceeds by a double induction on degree and rank.

The key lemmas are: that one can reduce the degree of a cut whose rank is zero; that one can reduce the rank of a cut without increasing the degree, though possibly increasing the number of cuts; and, that cuts with rank zero and degree zero may be eliminated.

For example, if $q = (p \cap r)$, but the rank is zero—i.e., there are no appearances of q in T_1 or T_2, then our proof must be either

$$\dfrac{\dfrac{\dfrac{T_0}{\vec{a} \vdash \vec{x}, p} \quad \dfrac{T_1}{\vec{a} \vdash \vec{x}, r}\ (\text{MS}) \quad \dfrac{\dfrac{T_2}{r, \vec{b} \vdash \vec{y}}\ (\text{MAL})}{(p \cap r), \vec{b} \vdash \vec{y}}}{\vec{a} \vdash \vec{x}, (p \cap r)}\ (\text{X})}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}}$$

or

$$\dfrac{\dfrac{\dfrac{T_0}{\vec{a} \vdash \vec{x}, p} \quad \dfrac{T_1}{\vec{a} \vdash \vec{x}, r}\ (\text{MS}) \quad \dfrac{\dfrac{T_2}{p, \vec{b} \vdash \vec{y}}\ (\text{MAR})}{(p \cap r), \vec{b} \vdash \vec{y}}}{\vec{a} \vdash \vec{x}, (p \cap r)}\ (\text{X})}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}}\ .$$

In the former case, we substitute

$$\dfrac{\dfrac{T_1}{\vec{a} \vdash \vec{x}, r} \quad \dfrac{T_2}{r, \vec{b} \vdash \vec{y}}\ (\text{X})}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}}$$

and in the latter,

$$\dfrac{\dfrac{T_0}{\vec{a} \vdash \vec{x}, p} \quad \dfrac{T_2}{p, \vec{b} \vdash \vec{y}}\ (\text{X})}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}}$$

By contrast, faced with

$$\dfrac{\dfrac{T_0}{\vec{a} \vdash \vec{x}, q} \quad \dfrac{\dfrac{T_1}{q, \vec{b} \vdash \vec{y}, z} \quad \dfrac{T_2}{q, \vec{b} \vdash \vec{y}, t}\ (\text{MS})}{q, \vec{b} \vdash \vec{y}, (z \cap t)}\ (\text{X})}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}, (z \cap t)}$$

which is a proof of rank greater than or equal to two, we substitute

$$\dfrac{\dfrac{\dfrac{T_0}{\vec{a} \vdash \vec{x}, q} \quad \dfrac{T_1}{q, \vec{b} \vdash \vec{y}, z}}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}, z}\,(\textsc{x}) \qquad \dfrac{\dfrac{T_0}{\vec{a} \vdash \vec{x}, q} \quad \dfrac{T_2}{q, \vec{b} \vdash \vec{y}, t}}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}, t}\,(\textsc{x})}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}, (z \cap t)}\,(\textsc{ms})$$

which has two cuts of strictly smaller rank.

Thus it suffices to treat the sub-proofs

$$\dfrac{\dfrac{T_0}{\vec{a} \vdash \vec{x}, q} \quad \dfrac{T_1}{q, \vec{b} \vdash \vec{y}, z}}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}, z}\,(\textsc{x}) \qquad \text{and} \qquad \dfrac{\dfrac{T_0}{\vec{a} \vdash \vec{x}, q} \quad \dfrac{T_2}{q, \vec{b} \vdash \vec{y}, t}}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}, t}\,(\textsc{x})$$

separately.

Proofs with rank zero and degree zero include

$$\dfrac{\dfrac{}{q \vdash q}\,(\textsc{t}) \quad \dfrac{T_2}{q, \vec{b} \vdash \vec{y}}}{q, \vec{b} \vdash \vec{y}}\,(\textsc{x}) \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{T_0}{\vec{a} \vdash \vec{x}}}{\vec{a} \vdash \vec{x}, q}\,(\textsc{ds}) \quad \dfrac{T_2}{q, \vec{b} \vdash \vec{y}}}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}}\,(\textsc{x})$$

for which we may substitute

$$\dfrac{T_2}{q, \vec{b} \vdash \vec{y}} \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{T_0}{\vec{a} \vdash \vec{x}}}{\vdots}}{\vec{a}, \vec{b} \vdash \vec{x}, \vec{y}}$$

respectively.

Corollary 2.5.2

One cannot derive $\vdash$ (the sequent with no premisses and no conclusions) in either [LK] or [LL].

Classically, this means that [LK] is *consistent*—i.e., one can not derive a contradiction from nothing.

2.6 Denotational Semantics

We return, for a moment, to sequents of the form $a \vdash x$. That is, we return to the concept of $\vdash$ as a (reflexive and transitive) binary relation(—i.e., a *pre-ordering*—)on $\mathbb{L}(= \mathbb{L}_c$ or $\mathbb{L}_l$, as appropriate).

But any pre-ordered set $(x, \lesssim)$ can be turned into ordered set by taking $\approx$-equivalence classes, where $\alpha \approx \beta$ is defined to mean $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$. It is easy an easy exercise to show that

$$[\alpha] \leq [\beta] \iff \alpha \lesssim \beta$$

is a well-defined (reflexive, transitive, and anti-symmetric) binary relation(—i.e., an *ordering*—)on $x/\approx$.

As it happens, we have already defined $\equiv$ to mean $a \vdash x$ and $x \vdash a$. Thus $\mathbb{L}/\equiv$ is an ordered set. Now, in the case of [LK], $\mathbb{L}_c/\equiv$ is not merely an ordered set, but a boolean algebra. For example,

$$\frac{\dfrac{}{c \vdash c}\,(\mathrm{T})}{b \cap c \vdash c}\,(\mathrm{MAL}) \qquad \text{and} \qquad \frac{a \vdash b \quad a \vdash c}{a \vdash b \cap c}\,(\mathrm{MS}) \qquad \text{and} \qquad \frac{\dfrac{}{b \vdash b}\,(\mathrm{T})}{b \cap c \vdash b}\,(\mathrm{MAR})$$

demonstrate that

$$[b] \cap [c] := [b \cap c]$$

is the greatest lower bound of $[b]$ and $[c]$ in $\mathbb{L}_c/\equiv$.

$\mathbb{L}_c/\equiv$ is called a *Lindenbaum-Tarski algebra*.

Theorem 2.6.1

[LK] is complete with respect to classical logic. I.e., one can derive $a_1, \ldots, a_n \vdash x_1, \ldots, x_m$ if and only if $(a_1 \cap \cdots \cap a_n) \Rightarrow (x_1 \cup \cdots \cup x_m)$ is valid in all Boolean algebras.

Applying the same process to $\mathbb{L}_l$ one does not obtain a boolean algebra, but rather a $*$-autonomous poset with binary meet and join.

Theorem 2.6.2

[LL] is complete with respect to phase space semantics. I.e., one can derive $a_1, \ldots, a_n \vdash x_1, \ldots, x_m$ if and only if $(a_1 \cap \cdots \cap a_n) \Rightarrow (x_1 \cup \cdots \cup x_m)$ is valid in all Girard quantales.

Posetal semantics, such as that above, suffices to answer the mathematical question of whether a sequent is derivable, or not. What it fails to do is to distinguish between proofs.

One solution is *categorical semantics*. Here the idea is to define a category whose objects are elements of $\mathbb{L}$ and whose arrows $a \to x$ correspond to proofs of $a \vdash x$. Composition of arrows should be given by the cut rule, and identity arrows by (T).

The key problem here is deciding when two proofs should be considered the same. For if we take literal equality of trees, the result is not a category, because composition of arrows will not be associative, and the identity arrows will not satisfy the appropriate unit laws. I.e., if

$$\frac{T_0}{w \vdash x} \quad \text{and} \quad \frac{T_1}{x \vdash y} \quad \text{and} \quad \frac{T_2}{y \vdash z}$$

then the "composite proof" $(T_0 \circ T_1) \circ T_2$

$$\dfrac{\dfrac{\dfrac{T_0}{w \vdash x} \quad \dfrac{T_1}{x \vdash y}}{w \vdash y} \quad \dfrac{T_2}{y \vdash z}}{w \vdash z}$$

does not equal the composite proof $T_0 \circ (T_1 \circ T_2)$

$$\dfrac{\dfrac{T_0}{w \vdash x} \quad \dfrac{\dfrac{T_1}{x \vdash y} \quad \dfrac{T_2}{y \vdash z}}{x \vdash z}}{w \vdash z}$$

It is also desirable that the various binary connectives, $\cap$, etc., yield monoidal structures on our category. But it is not *a priori* clear that the canonical proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x \vdash x} \quad \dfrac{\dfrac{}{y \vdash y} \quad \dfrac{}{z \vdash z}}{y, z \vdash y \boxtimes z}}{x, y, z \vdash (x \boxtimes (y \boxtimes z))}}{(x \boxtimes y), z \vdash (x \boxtimes (y \boxtimes z))}}{((x \boxtimes y) \boxtimes z) \vdash (x \boxtimes (y \boxtimes z))}$$

should be invertible.

Theorem 2.6.3

There exists an equivalence relation on the set of all [LL]-proofs such that

1. every proof is equivalent to a cut-free proof
2. $\mathbb{L}_l$ becomes a $*$-autonomous category when equipped with equivalence classes of proofs of the form $a \vdash x$.

Chapter 3

Abstract homotopy theory

(Concrete) Homotopy theory is a branch of algebraic topology—the process of assigning algebraic invariants to topological spaces.

Following the introduction of the Eilenberg-Steenrod axioms for homology [42], there was a perceived need for an analogous axiomatisation of homotopy theory. In his daringly titled monograph, *Homotopical Algebra* [36], Quillen claimed to have achieved just that. But Quillen's axiomatic setting for homotopy was never acclaimed in quite the same way as its homological cousin; only recently has it come to be regarded as part of the canon.

This is surely because of its great generality: its axioms do not, at first, seem particularly topological or algebraic. Of course, this may also be regarded as a great strength. Just as once ridiculed notion of non-Hausdorff space has born fruit in (for example) algebraic geometry and information theory, the notion of Quillen model structure has been applied to fields only indirectly related to topology—see, for example, [32].

The great advantage of Quillen's axiomatisation, to us, is the fact that it takes as one of its primitives a class of maps, $\mathcal{W} \subseteq \mathsf{mor}[\mathcal{K}]$, to be formally inverted. In order to formally invert a completely arbitrary class of maps, it is necessary to stretch the definition of category to allow a proper class of arrows between two fixed objects. But this is not necessary in Quillen's setting: his remarkable first main theorem describes an equivalence between the result of formally inverting $\mathcal{W}$, and quotient of a full

subcategory of $\mathcal{K}$. Monoidal closed structures are also common in topology, so it is of little surprise that their interaction with Quillen model structures has also been intensively studied.

One disadvantage is that Quillen assumes the existence of arbitrary finite limits and colimits, whereas denotational models of linear logic typically possess only finite products and co-products. We have been able to surmount this difficulty by re-proving one key lemma (3.3.5) and adjusting a few definitions (most notably, 3.4.6).

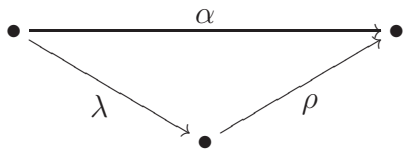
3.1 Factorisation schemes

We begin this chapter with an auxiliary definition. It is an important generalisation of the familiar notion of a factorisation system [10] on a category, but one which seems to have been largely ignored by the categorical mainstream.

Definition 3.1.1

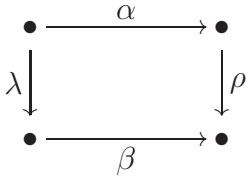
A *factorisation scheme* for a category $\mathcal{K}$ is an ordered pair of classes $\mathcal{L}, \mathcal{R} \subseteq \mathsf{mor}[\mathcal{K}]$ such that

1. (FACTORING) every arrow in $\mathcal{K}$ can be factored as an arrow in $\mathcal{L}$ followed by an arrow in $\mathcal{R}$—i.e., for every $\alpha \in \mathsf{mor}[\mathcal{K}]$, there exist $\lambda \in \mathcal{L}$ and $\rho \in \mathcal{R}$ such that

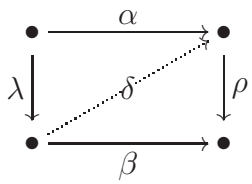


   commutes;

2. (LIFTING) for every commutative square



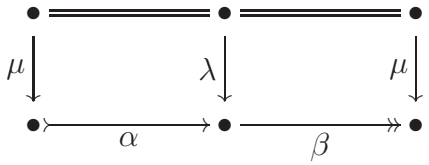
with $\lambda \in \mathcal{L}$ and $\rho \in \mathcal{R}$, there should exist a map δ



which makes both triangles commute; and

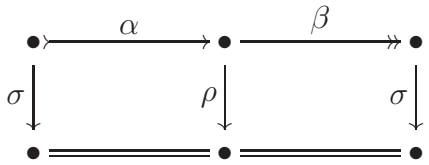
3. (CLOSURE) for every retract, , in $\mathcal{K}$:

   (a) if $\lambda \in \mathcal{L}$ and



   commutes, then $\mu \in \mathcal{L}$;

   (b) if $\rho \in \mathcal{R}$ and



   commutes, then $\sigma \in \mathcal{R}$.

We recall that a *factorisation system* is defined similarly, except that the map δ in axiom (LIFTING) is required to be unique, and that the axiom (CLOSURE) is only required to hold for isomorphisms.

It is traditional, with factorisation systems, to use the letters $\mathcal{E}$ (for epi-like) and $\mathcal{M}$ (for mono-like) in place of $\mathcal{L}$ and $\mathcal{R}$, but the following example shows why a less prejudicial notation is preferable for factorisation schemes.

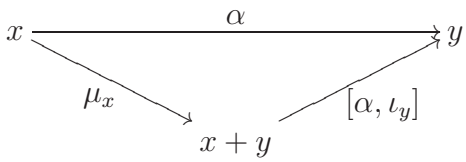
Theorem 3.1.2

Let $\mathcal{K}$ be any extensive category [13], let $\mathcal{J}$ denote the class of complemented maps, and let $\mathcal{S}$ denote the class of split epis in $\mathcal{K}$. Then $\mathcal{J}, \mathcal{S}$ form a factorisation scheme for $\mathcal{K}$.

Moreover, if $\mathcal{K}$ is a topos satisfying the axiom of choice [27], then both $(\mathcal{J}, \mathcal{S})$ and $(\mathcal{S}, \mathcal{J})$ form factorisation schemes for $\mathcal{K}$.

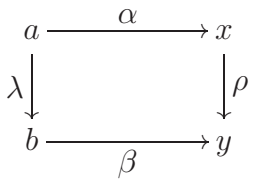
Proof

Observe that we can factor any map $x \xrightarrow{\alpha} y$ through its *co-graph*.



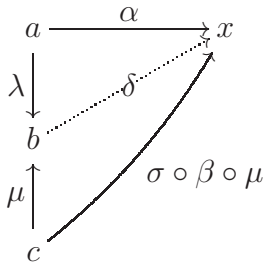
Clearly, $\mu_x \in \mathcal{J}$; and $[\alpha, \iota_y] \in \mathcal{S}$ because it is split by μ_y.

Moreover, given any commutative square

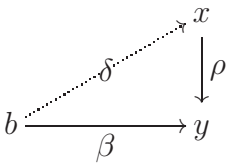


with $\lambda \in \mathcal{J}$ and $\rho \in \mathcal{S}$, let $c \xrightarrow{\mu} b$ be a complement of α and let $y \xrightarrow{\sigma} x$ be a split for ρ.

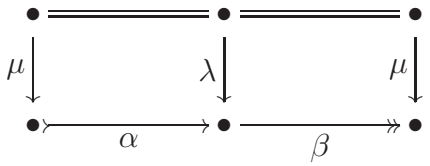
Then the existence part of the universal property of $a \xrightarrow{\lambda} b \xleftarrow{\mu} c$, we can define δ as follows:



and the uniqueness part allows us to conclude

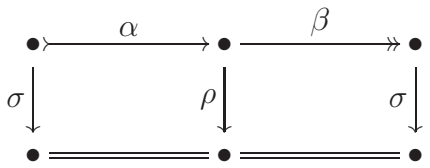


Finally, if



commutes, then the left hand square is a pullback; and, in an extensive category, the pullback of a complemented map is again a complemented map.

Similarly, if ρ is split by γ and



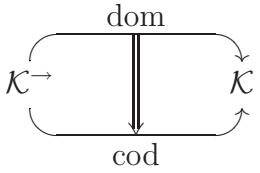
then σ is split by $\beta \circ \gamma$.

Now any topos is extensive, but in a topos satisfying choice, every epi is split, and a map is complemented if and only if it is a mono; epis and monos form a factorisation system in any topos [27]. Q.E.D.

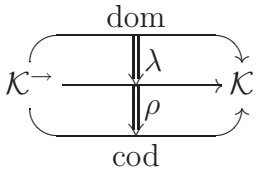
Note that, in this example, we were able to witness the axiom (LIFTING) in a natural way. This seems to be typical, and it is sometimes useful to have the extra strength of this condition. These observations motivate the following definition.

Definition 3.1.3

A *functorial factorisation scheme* is a factorisation scheme together with a factorisation of the "obvious" natural transformation



(where $\mathrm{dom}, \mathrm{cod}\colon \mathcal{K}^{\to} \to \mathcal{K}$ represent the domain and codomain functors, respectively) as



where all the components of λ, ρ lie in $\mathcal{L}, \mathcal{R}$ respectively.

Observe that a factorisation system can always be made functorial, if one assumes the axiom of choice for classes. This is because any two factorisations of a given arrow are necessarily isomorphic —but that there is no reason to believe the analogous statement for factorisation schemes.

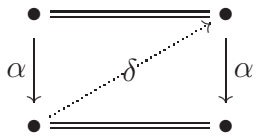
Given the extent to which factorisation schemes have been ignored in favour of factorisation systems, it is somewhat surprising to notice how many of the well-known results for factorisation systems generalise to factorisation schemes.

Lemma 3.1.4

Any arrow which belongs to both $\mathcal{L}$ and $\mathcal{R}$ is invertible.

Proof

Suppose $\alpha \in \mathcal{L} \cap \mathcal{R}$. Then we can find a $\delta \in \mathsf{mor}[\mathcal{K}]$ such that



Q.E.D.

Much more important is the following lemma. Indeed, this lemma represents the true purpose for taking (CLOSURE) as an axiom.

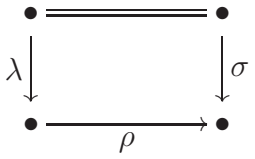
Lemma 3.1.5

If $(\mathcal{L}, \mathcal{R})$ is a factorisation scheme for $\mathcal{K}$, then $\mathcal{R}$ is maximal among those $\mathcal{S} \subseteq \mathsf{mor}[\mathcal{K}]$ such that $(\mathcal{L}, \mathcal{S})$ satisfies (LIFTING).

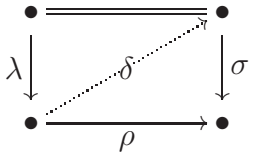
Dually, $\mathcal{L}$ is maximal among those $\mathcal{M} \subseteq \mathsf{mor}[\mathcal{K}]$ such that $(\mathcal{M}, \mathcal{R})$ satisfies (LIFTING).

Proof

Suppose $(\mathcal{L}, \mathcal{R})$ is a factorisation scheme for $\mathcal{K}$, and that $(\mathcal{L}, \mathcal{S})$ satisfies (LIFTING); let $\sigma \in \mathcal{S}$. According to (FACTORING) we can factor σ as a map $\lambda \in \mathcal{L}$ followed by a map $\rho \in \mathcal{R}$. Therefore we have a commutative square

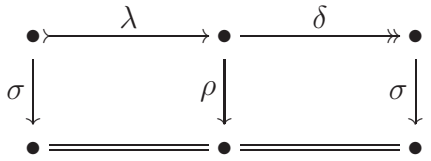


which can be lifted, by hypothesis, as follows:



hence we have a retract

, and



Therefore we have $\sigma \in \mathcal{R}$, as desired. Q.E.D.

This is a desirable property for factorisation schemes to possess, since otherwise it would be possible to have distinct factorisation schemes which were not *appreciably* distinct. For example, given $\mathcal{L} = \mathsf{mor}[\mathcal{K}]$ one might take $\mathcal{R}$ to be the set of all invertible maps in $\mathcal{K}$ or just the class of identity arrows.

Theorem 3.1.6

If $(\mathcal{L}, \mathcal{R})$ is a factorisation scheme for $\mathcal{K}$, then

1. $\mathcal{L} \cap \mathcal{R}$ equals the class $\mathcal{I}$ of isomorphisms in $\mathcal{K}$;

2. $\mathcal{L}$ and $\mathcal{R}$ are both subcategories of $\mathcal{K}$;

3. $\mathcal{L}$ and $\mathcal{R}$ are both closed under all retracts in $\mathcal{K}^{\to}$;

4. $\mathcal{L}$ is closed under pushouts, and $\mathcal{R}$ is closed under pullbacks—insofar as pushouts and pullbacks exist in $\mathcal{K}$;

5. $\mathcal{L}$ is closed under filtered colimits, and $\mathcal{R}$ is closed under cofiltered limits; and

6. if $(\mathcal{M}, \mathcal{S})$ is another weak factorisation systems for $\mathcal{K}$, then $\mathcal{L} \subseteq \mathcal{M}$ if and only if $\mathcal{S} \subseteq \mathcal{R}$.

Proof

All of these follow easily from lemma 3.1.5—we shall prove only case 4, as a demonstration.

Let $\mathcal{M}$ be the class of pushouts of elements of $\mathcal{L}$. We aim to show that $(\mathcal{M}, \mathcal{R})$ satisfies (LIFTING).

Suppose

$$\begin{array}{ccc} q & \xrightarrow{\omega} & a \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\mu} \\ r & \xrightarrow[\psi]{} & b \end{array}$$

is a pushout square with $\lambda \in \mathcal{L}$ and that

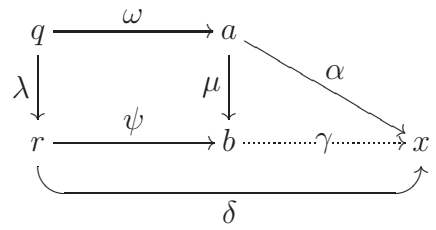
$$\begin{array}{ccc} a & \xrightarrow{\alpha} & x \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\rho} \\ b & \xrightarrow[\beta]{} & y \end{array}$$

commutes, with $\rho \in \mathcal{R}$.

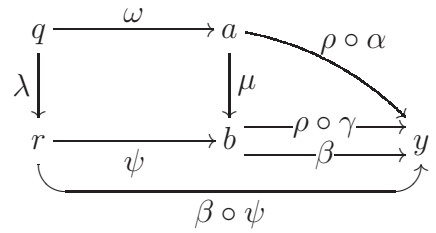
Then, we can find a lift for the square

$$\begin{array}{ccc} q & \xrightarrow{\alpha \circ \omega} & x \\ {\scriptstyle\lambda}\downarrow & \overset{\delta}{\nearrow} & \downarrow{\scriptstyle\rho} \\ r & \xrightarrow[\beta \circ \psi]{} & y \end{array}$$

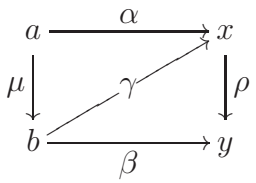
But the universal property of the pushout square implies the existence of a map



as well as the equality of the parallel maps in



Hence



as desired.

Hence $\mathcal{M} \subseteq \mathcal{L}$—i.e., $\mathcal{L}$ is closed on under pushouts. Q.E.D.

We end this section with a special case of the so-called *small-object argument*.

Theorem 3.1.7

If $\mathcal{K}$ is cocomplete and $\mathcal{G}$ is a set of arrows with finitely presented [5] domain, then there exists a minimal/canonical functorial factorisation scheme $(\mathcal{L}, \mathcal{R})$ with $\mathcal{G} \subseteq \mathcal{L}$.

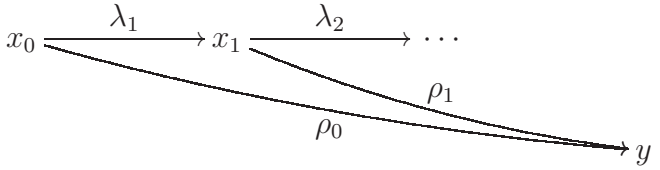
Proof

We define $\mathcal{R}$ to be the class of arrows satisfying (LIFTING) with respect to $\mathcal{G}$, and $\mathcal{L}$ to be the class of arrows satisfying (LIFTING) with respect to $\mathcal{R}$. Then $\mathcal{L}, \mathcal{R}$ satisfy (LIFTING) and (CLOSURE), and $\mathcal{G} \subseteq \mathcal{L}$. So it just remains to show (FACTORING).

Given an arrow $x \xrightarrow{\psi} y$ in $\mathcal{K}$, we define a sequence

$$x_0 \xrightarrow{\lambda_1} x_1 \xrightarrow{\lambda_2} \cdots$$

and a co-cone



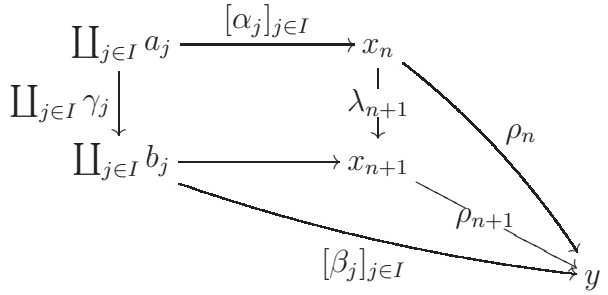
as follows: $x_0 = x$ and $\rho_0 = \psi$; if x_n, ρ_n have been defined, let $I(= I_n)$ equal the set of all diagrams

$$\begin{array}{ccc} a & \xrightarrow{\alpha} & x_n \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\rho_n} \\ b & \xrightarrow[\beta]{} & y \end{array}$$

with $\gamma \in \mathcal{G}$. We then define x_{n+1} to be the pushout of

$$\begin{array}{ccc} & \coprod_{j\in I} a_j & \xrightarrow{[\alpha_j]_{j\in I}} x_n \\ \coprod_{j\in I}\gamma_j & \downarrow & \\ & \coprod_{j\in I} b_j & \end{array}$$

and $\lambda_{n+1}, \rho_{n+1}$ to be the maps

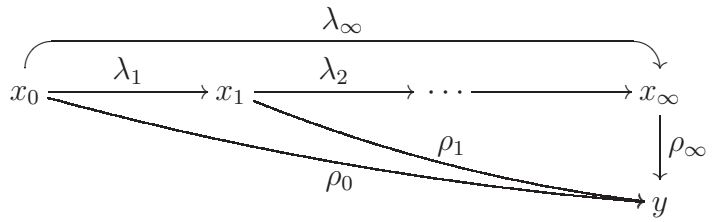


Note that all the $\lambda_n \in \mathcal{L}$ for every $n > 0$, by construction.

Unsurprisingly, we now define x_∞ to be the colimit of the sequence

$$x_0 \xrightarrow{\lambda_1} x_1 \xrightarrow{\lambda_2} \cdots$$

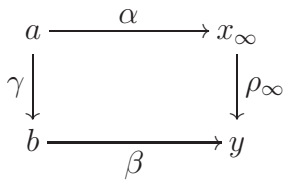
and $\lambda_\infty, \rho_\infty$ to be



[λ_∞, which is actually a coprojection, is commonly referred to, in topological literature, as the *(transfinite) composite* of the sequence $x_0 \xrightarrow{\lambda_1} x_1 \xrightarrow{\lambda_2} \cdots$.]

We claim that $\lambda_\infty \in \mathcal{L}$ and $\rho_\infty \in \mathcal{R}$. But $\mathcal{G} \subseteq \mathcal{L}$ and λ_∞ has been built up from elements of $\mathcal{G}$ using operations under which λ_∞ is known to be closed (Theorem 3.1.6): pushouts, etc. So we can be sure that $\lambda_\infty \in \mathcal{L}$.

As for ρ_∞—to show it is in $\mathcal{R}$, let $a \xrightarrow{\gamma} b$ be an arbitrary element of $\mathcal{G}$, and let

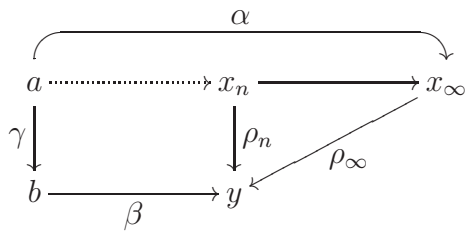


be a commutative square.

Since a is finitely presented and x_∞ is the (filtered) colimit of

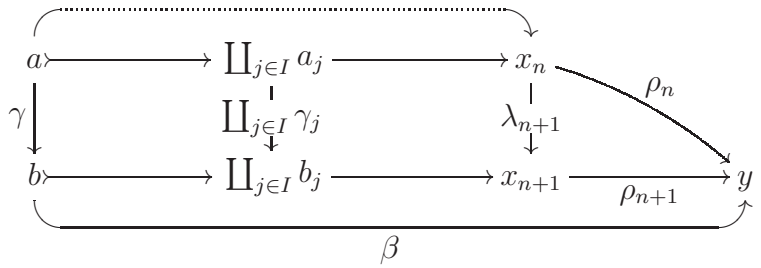
$$x_0 \xrightarrow{\lambda_1} x_1 \xrightarrow{\lambda_2} \cdots$$

we can be sure that α factors through one of the x_n's ($n < \infty$). I.e., we have

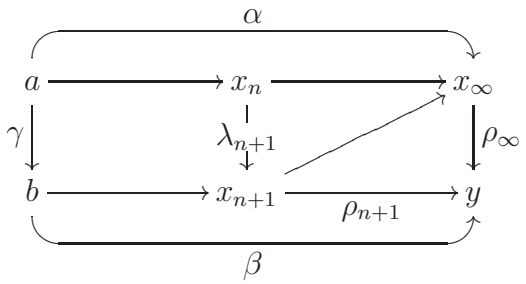


Now the left-hand square is an element of I_n.

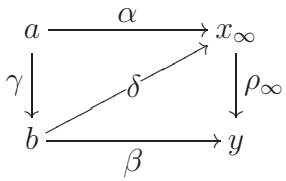
By the definition of x_{n+1}, λ_{n+1} and ρ_{n+1}, we obtain



The net result of which is



So by choosing, δ to be the composite $b \longrightarrow x_{n+1} \longrightarrow x_\infty$ we obtain



as desired. Q.E.D.

Examining the proof of 3.1.7, we are led to the following definition.

Definition 3.1.8

Given a set (or class) of arrows $\mathcal{G}$ in $\mathcal{K}$, the closure of $\mathcal{G}$ under coproducts, pushouts and 'transfinite composition', will be called $\tilde{\mathcal{G}}$. Elements of $\tilde{\mathcal{G}}$ are called $\mathcal{G}$-cells.

What the proof of 3.1.7 shows is that every arrow in $\mathcal{K}$ can be factored as a $\mathcal{G}$-cell followed by an element of $\mathcal{R}$. It follows from the proof of 3.1.5 that every element of $\mathcal{L}$ is the retract of a $\mathcal{G}$-cell.

Now, it should be clear how this theorem can be generalised. Firstly, it suffices that the domains of the arrows in $\mathcal{G}$ be κ-presentable for some regular cardinal κ. More interestingly, it suffices that the domains of the arrows in $\mathcal{G}$ be κ-*generated* with respect to the arrows in $\tilde{\mathcal{G}}$.

This might seem a little circular, but can be quite effective in practice.

Example 3.1.9

Let **Hdf** denote the category of Hausdorff topological spaces[1] and continuous maps, and $\mathcal{G}$ the set of inclusions $D_n \times \{0\} \rightarrowtail D_n \times [0,1]$. [where $D_n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.] We claim that $\mathcal{L}, \mathcal{R}$ defined as in 3.1.7 form a factorisation scheme. Elements of $\mathcal{R}$ are traditionally called *Serre fibrations*.

Proof

Only finite spaces are finitely presented in **Hdf** [5], but compact spaces are finitely generated with respect to the class of closed inclusions [26]. All the D_n's are compact.

So to show that $\mathcal{L}, \mathcal{R}$ form a factorisation scheme for **Hdf**, it suffices to show $\tilde{\mathcal{G}} \subseteq \{\text{closed inclusions}\}$. Now clearly $\mathcal{G} \subseteq \{\text{closed inclusions}\}$, so it suffices to show that the class of closed inclusions is closed under pushout and 'transfinite composition'. This is done in [26]. Q.E.D.

Incidentally, closed inclusions are precisely the regular monos in **Hdf**. Predictably, they form a factorisation system together with the class of epis in **Hdf**, which are precisely the dense maps—i.e., maps with dense range.

3.2 Quillen model categories

Armed with our auxiliary notion of factorisation scheme, we are now well-placed to give a shorter-than-usual definition of Quillen model category.

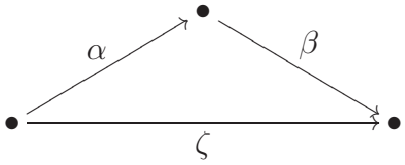
[1] The Hausdorff condition can be avoided here—and in Example 3.2.3—but only at the cost of introducing the notion of a T_1-inclusion.

Definition 3.2.1

A *Quillen model category* is a finitely complete and cocomplete category $\mathcal{K}$ together with a choice of three subclasses of $\mathsf{mor}[\mathcal{K}]$—$\mathcal{C}$, whose elements are called *cofibrations*; $\mathcal{W}$, whose elements are called *weak equivalences*; and $\mathcal{F}$, whose elements are called *fibrations*—which satisfy the following axioms.

1. Both $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ form factorisation schemes.

2. $\mathcal{W}$ has the *2-out-of-3* property; that is, if any two of the maps in a commutative triangle



are weak equivalences, then so is the third.

The elements of $\mathcal{W}$ are also called *acyclic*; thus in particular, elements of $\mathcal{C} \cap \mathcal{W}$ and $\mathcal{W} \cap \mathcal{F}$ will be also called *acyclic cofibrations* and *acyclic fibrations* respectively.

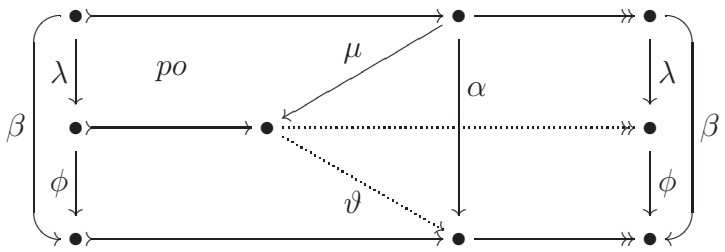
To show that our definition agrees with that of [36], it is necessary to prove the following:

Lemma 3.2.2

Under the assumptions given, $\mathcal{W}$ is closed under all retracts in $\mathcal{K}^{\to}$.

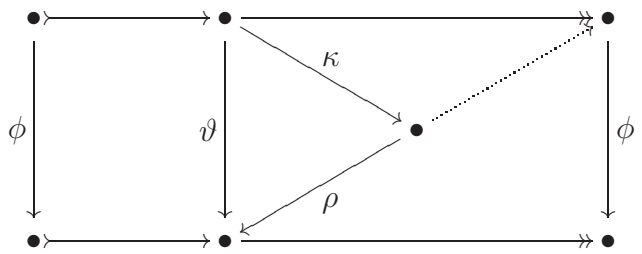
Proof

Let α be a weak equivalence, and β a retract of α. Then we can factor β as an acyclic cofibration, λ, followed by a fibration, ϕ, and form the following pushout.



Observe that μ is an acyclic cofibration, since λ is. Hence ϑ is a weak equivalence, by the 2-out-of-3 axiom. But ϕ is a retract of ϑ. So it suffices to show that ϕ, which is both a fibration and a retract of ϑ, is acyclic.

But ϑ can be factored as an acyclic cofibration, κ, followed by an acyclic fibration, ρ. We can then apply the lifting property, as follows:



—which shows that ϕ is also a retract of ρ, and therefore also a acyclic fibration.

Q.E.D.

Example 3.2.3

We recall that an arrow $X \xrightarrow{f} Y$ in **Hdf** is called a *weak homotopy equivalence* if "every $\pi_n(f)$ is invertible"—meaning that: $\pi_0(X) \xrightarrow{\pi_0(f)} \pi_0(Y)$ is a bijection; and, for every $x \in X$ and every $n > 0$, $\pi_n(X, x) \xrightarrow{\pi_n(f)} \pi_n(Y, f(x))$ is an isomorphism.

Then **Hdf** can be made into a Quillen model category by choosing $\mathcal{F}$ to be the class of Serre fibrations (as defined in 3.1.9), $\mathcal{W}$ to be the class of weak homotopy equivalences, and $\mathcal{C}$ to be the class of maps which satisfy (LIFTING) with respect to $\mathcal{W} \cap \mathcal{F}$ [26].

Many authors, e.g. Hovey, require a Quillen model category to have arbitrary limits and colimits, not just finite ones. But the categories with which linear logicians are wont to work often fail to have equalisers and coequalisers. We are therefore led in the opposite direction: requiring fewer limits and colimits, instead of more.

Definition 3.2.4

An *almost-Quillen model category* is a category with finite products and coproducts together with a choice of three classes $\mathcal{C}, \mathcal{W}, \mathcal{F} \subseteq \mathsf{mor}[\mathcal{K}]$ satisfying the same conditions as above.

Note that it is no longer the case that a retract of an arbitrary weak equivalence is again a weak equivalence. Hence our definition of almost-Quillen model category does not imply that $\mathcal{C}, \mathcal{W}, \mathcal{F}$ form a Quillen model structure in the sense of [26].

But examining the proof of lemma 3.2.2, one observes that the 'second half' of the argument holds—i.e., the part which shows that a fibration which is a retract of a weak equivalence is acyclic. It turns out that this, and the dual statement that any cofibration which is a retract of a weak equivalence is acyclic, will suffice to prove the theorems we need.

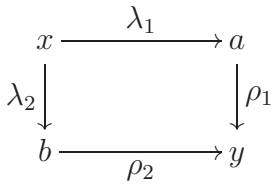
One lemma we shall need in the next section is the following.

Lemma 3.2.5

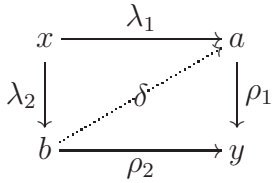
In a Quillen model category (or almost-Quillen model category), factorisations are 'unique up to weak equivalence'.

Proof

Suppose we have two factorisations of $x \xrightarrow{\psi} y$ as a cofibration followed by an acyclic fibration. Then we get a commutative square



which can be lifted



But since both ρ_1, ρ_2 are weak equivalences, we can conclude that δ is too.

Dually, if we have two factorisations of ψ as an acyclic fibration followed by a fibration, then they are also 'the same up to weak equivalence'. Q.E.D.

This is a nice result, because weak equivalences are to be thought of as maps that 'only just' fail to be isomorphisms. Indeed, the main objective of this chapter will be to describe the result of formally inverting them.

Definition 3.2.6

Let $\mathcal{K}$ be a category and $\mathcal{W}$ a subcategory of $\mathcal{K}$. A functor $F : \mathcal{K} \to \mathcal{A}$ is said to *invert* $\mathcal{W}$ if it maps every element of $\mathcal{W}$ to an isomorphism in $\mathcal{A}$. If $F : \mathcal{K} \to \mathcal{A}$ is universal among functors that invert $\mathcal{W}$—i.e., if for every other such functor $G : \mathcal{K} \to \mathcal{B}$ there exists a unique functor $H : \mathcal{A} \to \mathcal{B}$ such that $G = H \circ F$—then $\mathcal{A}$ is called *a category of fractions*.

Theorem 3.2.7

It is always possible to construct a category of fractions, so long as one is willing to countenance locally-large categories.

Proof

Given $\mathcal{K}$ and $\mathcal{W}$ as above, we define a category $\mathcal{K}[\mathcal{W}^{-1}]$ as follows: $\mathsf{obj}[\mathcal{K}[\mathcal{W}^{-1}]] = \mathsf{obj}[\mathcal{K}]$ but $\mathsf{mor}[\mathcal{K}[\mathcal{W}^{-1}]]$ is defined to be the class of 'zig-zags'

$$x_0 \xleftarrow{\omega_1} r_1 \xrightarrow{\kappa_1} \cdots \xleftarrow{\omega_n} r_n \xrightarrow{\kappa_n} x_n$$

with $\omega_1, \ldots, \omega_n \in \mathcal{W}$ and $\kappa_1, \ldots, \kappa_n \in \mathsf{mor}[\mathcal{K}]$, modulo a simple equivalence relation [18].

The domain of

$$x_0 \xleftarrow{\omega_1} r_1 \xrightarrow{\kappa_1} \cdots \xleftarrow{\omega_n} r_n \xrightarrow{\kappa_n} x_n$$

is defined to be x_0 and its codomain is defined to be x_n. Composition is defined by juxtaposition, and identities are given by zig-zags of length 0.

The functor $F : \mathcal{K} \to \mathcal{K}[\mathcal{W}^{-1}]$ is the identity on objects and maps each arrow $x \xrightarrow{\alpha} y$ to the zig-zag $x \xleftarrow{\iota x} x \xrightarrow{\alpha} y$. If $G : \mathcal{K} \to \mathcal{B}$ inverts $\mathcal{W}$, then we define $H : \mathcal{K}[\mathcal{W}^{-1}] \to \mathcal{B}$ as follows: H agrees with G on objects, and H maps

$$x_0 \xleftarrow{\omega_1} r_1 \xrightarrow{\kappa_1} \cdots \xleftarrow{\omega_n} r_n \xrightarrow{\kappa_n} x_n$$

to the composite of

$$G(x_0) \xrightarrow{G(\omega_1)^{-1}} G(r_1) \xrightarrow{G(\kappa_1)} \cdots \xrightarrow{G(\omega_n)^{-1}} G(r_n) \xrightarrow{G(\kappa_n)} G(x_n)$$

Q.E.D.

Definition 3.2.8

The *homotopy category* of a Quillen model category $\mathcal{K}$ is defined to be the category of fractions $\mathcal{K}[\mathcal{W}^{-1}]$, which we also write as $\mathsf{Ho}[\mathcal{K}]$.

3.3 The homotopy relation

Throughout this section we shall assume $\mathcal{K}$ to be an almost-Quillen model category, and we will show how it is possible to describe its homotopy category $\mathsf{Ho}[\mathcal{K}] = \mathcal{K}[\mathcal{W}^{-1}]$ as a quotient of a full subcategory of $\mathcal{K}$.

First, we define the subcategory in question.

Definition 3.3.1

An object x of $\mathcal{K}$ is said to be *fibrant* if the unique map $x \longrightarrow 1$ is a fibration. A *fibrant replacement* of an arbitrary $x \in \mathsf{obj}[\mathcal{K}]$ is a fibrant object r together with a weak equivalence $x \longrightarrow r$.

Dually, an object y of $\mathcal{K}$ is said to be *cofibrant* if the unique map $0 \longrightarrow y$ is a cofibration. A *cofibrant replacement* of an arbitrary $y \in \mathsf{obj}[\mathcal{K}]$ is a cofibrant object q together with a weak equivalence $q \longrightarrow y$.

We write $\mathcal{K}_{cf}$ for the full subcategory of fibrant and cofibrant objects.

Theorem 3.3.2

The obvious inclusion of $\mathcal{K}_{cf}[\mathcal{W}^{-1}]$ into $\mathcal{K}[\mathcal{W}^{-1}]$ is an equivalence.

Proof

Given any object x of $\mathcal{K}[\mathcal{W}^{-1}]$ we must show there exists an object y of $\mathcal{K}_{cf}[\mathcal{W}^{-1}]$ and an isomorphism from x to y in $\mathcal{K}[\mathcal{W}^{-1}]$.

First, we find a cofibrant replacement of x by factoring the unique map $0 \longrightarrow x$ as a cofibration followed by an acyclic fibration.

$$0 \longrightarrow q \xrightarrow{\rho} x$$

Next, we find a fibrant replacement of q, by factorising the unique map $q \longrightarrow 1$ as an acyclic cofibration followed by a fibration.

$$q \xrightarrow{\lambda} y \longrightarrow 1$$

Then y is both fibrant and cofibrant (since cofibrations are closed under composition) and the span $y \xleftarrow{\lambda} q \xrightarrow{\rho} x$ constitutes an isomorphism in $\mathcal{K}[\mathcal{W}^{-1}]$, since $\lambda, \rho \in \mathcal{W}$. Q.E.D.

Note that, if $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are functorial factorisation schemes, then there exist canonical fibrant and cofibrant replacements for every object x, which we denote Rx and Qx, respectively.

We should now specify a congruence on $\mathcal{K}_{cf}$, but for generality's sake we instead define a pair of relations on the homsets of $\mathcal{K}$. These will turn out to coincide on $\mathcal{K}_{cf}$.

Definitions 3.3.3

Let x and y be objects of $\mathcal{K}$; and let α and β be parallel arrows $x \longrightarrow y$.

1. A *cylindrification* of x is any factorisation of the co-diagonal map $x + x \longrightarrow x$ as a cofibration followed by a weak equivalence. If $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ is a functorial factorisation scheme, then we have a canonical cylindrification for every object x, which we denote:

   $$x + x \xrightarrow{\mu} \mathrm{c}(x) \xrightarrow{\sigma} x$$

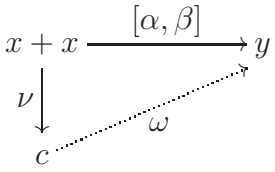
   with $\sigma \in \mathcal{W} \cap \mathcal{F}$. We shall write μ_0, μ_1 for the two co-components of μ.

2. A *pathification* of y is any factorisation of the diagonal map $y \longrightarrow y \times y$ as a weak equivalence followed by a fibration. Again, if $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a functorial, then we have a canonical pathification for every object y, which we denote:

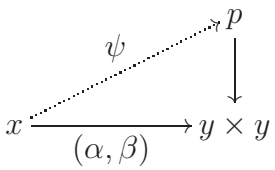
   $$y \xrightarrow{\kappa} \mathrm{p}(y) \xrightarrow{\varpi} y \times y$$

   with $\kappa \in \mathcal{C} \cap \mathcal{W}$. We write ϖ_0, ϖ_1 for the two components of ϖ.

3. α and β are said to be *left-homotopic*, written $\alpha \overset{\ell}{\sim} \beta$, if there exists a cylindrification $x + x \xrightarrow{\nu} c \xrightarrow{\zeta} x$ of x and a map ω satisfying



4. α and β are said to be *right-homotopic*, written $\alpha \overset{r}{\sim} \beta$, if there exists a pathification $y \longrightarrow p \longrightarrow y \times y$ of y and a map ψ satisfying



From now on, we assume both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ to be functorial.

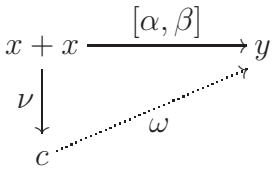
Lemma 3.3.4

Let α and β be parallel arrows $x \longrightarrow y$. If y is fibrant and $\alpha \overset{\ell}{\sim} \beta$ holds, then we can witness the latter statement with the canonical cylindrification. Dually, if x is cofibrant, then $\alpha \overset{r}{\sim} \beta$ (if it holds) can always be witnessed using the canonical pathification.

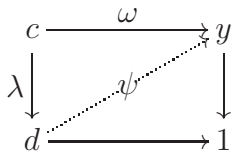
Proof

Suppose $x + x \xrightarrow{\nu} c \xrightarrow{\zeta} x$ is a cylinder object. Then we can factor ζ as a cofibration followed by an acyclic fibration: $c \xrightarrow{\lambda} d \xrightarrow{\rho} x$. By two-out-of-three, λ is also acyclic.

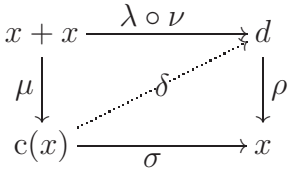
So if we have



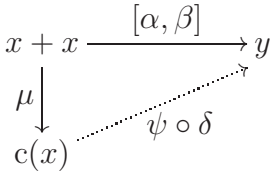
then we can extend ω to a map



By the uniqueness of factorisations up to weak equivalence (Lemma 3.2.5), we obtain a map $\mathrm{c}(x) \xrightarrow{\delta} d$ such that



So, in particular, we have



Q.E.D.

Note that, despite the notation, neither $\overset{\ell}{\sim}$ nor $\overset{r}{\sim}$ are, in general, equivalence relations; They are reflexive and symmetric, but not transitive in general.

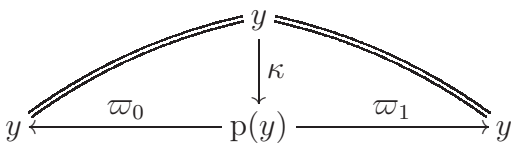
The usual proof of transitivity uses pullbacks and/or pushouts, which is unacceptable in our setting. But the usual proof that $\overset{\ell}{\sim}$ and $\overset{r}{\sim}$ coincide on $\mathcal{K}_{cf}$ can be adapted as follows.

Lemma 3.3.5

Let α, β and γ be parallel arrows $x \longrightarrow y$, and suppose $\alpha \overset{\ell}{\sim} \beta \overset{r}{\sim} \gamma$. Then y fibrant implies $\alpha \overset{\ell}{\sim} \gamma$; dually, x cofibrant implies $\alpha \overset{r}{\sim} \gamma$.

Proof

Firstly note that both ϖ_0 and ϖ_1 are weak equivalences, by the 2-out-of-3 axiom, since we have commutative triangles



and κ is a weak equivalence.

Moreover,

$$\begin{array}{ccc} y \times y & \xrightarrow{\pi_1} & y \\ {\scriptstyle \pi_0}\downarrow & & \downarrow \\ y & \longrightarrow & 1 \end{array}$$

is a pullback square, so if y is fibrant, then both π_0 and π_1 are fibrations. Hence $\varpi_0 = \pi_0 \circ \varpi$ and $\varpi_1 = \pi_1 \circ \varpi$ are also fibrations.

Now suppose ω and ψ witness our hypothesis, as follows:

$$\begin{array}{ccc} x + x & \xrightarrow{[\alpha,\beta]} & y \\ {\scriptstyle \mu}\downarrow & \overset{\omega}{\nearrow} & \\ \mathrm{c}(x) & & \end{array} \qquad \begin{array}{ccc} & & \mathrm{p}(y) \\ & \overset{\psi}{\nearrow} & \downarrow{\scriptstyle \varpi} \\ x & \xrightarrow[(\beta,\gamma)]{} & y \times y \end{array}$$

Then we can factor $[\alpha, \gamma]$ through ϖ_1 as follows:

$$\begin{aligned} [\alpha, \gamma] &= [(\varpi_1\kappa)\alpha, \varpi_1\psi] \\ &= \varpi_1[\kappa\alpha, \psi] \end{aligned}$$

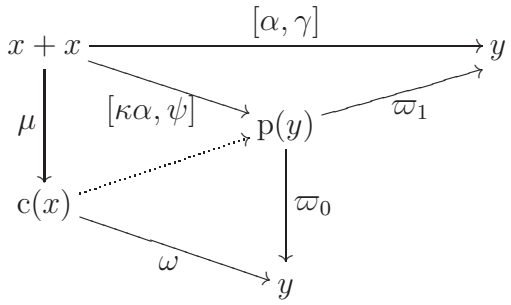
—i.e.,

$$\begin{array}{ccccc} x + x & & \xrightarrow{[\alpha,\gamma]} & & y \\ & {\scriptstyle [\kappa\alpha,\psi]}\searrow & & \nearrow{\scriptstyle \varpi_1} & \\ & & \mathrm{p}(y) & & \end{array}$$

Moreover,

$$\begin{aligned} \varpi_0[\kappa\alpha, \psi] &= [\varpi_0\kappa\alpha, \varpi_0\psi] \\ &= [\alpha, \beta] \\ &= \omega\mu \end{aligned}$$

Hence (ϖ_0 being an acyclic fibration) we can find a lift



which exhibits $\alpha \overset{\ell}{\sim} \gamma$. Q.E.D.

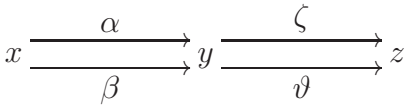
A quick corollary of the last lemma is that $\overset{\ell}{\sim}$ and $\overset{r}{\sim}$ coincide on $\mathcal{K}_{cf}$. We therefore write just $\sim$.

Theorem 3.3.6

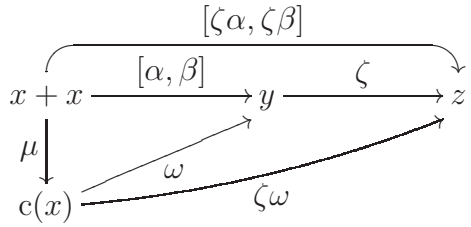
$\sim$ is a congruence on $\mathcal{K}_{cf}$.

Proof

We have already shown that $\sim$ is an equivalence relation on each hom-set of $\mathcal{K}_{cf}$. To show that it is a congruence, we need to show it is compatible with $\circ$. So let x, y, z be objects in $\mathcal{K}_{cf}$, and let



be arrows in $\mathcal{K}$ such that $\alpha \sim \beta$ and $\zeta \sim \vartheta$. Then, choosing a witness ω for $\alpha \overset{\ell}{\sim} \beta$, we see that $\zeta \circ \omega$ is a witness for $\zeta \circ \alpha \overset{\ell}{\sim} \zeta \circ \beta$. Since



Similarly, if ψ witnesses $\zeta \overset{r}{\sim} \vartheta$, then $\psi \circ \beta$ witnesses $\zeta \circ \beta \overset{r}{\sim} \vartheta \circ \beta$.

So, by lemma, we have $\zeta \circ \alpha \sim \vartheta \circ \beta$. Q.E.D.

Definitions 3.3.7

Let x and y be objects in $\mathcal{K}$. An arrow $x \xrightarrow{\alpha} y$ is said to be *homotopy-invertible* if there exists an arrow $y \xrightarrow{\beta} x$ such that $\beta \circ \alpha \sim \iota_x$ and $\alpha \circ \beta \sim \iota_y$.

If, in addition, $\beta \circ \alpha$ actually equals ι_x, then α is called a *strong section* and β a *strong retraction*.

Theorem 3.3.8

Let x and y be objects of $\mathcal{K}_{cf}$. Then the following are equivalent for a fibration $x \xrightarrow{\rho} y$:

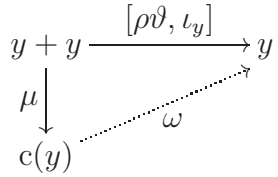
1. ρ is homotopy invertible,
2. ρ is a strong retraction, and
3. ρ is an acyclic fibration.

Dually, the following are equivalent for a cofibration $x \xrightarrow{\lambda} y$:

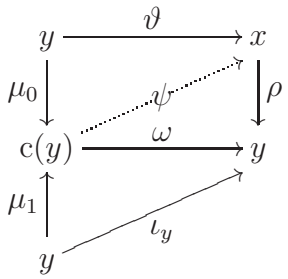
4. λ is homotopy invertible,
5. λ is a strong section, and
6. λ is an acyclic cofibration.

Proof

$(1 \Rightarrow 2)$ Suppose ϑ is a homotopy inverse for ρ. Then we can find a [left] homotopy ω such that



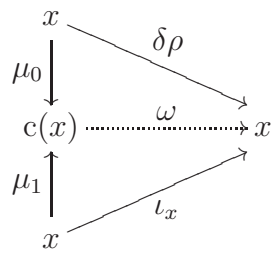
The fact that ρ is a fibration means that we can find a lift ψ as in the diagram below:



If we define δ to be the composite $\psi\mu_1$, then we may regard ψ as a [left] homotopy between ϑ and δ. So $\delta \sim \vartheta$ implies $\delta\rho \sim \vartheta\rho \sim \iota_x$. But $\rho\delta = \iota_y$ by construction.

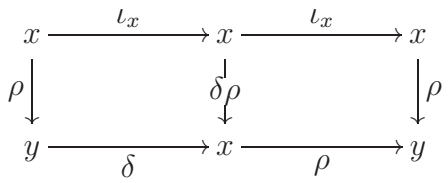
(2 $\Rightarrow$ 3) Suppose δ is a strong section for ρ. By repeated 2-out-of-3, $\delta\rho \sim \iota_x$ implies that $\delta\rho$ is a weak equivalence.

[We can find an ω



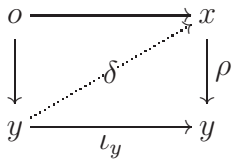
so ι_x, μ_1 weak equivalences implies ω is too. Then ω, μ_0 weak equivalences implies $\delta\rho$ is too.]

But ρ is a retract of $\delta\rho$:



So $\delta\rho$ a weak equivalence implies that ρ is too.

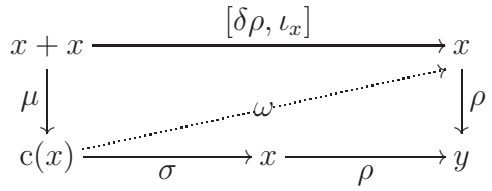
(3 $\Rightarrow$ 2) Since y is cofibrant we can find a lift



Now $\rho\delta = \iota_y$ implies that

$$\begin{aligned} \rho[\delta\rho, \iota_x] &= [\rho\delta\rho, \rho] \\ &= [\rho, \rho] \\ &= \rho[\iota_y, \iota_y] \\ &= \rho\sigma\mu \end{aligned}$$

Hence we can find a lift



which proves that $\delta\rho \sim \iota_x$.

(2 $\Rightarrow$ 1) Trivial. Q.E.D.

Corollary 3.3.9

Let x and y be objects of $\mathcal{K}_{cf}$. Then $x \xrightarrow{\alpha} y$ is a weak equivalence if and only if it has a homotopy inverse.

Proof

($\Rightarrow$) Factorise α as an acyclic cofibration, λ, followed by an acyclic fibration, ρ. Then both λ and ρ are homotopy invertible; hence also α.

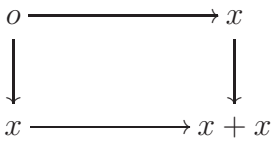
($\Leftarrow$) Factorise α as a cofibration, λ, followed by an acyclic fibration, ρ; and suppose β is a homotopy inverse to α. Then $\beta \circ \rho$ is a homotopy inverse for λ. It follows that λ is acyclic; hence also α. Q.E.D.

The following example demonstrates the historic importance of the more general notion of factorisation scheme.

Example 3.3.10

Suppose that $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a Quillen model structure on $\mathcal{K}$ such that $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ is a unique factorisation scheme—i.e., a factorisation system, as defined in the comment following Definition 3.1.1.

Then, for any cofibrant object x, the fact that



is a pushout implies that the coproduct injections $x \longrightarrow x + x$ are cofibrations. Thus the two co-components of $x + x \xrightarrow{\mu} \mathrm{c}(x)$, being the composite of μ with the two coproduct injections, are also cofibrations.

Hence both $x \xrightarrow{\mu_0} \mathrm{c}(x) \xrightarrow{\sigma} x$ and $x \xrightarrow{\mu_0} \mathrm{c}(x) \xrightarrow{\sigma} x$ are $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$-factorisation of the identity on x. Since we assumed that such factorisations are unique, they must coincide (upto isomorphism) with each other and with $x \xrightarrow{\iota_x} x \xrightarrow{\iota_x} x$. In particular, this means that σ is an isomorphism and that $\mu_0 = \sigma^{-1} = \mu_1$. This, in turn, entails that $\alpha \overset{\ell}{\sim} \beta$ implies $\alpha = \beta$, for any parallel pair of arrows $x \longrightarrow y$.

Thus, $\mathsf{Ho}[\mathcal{K}]$ is equivalent to $\mathcal{K}_{cf}/\!\sim = \mathcal{K}_{cf}$.

3.4 Monoidal QMCs

Definition 3.4.1

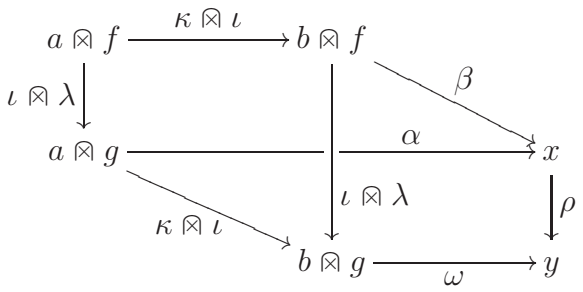
A (symmetric) *monoidal Quillen model category* is a (symmetric) monoidal closed category $(\mathcal{K}, e, \boxtimes)$ equipped with a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ which satisfies the following conditions:

1. e (is either cofibrant or) admits a cofibrant replacement $q \longrightarrow e$ with the property that for every cofibrant object x,

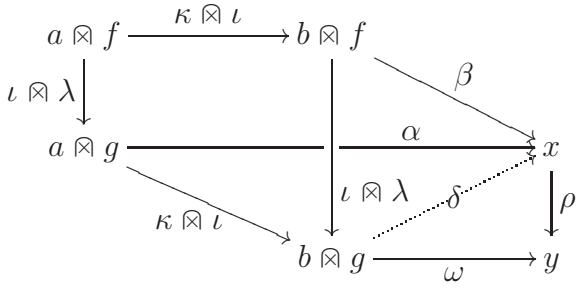
$$q \boxtimes x \longrightarrow e \boxtimes x$$

   is a weak equivalence; and

2. for every commutative diagram of the form



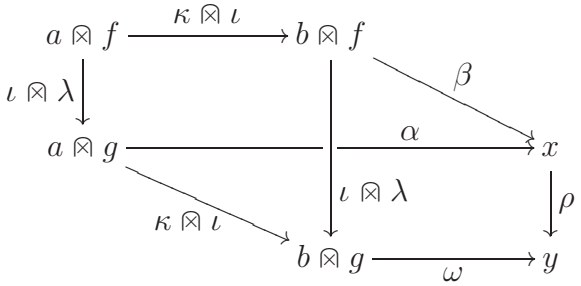
with $\kappa, \lambda \in \mathcal{C}$, $\rho \in \mathcal{F}$, and at least one of κ, λ, ρ belonging to $\mathcal{W}$, there exists a $b \boxtimes g \xrightarrow{\delta} x$ such that



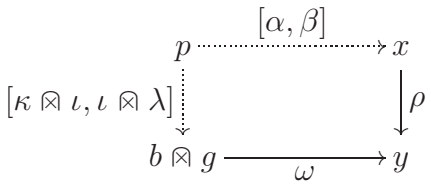
commutes.

Remark 3.4.2

If $\mathcal{K}$ has pushouts, then a commutative diagram of the form



induces a commutative square



(where p is the pushout of $a \boxtimes g \xleftarrow{\iota \boxtimes \lambda} a \boxtimes f \xrightarrow{\kappa \boxtimes \iota} b \boxtimes f$), and conversely. Moreover, a lift of the first diagram is equivalent to a lift of the second.

Thus, if pushouts exist, the definition of monoidal model category can be rephrased as a set of sufficient conditions for the map $[\kappa \boxtimes \iota, \iota \boxtimes \lambda]$, henceforth abbreviated $\kappa \circledast \lambda$, to be a(n acyclic) cofibration.

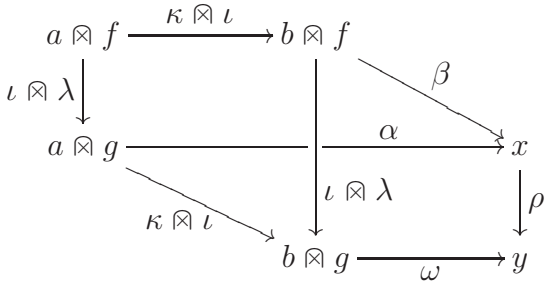
On the one hand, $\kappa \circledast \lambda$ should be a cofibration whenever κ and λ are—since it then has the lifting property with respect to all $\rho \in \mathcal{W} \cap \mathcal{F}$. On the other, $\kappa \circledast \lambda$

should be an acyclic cofibration if, in addition, either one of κ, λ is so—since it then has the lifting property with respect to all $\rho \in \mathcal{F}$. I.e., $\mathcal{C}$ should be closed under $\circledast$, and $\mathcal{C} \cap \mathcal{W}$ should be a 2-sided ideal in $(\mathcal{C}, \circledast)$.

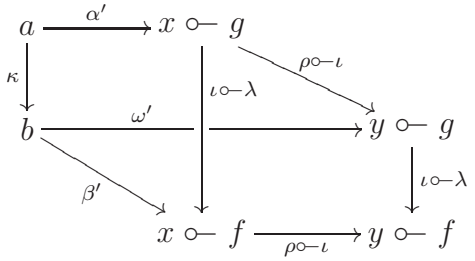
The following lemma implies further equivalent definitions of monoidal Quillen model categories.

Lemma 3.4.3

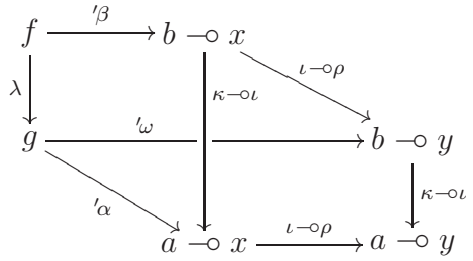
Given arrows $a \xrightarrow{\kappa} b$, $f \xrightarrow{\lambda} g$, $x \xrightarrow{\rho} y$, $a \circledast g \xrightarrow{\alpha} x$, $b \circledast f \xrightarrow{\beta} x$, and $b \circledast g \xrightarrow{\omega} y$ in $\mathcal{K}$, then the diagram



commutes if and only if either/both of



and

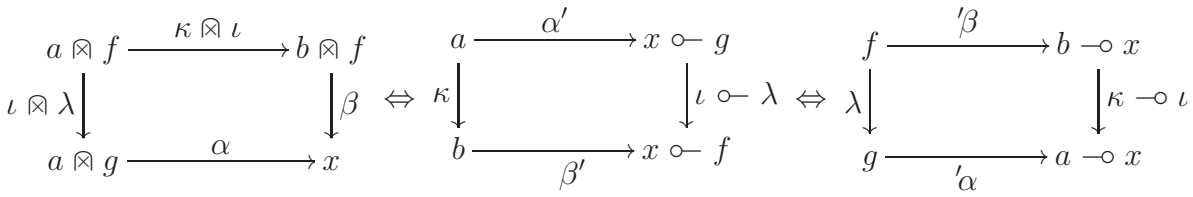


do too. Similarly, $b \circledast g \xrightarrow{\delta} x$ 'lifts' the first diagram (—i.e., makes all three resulting triangles commute) if and only if $b \xrightarrow{\delta'} x \circ\!\!- g$ lifts the second diagram if and only if $g \xrightarrow{'\delta} b \multimap x$ lifts the third.

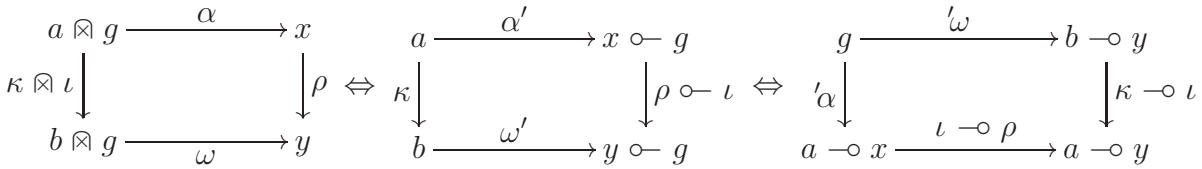
Proof

This is actually a very simple exercise: each of the three non-trivial components in the first diagram is equivalent to a non-trivial component in each of the second

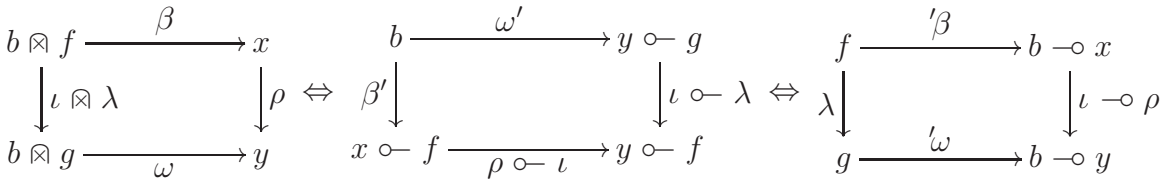
and the third. More specifically



and



and



Q.E.D.

Now we can define $-\!*$, $*\!-$ analogously to $\circledast$ and state the definition of a monoidal Quillen model category in terms of either of them.

Of course, all these definitions seem rather unwieldy; luckily, a weaker but more elegant definition suffices to prove the main theorem.

Definition 3.4.4

An adjunction $F \dashv G$ between Quillen model categories (or almost-Quillen model categories)

$$\mathcal{K}' \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{K}$$

is called a *Quillen adjunction* if either/both of the following conditions (which are equivalent by Theorem 3.1.6) hold:

1. F preserves cofibrations and acyclic cofibrations.

2. G preserves fibrations and acyclic fibrations.

Theorem 3.4.5

If $\mathcal{K}$ is a monoidal Quillen model category, then

$$\mathcal{K} \underset{x \boxtimes (-)}{\overset{x \multimap (-)}{\rightleftarrows}} \mathcal{K}$$

is a Quillen adjunction for every cofibrant object x,

$$\mathcal{K} \underset{(-) \boxtimes y}{\overset{(-) \circ\!\!- y}{\rightleftarrows}} \mathcal{K}$$

is a Quillen adjunction for every cofibrant object y, and

$$\mathcal{K}^{\mathsf{op}} \underset{(-) \circ\!\!- z}{\overset{(-) \multimap z}{\rightleftarrows}} \mathcal{K}$$

is a Quillen adjunction for every fibrant object z.

[Here $\mathcal{K}^{\mathsf{op}}$ is made into a almost-Quillen model category by reversing the fibrations and cofibrations.]

Proof

If x is cofibrant and $s \xrightarrow{\beta} t$ is a(n acyclic) cofibration, then

$$(0 \longrightarrow x) \circledast (s \xrightarrow{\beta} t) = (x \boxtimes s \xrightarrow{x \boxtimes \beta} x \boxtimes t)$$

—so $x \boxtimes \beta$ is a(n acyclic) cofibration too.

Similarly,

$$(s \xrightarrow{\beta} t) \circledast (0 \longrightarrow y) = (s \boxtimes y \xrightarrow{\beta \boxtimes y} t \boxtimes y)$$

The third adjunction is also proved in the same manner, but using $-\!\!\square$ (or $\square\!\!-$) and $z \longrightarrow 1$ instead. Q.E.D.

Definition 3.4.6

A (symmetric) *monoidal almost-Quillen model category* is an almost-Quillen model category $\mathcal{K}$ together with a closed (symmetric) tensor $\boxtimes$ such that:

1. e (is either cofibrant or) admits a cofibrant replacement $q \longrightarrow e$ with the property that for every cofibrant object x,

$$q \boxtimes x \longrightarrow e \boxtimes x$$

   is a weak equivalence; and

2. the conclusions of Theorem 3.4.5 holds.

Lemma 3.4.7 (Ken Brown)

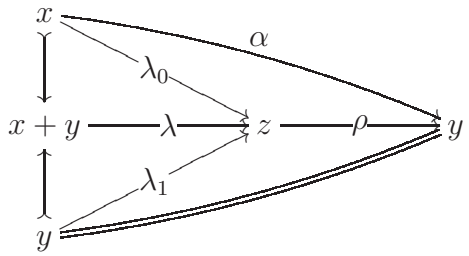
If

$$\mathcal{K}' \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{K}$$

is a Quillen adjunction, then F preserves weak equivalences between cofibrant objects, and dually G preserves weak equivalences between fibrant objects.

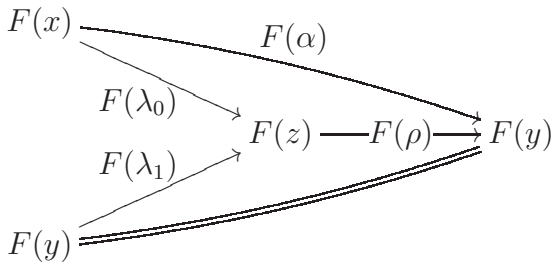
Proof

Let $x \xrightarrow{\alpha} y$ be a weak equivalence between cofibrant objects. Consider the diagram



where λ is a cofibration and ρ is an acyclic fibration. Then λ_0, λ_1 are acyclic cofibrations, by 2-out-of-3 and because x, y are cofibrant.

It then follows that $F(\lambda_0), F(\lambda_1)$ are also acyclic cofibrations and by repeatedly applying 2-out-of-3 to the diagram



we see that $F(\alpha)$ is a weak equivalence. Q.E.D.

Lemma 3.4.8

Let $\mathcal{K}$ be a monoidal almost-Quillen model category. Suppose $x \boxtimes y \overset{\alpha}{\underset{\beta}{\rightrightarrows}} z$ are arrows in $\mathcal{K}$, with x, y cofibrant and z fibrant. Then $\alpha \sim \beta$ if and only if $'\alpha \sim '\beta$.

Proof

If $'\alpha \sim '\beta$, then we can find ω such that

$$\begin{array}{ccc} y + y & \xrightarrow{['\alpha, '\beta]} & x \multimap z \\ {\scriptstyle\mu}\downarrow & \nearrow_{\omega} & \\ \mathrm{c}(y) & & \end{array}$$

But since $x \boxtimes (-)$ preserves cofibrations and weak equivalences between cofibrant objects,

$$(x \boxtimes y) + (x \boxtimes y) \xrightarrow{\sim} x \boxtimes (y + y) \xrightarrow{\iota_x \boxtimes \mu} x \boxtimes \mathrm{c}(y) \xrightarrow{\iota_x \boxtimes \sigma} x \boxtimes y$$

defines a cylindrification for $x \boxtimes y$. Therefore,

$$\begin{array}{ccccc} (x \boxtimes y) + (x \boxtimes y) & & & \xrightarrow{[\alpha, \beta]} & \\ {\scriptstyle\sim}\downarrow & & & & \downarrow \\ x \boxtimes (y + y) & \xrightarrow{\iota_x \boxtimes ['\alpha, '\beta]} & x \boxtimes (x \multimap z) & \xrightarrow{\vec{\varepsilon}} & z \\ {\scriptstyle\iota_x \boxtimes \mu}\downarrow & \nearrow_{\iota_x \boxtimes \omega} & & & \\ x \boxtimes \mathrm{c}(y) & & & & \end{array}$$

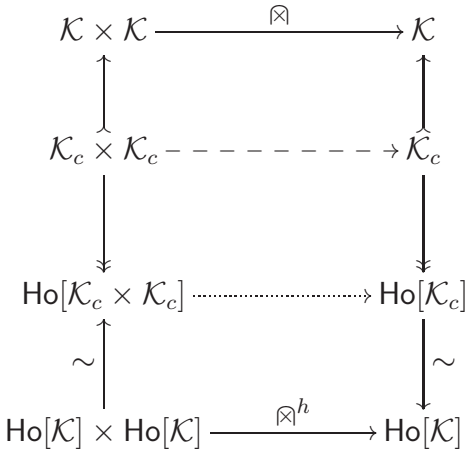
witnesses $\alpha \sim \beta$.

The reverse argument is exactly dual, using pathifications in place of cylindrifications and the fact that $x \multimap (-)$ preserves fibrations and weak equivalences between fibrant objects. Q.E.D.

Theorem 3.4.9

If $\mathcal{K}$ is a (symmetric) monoidal almost-Quillen model category, then $\mathsf{Ho}[\mathcal{K}]$ is a (symmetric) monoidal closed category.

Proof

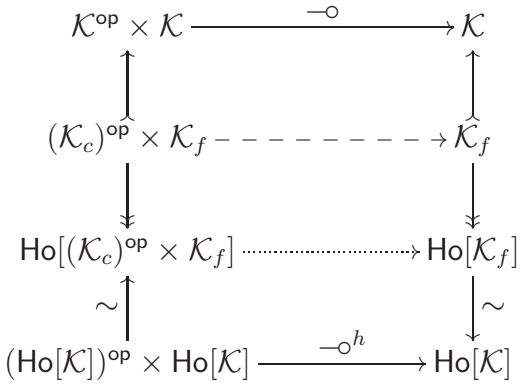


because $\boxtimes$ preserves cofibrant objects

because $\boxtimes$ preserves weak equivalences between cofibrant objects

So if $\mathcal{K}$ admits a functorial cofibrant replacement Q, then we can write $x \boxtimes^h y$ as $Qx \boxtimes Qy$.

Similarly,



So if $\mathcal{K}$ also admits a functorial fibrant replacement R, then $y \multimap^h z$ can be written as $Qy \multimap Rz$.

Now it remains to show that

$$[x \boxtimes^h y, z]_{\mathsf{Ho}[\mathcal{K}]} \cong [x, y \multimap^h z]_{\mathsf{Ho}[\mathcal{K}]}$$

but

$$[x \boxtimes^h y, z]_{\mathsf{Ho}[\mathcal{K}]} \cong [x \boxtimes^h y, Rz]_{\mathsf{Ho}[\mathcal{K}]} \cong [Qx \boxtimes Qy, Rz]/\!\sim$$

since $x \boxtimes^h y$ is cofibrant and Rz is fibrant. Similarly,

$$[x, y \multimap^h z]_{\mathsf{Ho}[\mathcal{K}]} \cong [Qx, Qy \multimap Rz]/\!\sim$$

But Qx and Qy are cofibrant and Rz is fibrant, so by the preceding lemma,

$$[Qx \boxtimes Qy, Rz]/\!\sim \cong [Qx, Qy \multimap Rz]/\!\sim$$

Now for the unit. Observe that

$$e \boxtimes^h x = Qe \boxtimes Qx \longrightarrow\!\!\!\!\!\!\rightarrow e \boxtimes Qx \cong Qx \longrightarrow\!\!\!\!\!\!\rightarrow x$$

The rightmost arrow is a weak equivalence—hence an iso in $\mathsf{Ho}[\mathcal{K}]$—and the middle arrow is a weak equivalence by hypothesis. Q.E.D.

Chapter 4

Chu categories—an example

In an appendix to [9], Barr's student Po-Hsiang Chu describes a general procedure for constructing $*$-autonomous categories.

One important motivation for this construction, which has come to bear Chu's name, comes from topological algebra; specifically, from the failure of Example 1.3.2 to generalise to arbitrary (discrete or topological, real or complex) vector spaces.

Roughly speaking, the discrete case fails because there are too many arrows: v^{**} is much larger than v, except when v is finite-dimensional. Using topological vector space allows us to whittle down the number of arrows; but the topological case—even when restricted to, say, locally convex topologies—also fails. This time, again roughly speaking, there are too many objects: two distinct topologies on the same vector space may yield the same space of continuous linear functionals.

The above situation might be remedied as follows: consider only those topologies which are largest among those inducing the same space of continuous linear functionals; or, dually, consider only those topologies which are smallest among the same family. A more profound solution is to identify all those topological vector spaces having the same space of continuous linear functionals.

This is effected by endowing our original vector space, not with a topology but, instead, with a space of linear functionals—to be thought of as *the* space of continuous linear functionals. In order for a linear transformation to be deemed continuous, we require that it preserve continuous linear functionals.

4.1 A primer on Chu spaces

Definition 4.1.1

Let $(\mathcal{V}, e, \boxtimes)$ be a symmetric monoidal category and d an arbitrary object in $\mathcal{V}$.

A *Chu space*, with respect to $(\mathcal{V}, d)$, is a triple $A = (a^+, a^-, \alpha)$ where a^+, a^- are objects in $\mathcal{V}$, and α is a morphism $a^+ \boxtimes a^- \longrightarrow d$ in $\mathcal{V}$. A *Chu morphism* $A \xrightarrow{\Theta} B$ consists of a pair of arrows $a^+ \xrightarrow{\vartheta^+} b^+$ and $b^- \xrightarrow{\vartheta^-} a^-$ in $\mathcal{V}$ such that

$$\begin{array}{ccccc} a^+ \boxtimes a^- & \xleftarrow{\iota_{a^+} \boxtimes \vartheta^-} & a^+ \boxtimes b^- & \xrightarrow{\vartheta^+ \boxtimes \iota_{b^-}} & b^+ \boxtimes b^- \\ \alpha \downarrow & & & & \downarrow \beta \\ d & = & = & = & d \end{array}$$

commutes. The category of Chu spaces and Chu morphisms will be denoted by $\mathbf{Chu}(\mathcal{V}, d)$.

Note that, even in the completely arbitrary case, we are guaranteed at least two Chu spaces: $(e, d, \vec{v}_d)$ and $(d, e, \overleftarrow{v}_d)$. These will be denoted E and D, respectively.

While the definition of Chu space pre-supposes only a symmetric monoidal structure, most theorems about Chu spaces require a symmetric monoidal closed structure and pullbacks. So, for the remainder of this section, $(\mathcal{V}, e, \boxtimes, \multimap)$ will denote a symmetric monoidal closed category with pullbacks, and d (as before) an arbitrary object in $\mathcal{V}$. [The example to keep in mind, in light of the remarks preceding this section, is that of 1.1.5.2 together with the object $d = k$.]

With this additional structure, the notion of Chu morphism can be at least partially justified by the following lemma.

Lemma 4.1.2

The following are equivalent:

1. $A \xrightarrow{\Theta} B$ is a Chu morphism,
2. $$\begin{array}{ccc} a^+ & \xrightarrow{\quad\vartheta^+\quad} & b^+ \\ \alpha' \downarrow & & \downarrow \beta' \\ (a^- \multimap d) & \xrightarrow[(\vartheta^- \multimap \iota_d)]{} & (b^- \multimap d) \end{array}$$ commutes, and

3.
$$\begin{array}{ccc} a^- & \xleftarrow{\quad\vartheta^-\quad} & b^- \\ {\scriptstyle '\alpha}\downarrow & & \downarrow{\scriptstyle '\beta} \\ (a^+ \multimap d) & \xleftarrow[(\vartheta^+ \multimap \iota_d)]{} & (b^+ \multimap d) \end{array} \quad \text{commutes}$$

—where, as before, α', $'\alpha$ denote the two transposes of α (cf. Remark 1.1.8).

Note that there is, so far, no restriction on the nature of the arrows α' and $'\alpha$; in particular, they need not be monic. Nevertheless, we may think of $'\alpha$ as mapping *co-vectors* (i.e., formal continuous linear functionals on the space of vectors) to genuine linear functionals on the space of vectors. In this spirit, Condition 3 can be viewed as stating that composition-by-ϑ^+ respects co-vectors.

Definition 4.1.3

Given Chu spaces $A = (a^+, a^-, \alpha)$ and $X = (x^+, x^-, \xi)$, we define $A \boxtimes X$ to be the Chu space $((a \cdot x)^+, (a \cdot x)^-, \alpha \cdot \xi)$, where

- $(a \cdot x)^+$ denotes $a^+ \boxtimes x^+$;
- $(a \cdot x)^-$ denotes the pullback of

$$\begin{array}{ccccc} & & & & (a^+ \multimap x^-) \\ & & & & \downarrow{\scriptstyle (\iota_{a^+} \multimap {'\xi})} \\ & & & & (a^+ \multimap (x^+ \multimap d)) \\ & & & & \downarrow{\scriptstyle \sim} \\ (x^+ \multimap a^-) & \xrightarrow{(\iota_{x^+} \multimap {'\alpha})} & (x^+ \multimap (a^+ \multimap d)) & \xrightarrow{\quad\sim\quad} & \big((a \cdot x)^+ \multimap d\big) \end{array}$$

- $\alpha \cdot \xi$ denotes the transpose of the common composite

$$\begin{array}{ccc} (a \cdot x)^- & \longrightarrow & (a^+ \multimap x^-) \\ \downarrow & & \downarrow \\ (x^+ \multimap a^-) & \longrightarrow & \big((a \cdot x)^+ \multimap d\big) \end{array}$$

Similarly, we define $A \multimap X$ to be the Chu space $((a \triangleright x)^+, (a \triangleright x)^-, \alpha \triangleright \xi)$, where

- $(a \triangleright x)^+$ denotes the pullback of:

$$\begin{array}{ccccc} & & & & (a^+ \multimap x^+) \\ & & & & \downarrow{\scriptstyle (\iota_{a^+} \multimap \xi')} \\ & & & & (a^+ \multimap (x^- \multimap d)) \\ & & & & \downarrow{\scriptstyle \sim} \\ (x^- \multimap a^-) & \xrightarrow{(\iota_{a^-} \multimap {}'\alpha)} & (x^- \multimap (a^+ \multimap d)) & \xrightarrow{\quad\sim\quad} & (a^+ \otimes x^- \multimap d) \end{array}$$

- $(a \triangleright x)^-$ denotes $a^+ \otimes x^-$; and,

- $\alpha \triangleright \xi$ denotes the transpose of the common composite

$$\begin{array}{ccc} (a \triangleright x)^+ & \longrightarrow & (a^+ \multimap x^+) \\ \downarrow & & \downarrow \\ (x^- \multimap a^-) & \longrightarrow & \big((a \triangleright x)^- \multimap d\big) \end{array}$$

Theorem 4.1.4

$(\mathbf{Chu}, E, \otimes, \multimap, D)$ is a $*$-autonomous category.

Lemma 4.1.5

$A \multimap D$ is isomorphic to $(a^-, a^+, \tilde{\alpha})$, where $\tilde{\alpha}$ denotes the composite

$$a^- \otimes a^+ \xrightarrow{\chi_{a^-,a^+}} a^+ \otimes a^- \xrightarrow{\quad\alpha\quad} d$$

Now, we finally address the issue raised following Lemma 4.1.2: restricting the nature of the arrows α' and ${}'\alpha$.

Definition 4.1.6

Suppose $(\mathcal{E}, \mathcal{M})$ is a factorisation system on $\mathcal{V}$. Then, a Chu space $A = (a^+, a^-, \alpha)$ is said to be *separated* if $\alpha' \in \mathcal{M}$ and *extensional* if ${}'\alpha \in \mathcal{M}$. We write $\mathbf{Chu}_s$ for the full subcategory of **Chu** consisting of separated spaces, $\mathbf{Chu}_e$ for that of extensional spaces, and **chu** for their intersection.

In the motivating example, $(k\text{-}\mathbf{Lin}, k)$, one should choose $\mathcal{E}$ to be the class of surjective linear transformations and $\mathcal{M}$ that of injective linear transformations; then an extensional Chu space is one whose co-vectors can actually be thought of as linear functionals on the space of vectors and, in light of Lemma 4.1.5, a separated Chu space is one whose dual is extensional.

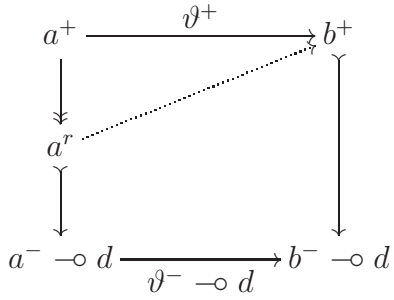
Theorem 4.1.7

$\mathbf{Chu}_s$ is a reflective subcategory of $\mathbf{Chu}$. Dually, $\mathbf{Chu}_e$ is a co-reflective subcategory of $\mathbf{Chu}$.

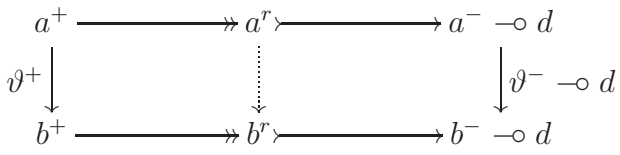
Proof

We write a^r for the $(\mathcal{E},\mathcal{M})$-factorisation of $a^+ \xrightarrow{\alpha'} a^- \multimap d$, α^r for the transpose of $a^r \longrightarrow a^- \multimap d$, and A^r for the Chu space (a^r, a^-, α^r).

Then for any separated Chu space $B = (b^+, b^-, \beta)$, and any Chu morphism $A \xrightarrow{\Theta} B$, there exists a unique factorisation



Thus, the reflection of $\mathbf{Chu}$ into $\mathbf{Chu}_s$ is given by $A \mapsto A^r$, $\Theta \mapsto (\vartheta^r, \vartheta^-)$, where ϑ^r is the unique lift in the diagram below.



Similarly the co-reflection of $\mathbf{Chu}$ into $\mathbf{Chu}_e$ is given by $A \mapsto A^q = (a^+, a^q, \alpha^q)$, $\Theta \mapsto (\vartheta^+, \vartheta^q)$, where a^q is the $(\mathcal{E},\mathcal{M})$-factorisation of $a^- \xrightarrow{'\alpha} a^+ \multimap d$, α^q is the

transpose of $a^q \longrightarrow a^+ \multimap d$, and ϑ^q is the unique lift depicted below.

$$\begin{array}{ccccc} a^- & \twoheadrightarrow & a^q & \rightarrowtail & a^+ \multimap d \\ {\scriptstyle \vartheta^-}\uparrow & & \vdots & & \uparrow{\scriptstyle \vartheta^+ \multimap d} \\ b^- & \twoheadrightarrow & b^q & \rightarrowtail & b^+ \multimap d \end{array}$$

Q.E.D.

Now, while the duality of **Chu** described in Lemma 4.1.5 clearly restricts to **chu**, the latter is not generally a monoidal subcategory of the former. Hence, it is not entirely clear whether the duality of **chu** is part of a $*$-autonomous structure.

Theorem 4.1.8

It is possible to induce a $*$-autonomous structure on **chu** if $(\mathcal{E}, \mathcal{M})$ satisfies either FS–1 and FS–2, or FS–2 and FS–3.

FS–1**.** $\mathcal{E} \subseteq \{\text{epis}\}$.

FS–2**.** For every x, the functor $x \multimap (-)$ preserves $\mathcal{M}$; equivalently,

$$\mathcal{V} \underset{x \boxtimes (-)}{\overset{x \multimap (-)}{\rightleftarrows}} \mathcal{V}$$

is a Quillen adjunction, where we choose $\mathcal{C} = \mathcal{E}$, $\mathcal{F} = \mathcal{M}$ and $\mathcal{W} = \mathsf{mor}[\mathcal{V}]$.

FS–3**.** For every z, the functor $(-) \multimap z$ maps $\mathcal{E}$ to $\mathcal{M}$; equivalently,

$$\mathcal{V}^{\mathsf{op}} \underset{(-) \multimap z}{\overset{(-) \multimap z}{\rightleftarrows}} \mathcal{V}$$

is a Quillen adjunction, with respect to the same choice of $\mathcal{C}$, $\mathcal{W}$ and $\mathcal{F}$.

4.2 A QMS for Chu spaces

The objective of this section is to bridge Barr's treatment of separated, extensional Chu spaces and the theory of monoidal almost-Quillen model categories developed between Definition 3.4.6 and Theorem 3.4.9.

Note that the theory of almost-Quillen model categories requires the presence of finite products and coproducts—so, for the remainder of this chapter, $\mathcal{V}$ will denote an arbitrary symmetric monoidal closed category with finite limits and coproducts. Further axioms on the factorisation system $(\mathcal{E}, \mathcal{M})$ will be introduced as necessary.

Definition 4.2.1

1. We define $\mathcal{W}_l$ to be the class of all Θ such that ϑ^- and ϑ^r are invertible.

2. We define $\mathcal{F}$ to be the class of all Θ such that

$$\begin{array}{ccc} a^+ & \twoheadrightarrow & a^r \\ {\scriptstyle \vartheta^+}\downarrow & & \downarrow{\scriptstyle \vartheta^r} \\ b^+ & \twoheadrightarrow & b^r \end{array}$$

   is a pullback square.

3. We define $\mathcal{W}_r$ to be the class of all Θ such that ϑ^+ and ϑ^q are invertible.

4. We define $\mathcal{C}$ to be the class of all Θ such that

$$\begin{array}{ccc} a^- & \twoheadrightarrow & a^q \\ {\scriptstyle \vartheta^-}\uparrow & & \uparrow{\scriptstyle \vartheta^q} \\ b^- & \twoheadrightarrow & b^q \end{array}$$

   is a pullback square.

5. We define $\mathcal{W}$ to be the class of all Θ such that ϑ^r and ϑ^q are invertible.

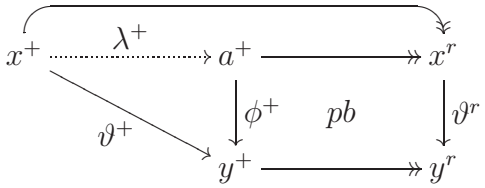
Theorem 4.2.2

If $\mathcal{E}$ is stable under pullbacks in $\mathcal{V}$, then both $(\mathcal{W}_l, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W}_r)$ are factorisation systems on **Chu**.

Proof

By symmetry, we need only show that $(\mathcal{W}_l, \mathcal{F})$ is a factorisation system.

The factorisation axiom is the easiest to verify: given Chu spaces X and Y and a Chu morphism $X \xrightarrow{\Theta} Y$, let a^+, λ^+ and ϕ^+ be as indicated:

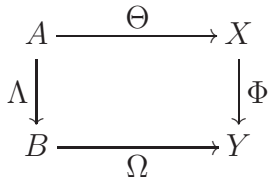


Let also $a^- = x^-$, λ^- the identity on x^-, and $\phi^- = \vartheta^-$. Then $A = (a^+, a^-, \alpha)$ is a Chu space where α is the transpose of the composite

$$a^+ \longrightarrow\!\!\!\!\!\!\rightarrow x^r \rightarrowtail (x^- \multimap d)$$

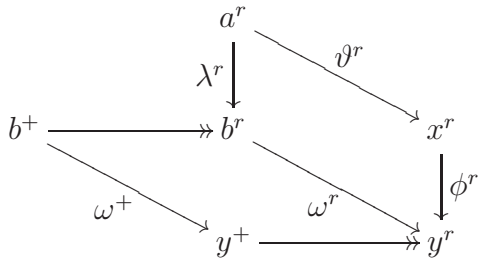
Then $\Lambda = (\lambda^+, \lambda^-)$ is a Chu morphism belonging to $\mathcal{W}_l$, and $\Phi = (\phi^+, \phi^-)$ is a Chu morphism belonging to $\mathcal{F}$.

Next the lifting axiom: given a commutative diagram

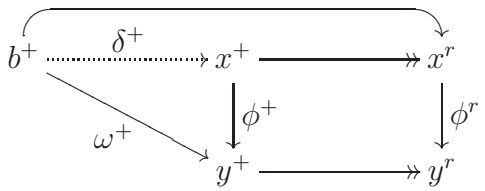


of Chu spaces and Chu morphisms, with $\Lambda \in \mathcal{W}_l$ and $\Phi \in \mathcal{F}$, we define $B \xrightarrow{\Delta} X$ as follows.

The fact that we can invert λ^r in the following diagram

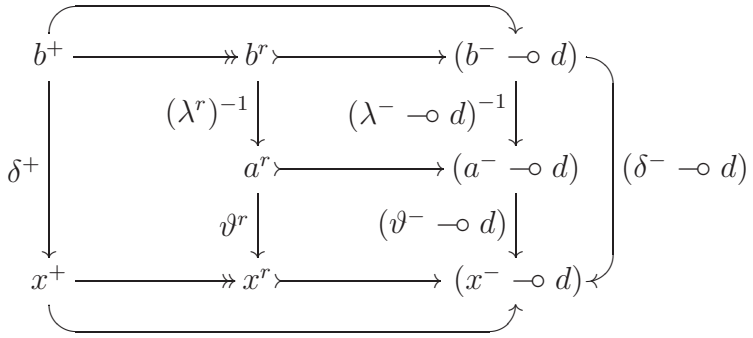


allows us to construct an arrow

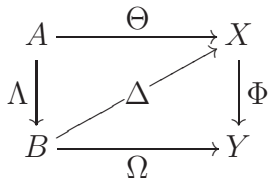


—since the right-hand square is, by hypothesis, a pullback.

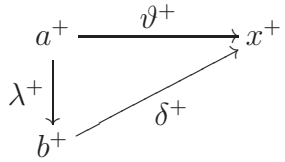
Now if we define δ^- to be the composite $x^- \xrightarrow{\vartheta^-} a^- \xrightarrow{(\lambda^-)^{-1}} b^-$, then $\Delta = (\delta^+, \delta^-)$ is a Chu morphism $B \longrightarrow X$, as demonstrated below.



Moreover, it is easy to check that Δ is the unique Chu morphism such that



For example,



follows from the universal property of the pullback above.

Finally, it should be clear that $\mathcal{W}_l$ and $\mathcal{F}$ are closed under isomorphisms and composition. Q.E.D.

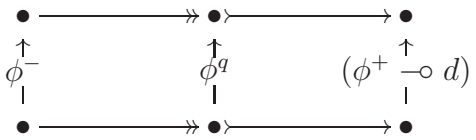
Theorem 4.2.3

If $\mathcal{E}$ is stable under pullbacks in $\mathcal{V}$ and the functor $(-) \multimap d$ maps $\mathcal{E}$ to $\mathcal{M}$, then $\mathcal{W}_r = \mathcal{W} \cap \mathcal{F}$ and $\mathcal{W}_l = \mathcal{C} \cap \mathcal{W}$. Moreover, $\mathcal{W}$ always satisfies 2-out-of-3. Hence, under these hypotheses, $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ form an almost-Quillen model structure on **Chu**.

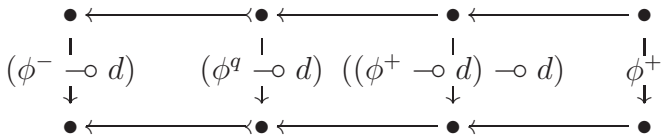
Proof

Firstly observe that $(\psi \circ \omega)^r = \psi^r \circ \omega^r$, by the uniqueness of lifts under the $(\mathcal{E}, \mathcal{M})$ factorisation. [I.e., $(-)^r$ is actually a functor $\mathbf{Chu} \longrightarrow \mathcal{V}$.] Hence if any two of ω^r, ψ^r, $(\psi \circ \omega)^r$ are invertible, then so is the third. Similarly $(\psi \circ \omega)^q = \omega^q \circ \psi^q$; hence $\mathcal{W}$ satisfies 2-out-of-3.

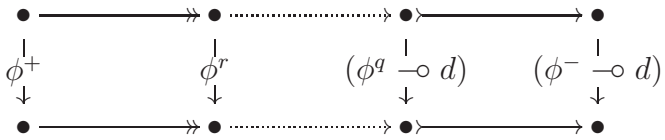
Now suppose $\Phi = (\phi^+, \phi^-) \in \mathcal{W}_r$. Then applying the functor $(-) \multimap d$ to the diagram



we obtain a diagram



By lifting, we get



By uniqueness of factorisation, we get that both dotted arrows are in $\mathcal{M}$.

But both ϕ^+ and $(\phi^q \multimap d)$ are invertible, so the uniqueness of factorisation [again!] implies that ϕ^r is invertible; i.e., $\phi \in \mathcal{W}$. But the leftmost square is also, trivially, a pullback; i.e. $\phi \in \mathcal{F}$.

Conversely, the pullback of an iso being an iso shows that $\mathcal{W} \cap \mathcal{F} \subseteq \mathcal{W}_r$. Q.E.D.

Theorem 4.2.4

A Chu space is separated if and only if it is fibrant with respect to the Quillen model structure given above, and extensional if and only if it cofibrant.

Proof

Let o and t denote the initial and terminal objects, respectively, of $\mathcal{V}$. Then the terminal Chu space is $T = (t, o, !)$ where $!$ is the unique map $t \boxtimes o \longrightarrow d$. Note that $o \multimap d \cong t$. So for $X \longrightarrow T$ to be a fibration means that we have a pullback square

$$\begin{array}{ccc} x^+ & \longrightarrow & x^r \\ \downarrow & & \downarrow \\ t & \xrightarrow{\sim} & t \end{array}$$

and the pullback of an iso is an iso. Q.E.D.

Now it is clear from all the definitions involved that the functor $(-)^*$ preserves and reflects $\mathcal{W}$ while swapping $\mathcal{C}$ and $\mathcal{F}$. As we shall see below, this observation greatly simplifies our proof of Corollary 4.2.6; it is also a sufficient, though far from necessary, condition for the homotopy category to be not merely symmetric monoidal closed but $*$-autonomous.

Also, note that E is cofibrant, since $\vec{\upsilon}_d$ is the canonical isomorphism $d \longrightarrow e \multimap d$. Similarly D is fibrant.

Theorem 4.2.5

If $(\mathcal{E}, \mathcal{M})$ satisfies axiom FS–2 and $\mathcal{M} \subseteq \{\text{monos}\}$, then X cofibrant implies $X \boxtimes (-)$ preserves cofibrations.

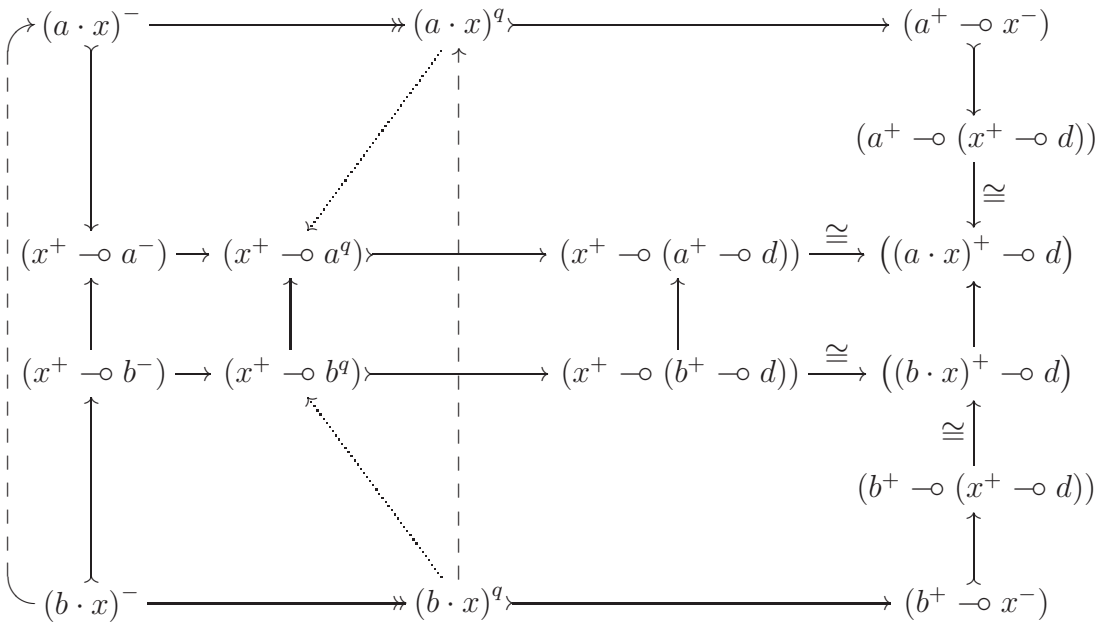
Proof

To understand $(a \cdot x)^q$, note that we have

$$\begin{array}{ccccc} (a \cdot x)^- & & \longrightarrow & & (a^+ \multimap x^-) \\ \downarrow & \searrow & & & \downarrow \\ & (a \cdot x)^q & \dashrightarrow & & (a^+ \multimap (x^+ \multimap d)) \\ & & \searrow & & \downarrow \cong \\ (x^+ \multimap a^-) & \longrightarrow & (x^+ \multimap (a^+ \multimap d)) & \xrightarrow{\cong} & \big((a \cdot x)^+ \multimap d\big) \end{array}$$

since $(x^- \longrightarrow x^+ \multimap d) \in \mathcal{M}$ and $a^+ \multimap (-)$ preserves $\mathcal{M}$.

Similarly, we obtain the dotted arrows below from the hypothesis that $x^+ \multimap (-)$ preserves $\mathcal{M}$.



We need to show that the large rectangle at the left (the one with dashed verticals) is a pullback. But the little square at the centre-left is a pullback by hypothesis, and because the functor $x^+ \multimap (-)$, being a right adjoint, preserves pullbacks.

Now it suffices that the two medium-sized trapezoids at the top left and bottom left are pullbacks.

But this follows from the fact that the large rectangles at the top and bottom are pullbacks by an easy diagram chase—*if the* $\rightarrowtail$*s are indeed monic.* Q.E.D.

Corollary 4.2.6

If $(\mathcal{E}, \mathcal{M})$ satisfies axiom FS–2 and the functor $(-) \multimap d$ maps $\mathcal{E}$ to $\mathcal{M}$ (which is a special case of FS–3), $\mathcal{M} \subseteq \{\text{monos}\}$, and $\mathcal{E}$ is stable under pullbacks in $\mathcal{V}$, then **Chu** is a monoidal almost-Quillen model category in the sense of Definition 3.4.6. Moreover, $\mathsf{Ho}[\mathbf{Chu}]$ is equivalent to **chu**.

Proof

What remains to show is that if X is cofibrant and Z is fibrant, then both of the adjunctions

$$\mathbf{Chu} \underset{X \boxtimes (-)}{\overset{X \multimap (-)}{\rightleftarrows}} \mathbf{Chu} \quad \text{and} \quad \mathbf{Chu}^{\mathsf{op}} \underset{(-) \multimap Z}{\overset{(-) \multimap Z}{\rightleftarrows}} \mathbf{Chu}$$

are Quillen adjunctions.

We have shown that the functor $X \boxtimes (-)$ preserves cofibrations, whenever X is cofibrant. It therefore follows that the functor $X \boxtimes (-)^*$ maps $\mathcal{F}$ to $\mathcal{C}$, and hence that $X \multimap (-) \cong (X \boxtimes (-)^*)^*$ maps $\mathcal{F}$ to $\mathcal{F}$. But this is equivalent to the assertion that $X \boxtimes (-)$ preserves $\mathcal{C} \cap \mathcal{W}$. Hence the first adjunction is a Quillen adjunction.

Similarly, if Z is fibrant, then Z^* is cofibrant. So $(Z^* \boxtimes (-))$ preserves both $\mathcal{C}$ and $\mathcal{C} \cap \mathcal{W}$, and hence $(-) \multimap Z \cong (Z^* \boxtimes (-))^*$ maps $\mathcal{C}$ to $\mathcal{F}$ and $\mathcal{C} \cap \mathcal{W}$ to $\mathcal{W} \cap \mathcal{F}$. So the second adjunction is also a Quillen adjunction.

The last statement of the corollary follows from Example 3.3.10. Q.E.D.

Chapter 5

Adherence Spaces

One of the most important categories in linear logic and theoretical computer science is **Rel**, [34]. This is partly because, from a computation-theoretic point of view, the standard notion of function is too restrictive: it is a sad fact of life that an arbitrary algorithm cannot be guaranteed to halt for every possible input. Thus one should consider partial functions, at least. Arbitrary relations can further be used to model non-determinism [1].

Another reason is that **Rel** has superb categorical features. Relations $r \subseteq (a \times b) \times c$ are evidently in natural bijection with relations $\hat{r} \subseteq a \times (b \times c)$. This implies that **Rel** is a symmetric monoidal closed category with $x \multimap y = x \times y$. In fact, **Rel** is compact closed, with $x^* \cong x$, naturally.

More general categories, in which objects are structured sets and morphisms are structure-preserving relations, are also common in linear logic. Indeed Schalk and de Paiva [40] have shown that many of the standard denotational models of linear logic, such as coherence spaces [22] and hypercoherence spaces [16], may be regarded as "dressed up" versions of **Rel**.

Their general construction depends on a monoidal functor $\mathbf{Rel} \xrightarrow{M} \mathbf{Rel}$, and a $*$-autonomous poset q, but we shall only consider the case where M is the identity functor. For it is this case of their construction at which I independently arrived.

But, whereas Schalk and de Paiva frequently note how completely their techniques depend on the degenerate structure of **Rel**, to the extent of writing

> "...it turns out that we use so much of the structure of **Rel** (the fact that it is compact closed as well as the fact that A negated is naturally isomorphic to A) that it does not seem worthwhile to give the most general version here."

my presentation focusses only on the nice 2-categorical aspects of **Rel**: the fact that its hom-sets are completely partially ordered, and that composition preserves suprema.

In particular, I was later able to show how my proofs could be extended to a more general construction on the category of all complete partially ordered sets and supremum-preserving maps, **Sup**, which is neither compact closed, nor does it have trivial negation. But there is a trade-off: the price of replacing **Rel** by **Sup** is that one must place extra restrictions on q.

5.1 Lattice-theoretic Preliminaries

We recall that a partially ordered set is called *complete* if it admits suprema (and therefore infima) of arbitrary subsets. A function between complete posets which preserves suprema is called a *sup-homomorphism*.

Definitions 5.1.1

Pos denotes the category of posets and monotone (i.e., order-preserving) functions, and **Sup** denotes the subcategory of complete posets (alias *sup-lattices*) and sup-homomorphisms.

We write 2 for the set of (boolean) truth-values $\{\phi, \tau\}$ with the usual ordering $(\phi < \tau)$.

Theorem 5.1.2

The forgetful functor $\mathbf{Sup} \xrightarrow{U} \mathbf{Pos}$ is monadic, as is the composite forgetful functor $\mathbf{Sup} \xrightarrow{U} \mathbf{Pos} \xrightarrow{|\ |} \mathbf{Set}$, [35].

We shall not prove the above, but do note that the free functor $\mathbf{Pos} \longrightarrow \mathbf{Sup}$ can be described as (the 2-enriched version of) the presheaf construction, $j \mapsto 2^{j^{\mathsf{op}}}$.

Under this description, the unit maps are just (the 2-enriched version of) the Yoneda embeddings $j \rightarrowtail 2^{j^{\mathsf{op}}}$.

More concretely, functors (i.e., monotone functions) $j^{\mathsf{op}} \longrightarrow 2$ are in bijective correspondence with down-closed subsets of j. So the free functor $\mathbf{Pos} \longrightarrow \mathbf{Sup}$ can be equivalently described by the formula

$$j \mapsto D(j) := \{\omega \in P(j) \mid (\forall \alpha, \beta \in j)((\alpha \leq \beta) \wedge (\beta \in \omega) \Rightarrow (\alpha \in \omega))\}$$

—the latter set being ordered by inclusion. Under this description, the unit maps $j \rightarrowtail D(j)$ are given by the formula

$$\alpha \mapsto {\downarrow}\alpha := \{\beta \in j \mid \beta \leq \alpha\}$$

Since the free functor $\mathbf{Set} \longrightarrow \mathbf{Pos}$ is given by endowing a set with the discrete order, it is easy to see that the composite free functor $\mathbf{Set} \xrightarrow{(\ ,=)} \mathbf{Pos} \xrightarrow{D} \mathbf{Sup}$ maps a set j to its full power-set $P(j)$. The unit maps $j \longrightarrow P(j)$ are given by $\alpha \mapsto \{\alpha\}$.

Theorem 5.1.3

Sup can be made into a symmetric monoidal closed category in such a way that the underlying set of $x \multimap y$ is exactly the set of all sup-homomorphisms $x \longrightarrow y$, and with 2 as the tensor unit.

Proof

There exists a general theory of when algebraic categories (categories for which the forgetful functor into **Set** is monadic) are symmetric monoidal closed [33], and **Sup** falls into this framework [9]. Moreover, the tensor unit must be (isomorphic to) the free object on one generator. Q.E.D.

Alternatively, this theorem can be proven directly, as suggested in [29]:

> "$x \otimes y$ can be constructed as a quotient of the free sup-lattice on $|x| \times |y|$—$P(|x| \times |y|)$—by the equivalence relation generated by the obvious set of conditions [8]." [Notation has been altered.]

Thus, in complete analogy with the situation of $(k\text{-}\mathbf{Lin}, k, \otimes)$, an arbitrary element of $x \invamp y$ can be written as a join of symbols $\alpha \invamp \beta$ with $\alpha \in x$ and $\beta \in y$, but not necessarily uniquely. In particular, we have

$$\alpha \invamp \bot_y = \bot_{x \invamp y} = \bot_x \invamp \beta$$

for all $\alpha \in x$ and $\beta \in y$.

Elements of the form $\alpha \invamp \beta$ with $\alpha \neq \bot_x$ and $\beta \neq \bot_y$ will be called *non-trivial pure tensors*. Evidently, every element of $x \invamp y$ can be written as a (possibly empty) join of non-trivial pure tensors.

Theorem 5.1.4

2 is a dualising object for **Sup**.

Proof

A sup-homomorphism $x \xrightarrow{\omega} 2$ is uniquely determined by its kernel, $\ker \omega = \{\alpha \in x \mid \omega(\alpha) = \phi\}$. Moreover, $\ker \omega$ has a maximum element $\sup \ker \omega$, which completely determines it.

But $\psi \leq \omega$ implies $\ker \psi \supseteq \ker \omega$ and hence $\sup \ker \psi \geq \sup \ker \omega$. This establishes an isomorphism $x \multimap 2 \xrightarrow{\sim} x^{\mathsf{op}}$.

Moreover, for any sup-homomorphism $x \xrightarrow{\omega} y$, the diagram

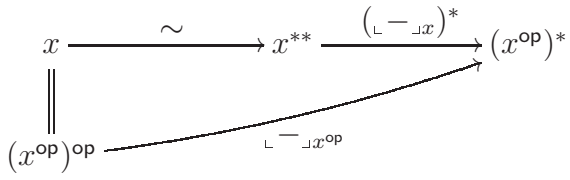
$$\begin{array}{ccc} y \multimap 2 & \xrightarrow{\omega \multimap \iota_2} & x \multimap 2 \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\ y^{\mathsf{op}} & \xrightarrow[(\omega^\sharp)^{\mathsf{op}}]{} & x^{\mathsf{op}} \end{array}$$

commutes—where $\omega^\sharp$ denotes the right adjoint of ω. [We recall that every sup-homomorphism $x \xrightarrow{\omega} y$ has a right adjoint $y \xrightarrow{\omega^\sharp} x$, which is an inf-homomorphism.]

Now it is clear that $x^{**} \cong (x^{\mathsf{op}})^{\mathsf{op}} = x$, but we still need to show that the canonical map $x \longrightarrow x^{**}$ is invertible. For this purpose it helps to write $x^{\mathsf{op}} \xrightarrow{\llcorner - \lrcorner} x^{**}$ for the inverse of $x^{**} \xrightarrow{\sup \ker} x^{\mathsf{op}}$. Explicitly, $\llcorner \alpha \lrcorner$ is the function

$$\beta \mapsto \begin{cases} \phi & \text{if } \beta \leq \alpha \\ \tau & \text{otherwise} \end{cases}$$

Then we have



since the top map is

$$\alpha \mapsto [\omega \Rrightarrow \omega(\alpha)] \mapsto [\beta \Rrightarrow \llcorner \beta_{\lrcorner x}(\alpha)]$$

whereas the bottom map is

$$\alpha \mapsto [\beta \mapsto \llcorner \alpha_{\lrcorner x^{\mathsf{op}}}(\beta)]$$

and $\beta \leq \alpha$ in x if and only if $\alpha \leq \beta$ in x^{op}. Q.E.D.

Corollary 5.1.5

(**Sup**, 2, $\otimes$, $\multimap$, 2) is a symmetric $*$-isomix category.

Theorem 5.1.6

For any posets j and k, we have $D(j^{\mathsf{op}}) \cong D(j)^{\mathsf{op}}$ and $D(j) \otimes D(k) \cong D(j \times k) \cong D(j) \parr D(k)$.

Proof

The complement of a down-closed subset of j is an up-closed subset of j and therefore a down-closed subset of j^{op}. Classically[1], this establishes an order-isomorphism between $D(j)^{\mathsf{op}}$ and $D(j^{\mathsf{op}})$—since a larger subset of j has a smaller complement! Moreover, any order-isomorphism between sup-lattices is a sup-isomorphism.

To establish the equation $D(j) \otimes D(k) \cong D(j \times k)$, let us consider j fixed and k arbitrary. Then we are trying to establish a natural isomorphism between functors:

$$\begin{array}{ccc} \mathbf{Sup} & \xleftarrow{D(j) \otimes (-)} & \mathbf{Sup} \\ {\scriptstyle D}\uparrow & \cong & \uparrow{\scriptstyle D} \\ \mathbf{Pos} & \xleftarrow[j \times (-)]{} & \mathbf{Pos} \end{array}$$

[1]Constructively, the statement $D(j^{\mathsf{op}}) \cong D(j)^{\mathsf{op}}$ is false.

—but each of these functors is a left adjoint. It therefore suffices to establish a natural isomorphism between the corresponding right adjoints:

$$\begin{array}{ccc} \mathbf{Sup} & \xrightarrow{D(j) \multimap (-)} & \mathbf{Sup} \\ {\scriptstyle U}\downarrow & \cong & \downarrow{\scriptstyle U} \\ \mathbf{Pos} & \xrightarrow[j \Rightarrow (-)]{} & \mathbf{Pos} \end{array}$$

I.e., we should have, for any sup-lattice x, $U(D(j) \multimap x) \cong j \Rightarrow U(x)$. But this is simply a re-statement of the adjunction $D \dashv U$.

Finally, by combining the two previous results, we obtain

$$\begin{aligned} D(j) \bowtie D(k) &\cong (D(j)^{\mathsf{op}} \invamp D(k)^{\mathsf{op}})^{\mathsf{op}} \\ &\cong (D(j^{\mathsf{op}}) \invamp D(k^{\mathsf{op}}))^{\mathsf{op}} \\ &\cong (D(j^{\mathsf{op}} \times k^{\mathsf{op}}))^{\mathsf{op}} \\ &\cong D((j^{\mathsf{op}} \times k^{\mathsf{op}})^{\mathsf{op}}) \\ &\cong D(j \times k) \end{aligned}$$

for all posets j and k. Q.E.D.

In fact, more is true: sup-lattices of the form $D(j)$ belong to the core of **Sup**, in the sense of Definition 1.4.1, as demonstrated in [38].

Lemma 5.1.7

Given

$$\begin{array}{ccc} & & x \\ & & \downarrow{\scriptstyle \omega} \\ q & \xrightarrow[\beta]{} & y \end{array}$$

there exists a largest $q \xrightarrow{\alpha} x$ such that $\omega \circ \alpha \leq \beta$, which we write as follows:

$$\begin{array}{ccc} & \overset{\alpha}{\nearrow} & x \\ & \swarrow\!\!\!| & \downarrow{\scriptstyle \omega} \\ q & \xrightarrow[\beta]{} & y \end{array}$$

Moreover, if q is of the form $D(j)$, then α can be calculated as the transpose of

$$j \xrightarrow{\eta_j} UD(j) \xrightarrow{U(\beta)} U(y) \xrightarrow{U(\omega)^\sharp} U(x)$$

Proof

Note that

$$q \multimap x \xrightarrow{\omega \circ (-)} q \multimap y$$

is a sup-homomorphism, and therefore

$$U(q \multimap x) \xrightarrow{U(\omega \circ (-))} U(q \multimap y)$$

has a right adjoint, (say) ψ—then $\alpha = \psi(\beta)$ has the desired property. More concretely, $\alpha = \sup\{\gamma \in (q \multimap x) \mid \omega \circ \gamma \leq \beta\}$

Moreover, in the case $q = D(j)$,

$$\begin{array}{ccc} U(q \multimap x) & \xrightarrow{U(\omega \circ (-))} & U(q \multimap y) \\ \sim\downarrow & & \downarrow\sim \\ j \Rightarrow U(x) & \xrightarrow[U(\omega) \circ (-)]{} & j \Rightarrow U(y) \end{array}$$

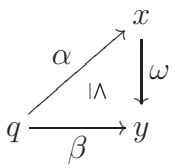
so it suffices to consider the right adjoint of $U(\omega) \circ (-)$, which is precisely $U(\omega)^\sharp \circ (-)$.

Q.E.D.

5.2 Categories of Adherence Spaces

Definition 5.2.1

Let q be an arbitrary sup-lattice. Then we write $q\backslash\backslash\mathbf{Sup}$ for the category whose objects are sup-homomorphisms of the form $(q \xrightarrow{\alpha} x)$ and whose arrows are "lax triangles" of the form



—i.e., an arrow from $(q \xrightarrow{\alpha} x)$ to $(q \xrightarrow{\beta} y)$ is a sup-homomorphism $x \xrightarrow{\omega} y$ satisfying $\omega \circ \alpha \leq \beta$.

We shall refer to $q\backslash\backslash\mathbf{Sup}$ as the category of *(q-)adherence spaces*.

Examples 5.2.2

In the case where $q = P(j)$, for some set j, an adherence space $(q \xrightarrow{\alpha} x)$ amounts to a sup-lattice x equipped with a j-indexed sequence of distinguished points—i.e., an ordinary function $j \longrightarrow x$.

Moreover, by naturality, a morphism of adherence spaces $(q \xrightarrow{\alpha} x) \xrightarrow{\omega} (q \xrightarrow{\beta} y)$ is just a sup-homomorphism $x \xrightarrow{\omega} y$ which maps each distinguished point of x to a point less-than-or-equal-to the corresponding distinguished point of y.

The case where j is a two-element set will feature prominently in the next chapter.

More generally, in the case where $q = D(j)$, for some poset j, an adherence space $(q \xrightarrow{\alpha} x)$ amounts to a sup-lattice x equipped with an monotone function $j \longrightarrow x$—which we shall still think of as a j-indexed sequence of distinguished points.

Theorem 5.2.3

If q is of the form $D(j)$, then there is a canonical equivalence $(q\backslash\backslash\mathbf{Sup})^{\mathsf{op}} \simeq q^*\backslash\backslash\mathbf{Sup}$.

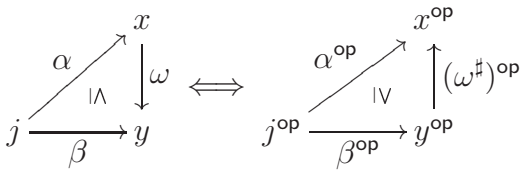
Proof

In effect, this amounts to the observation that any monotone function $j \xrightarrow{\alpha} x$ may be equally well viewed as an monotone function $j^{\mathsf{op}} \xrightarrow{\alpha} x^{\mathsf{op}}$.

Moreover, for any monotone functions $j \xrightarrow{\alpha} x$ and $j \xrightarrow{\beta} y$, and any sup-homomorphism $x \xrightarrow{\omega} y$,

$$\begin{aligned} \omega \circ \alpha \leq \beta &\iff \alpha \leq \omega^\sharp \circ \beta \\ &\iff \alpha^{\mathsf{op}} \geq (\omega^\sharp)^{\mathsf{op}} \circ \beta^{\mathsf{op}} \end{aligned}$$

—i.e., we have



More precisely, given an adherence space $(q \xrightarrow{\alpha} x)$, we take the opposite map of $j \xrightarrow{\eta_j} UD(j) \xrightarrow{U(\alpha)} U(x)$ and extend it along $\eta_{j^{\mathsf{op}}}$ to a sup-homomorphism which we shall call β.

$$\begin{array}{ccccc} j^{\mathsf{op}} & & \xrightarrow{\eta_j^{\mathsf{op}}} & & UD(j)^{\mathsf{op}} \xrightarrow{U(\alpha)^{\mathsf{op}}} U(x)^{\mathsf{op}} \\ {\scriptstyle \eta_{j^{\mathsf{op}}}}\downarrow & & & & \downarrow{\scriptstyle \sim} \\ UD(j^{\mathsf{op}}) & & \xrightarrow[U(\beta)]{} & & U(x^{\mathsf{op}}) \end{array}$$

The dual of $(q \xrightarrow{\alpha} x)$ is then defined to be $(q^* \xrightarrow{\sim} D(j^{\mathsf{op}}) \xrightarrow{\beta} x^{\mathsf{op}} \xrightarrow{\sim} x^*)$.

Then we have

$$\begin{array}{ccc} & x & \\ {\scriptstyle |\Lambda}\nearrow & \downarrow{\scriptstyle \omega} & \\ q & \longrightarrow & y \end{array} \iff \begin{array}{ccc} & x^* & \\ {\scriptstyle |\vee}\nearrow & \uparrow{\scriptstyle \omega^*} & \\ q^* & \longrightarrow & y^* \end{array}$$

so we can define the dual of a map $(q \longrightarrow x) \xrightarrow{\omega} (q \longrightarrow y)$ to be $(q^* \longrightarrow x^*) \xrightarrow{\omega^*} (q^* \longrightarrow y^*)$. Q.E.D.

It is important to distinguish η_j^{op} and $\eta_{j^{\mathsf{op}}}$, which both occur in the proof above. For example, in the case where j is a one-element set, η_j maps that element to the top element of $UD(j) \cong 2$. Therefore η_j^{op} maps the unique element of j to the bottom element of $UD(j)$ whereas $\eta_{j^{\mathsf{op}}} = \eta_j$.

Corollary 5.2.4

If q is of the form $D(j)$, then any order-isomorphism $j \cong j^{\mathsf{op}}$ (and there may be several) induces a duality $(q\backslash\backslash\mathbf{Sup})^{\mathsf{op}} \simeq q\backslash\backslash\mathbf{Sup}$.

In particular, if q is of the form $P(j)$, j a (discretely ordered) set, then any bijection on j induces a duality $(q\backslash\backslash\mathbf{Sup})^{\mathsf{op}} \simeq q\backslash\backslash\mathbf{Sup}$.

By naturality, we have

$$\begin{array}{ccc} j & \xrightarrow{\sim} & j^{\mathsf{op}} \\ {\scriptstyle \eta_j}\downarrow & & \downarrow{\scriptstyle \eta_{j^{\mathsf{op}}}} \\ UD(j) & \xrightarrow{\sim} & UD(j^{\mathsf{op}}) \end{array}$$

so, given an isomorphism $j \xrightarrow{\sim} j^{\mathsf{op}}$, we can calculate the dual of $(q \xrightarrow{\alpha} x)$ as the extension of

$$j \xrightarrow{\sim} j^{\mathsf{op}} \xrightarrow{\eta_j^{\mathsf{op}}} UD(j)^{\mathsf{op}} \xrightarrow{U(\alpha)^{\mathsf{op}}} U(x)^{\mathsf{op}} \xrightarrow{\sim} U(x^*)$$

along η_j.

Remark 5.2.5

One of the curious features of any category of adherence spaces is the presence of a "generic object", namely $(q \xrightarrow{\iota_q} q)$. We shall also write $\sigma^{(i)}$ for ι_q, when convenient.

It is important to note that the duality described in Theorem 5.2.3 does *not* preserve these objects. This is essentially due to the distinction between η_j^{op} and $\eta_{j^{\mathsf{op}}}$.

For example, in the case $q = 2 \cong D(1)$, the duality (which is given by the unique bijection on 1) maps $(2 \xrightarrow{\iota_2} 2)$ to $(2 \xrightarrow{\phi!} 2)$.

Lemma 5.2.6

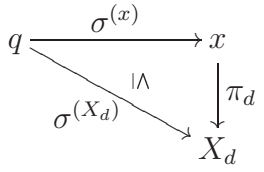
For any q, $q\backslash\backslash\mathbf{Sup}$ is complete.

Proof

The limit of a diagram $\mathcal{D} \xrightarrow{X} q\backslash\backslash\mathbf{Sup}$ is computed as follows: let x denote the limit, in **Sup**, of the diagram

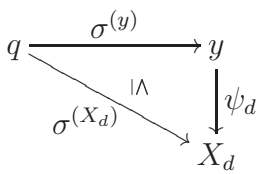
$$\mathcal{D} \xrightarrow{X} q\backslash\backslash\mathbf{Sup} \longrightarrow \mathbf{Sup}$$

and $\sigma^{(x)}$ the largest sup-homomorphism $q \longrightarrow x$ such that



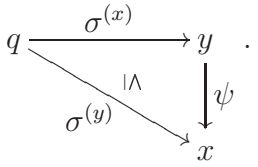
for all $d \in \mathsf{obj}[\mathcal{D}]$.

If $(\ q \xrightarrow{\sigma^{(y)}} y\ )$ is an adherence space, and



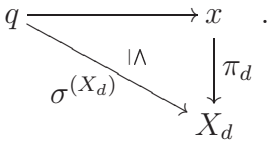
is a cone in $q\backslash\backslash\mathbf{Sup}$, then the arrows $y \xrightarrow{\psi_d} X_d$ form a cone in **Sup**. We therefore obtain a unique map $y \xrightarrow{\psi} x$ such that $\pi_d\psi = \psi_d$ for all $d \in \mathsf{obj}[\mathcal{D}]$.

It remains to show that



.

But, since $\pi_d \circ (\psi \circ \sigma^{(y)}) = \psi_d \circ \sigma^{(y)} \leq \sigma^{(X_d)}$, we have



.

for every $d \in \mathsf{obj}[\mathcal{D}]$. But $\sigma^{(x)}$ was defined to be largest sup-homomorphism with this property. Q.E.D.

In particular, note that $(1, *, *)$ is a terminal object for $q\backslash\backslash\mathbf{Sup}$.

5.3 Monoidal categories of Adherence Spaces

Since **Sup** is $*$-autonomous, it is possible to consider (co-)monoids with respect to $\otimes$, or with respect to $\bowtie$. In some sense (to be discussed below), the concept of a $\bowtie$-comonoid is more natural than its $\otimes$ counterpart.

Nevertheless, it is the notion of $\otimes$-comonoid which permits the proof of the following, new, theorem.

Theorem 5.3.1

Each $\otimes$-comonoid structure on q induces a monoidal closed structure on $q\backslash\backslash\mathbf{Sup}$.

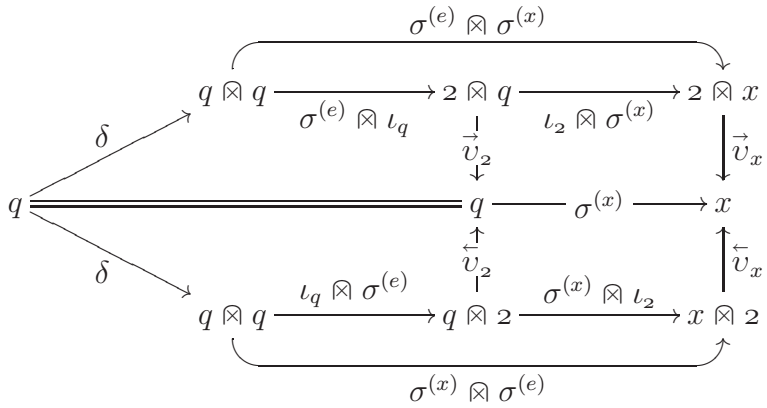
Proof

Let $2 \xleftarrow{\varepsilon} q \xrightarrow{\delta} q \otimes q$ be a comonoid in **Sup**. We define

$$(q \xrightarrow{\sigma^{(x)}} x) \otimes_q (q \xrightarrow{\sigma^{(y)}} y) := (q \xrightarrow{\sigma^{(x \otimes y)}} x \otimes y)$$

where $\sigma^{(x \boxtimes y)}$ denotes the composite $q \xrightarrow{\delta} q \boxtimes q \xrightarrow{\sigma^{(x)} \boxtimes \sigma^{(y)}} x \boxtimes y$. We also write $\sigma^{(e)}$ for ε, when convenient.

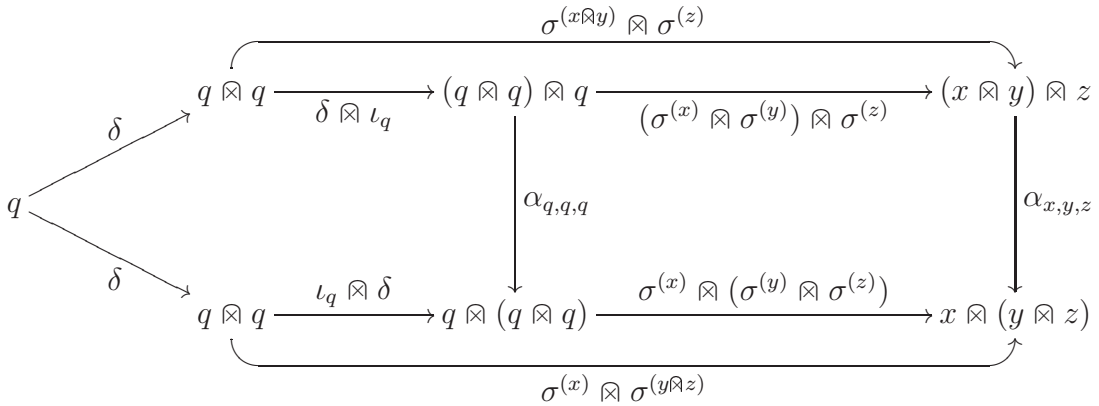
Then the diagrams



define natural isomorphisms

$$(\ q \xrightarrow{\sigma^{(e)}} 2 \ ) \boxtimes_q (\ q \xrightarrow{\sigma^{(x)}} x \ ) \xrightarrow{\vec{\upsilon}_x} (\ q \xrightarrow{\sigma^{(x)}} x \ ) \xleftarrow{\overleftarrow{\upsilon}_x} (\ q \xrightarrow{\sigma^{(x)}} x \ ) \boxtimes_q (\ q \xrightarrow{\sigma^{(e)}} 2 \ )$$

in $q\backslash\backslash\mathbf{Sup}$, and the diagram



defines a natural isomorphism

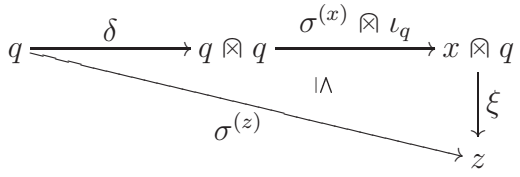
$$(\ q \xrightarrow{\sigma^{(x \boxtimes y)}} x \boxtimes y \ ) \boxtimes_q (\ q \xrightarrow{\sigma^{(z)}} z \ ) \xrightarrow{\alpha_{x,y,z}} (\ q \xrightarrow{\sigma^{(x)}} x \ ) \boxtimes_q (\ q \xrightarrow{\sigma^{(y \boxtimes z)}} y \boxtimes z \ )$$

—hence $(q\backslash\backslash\mathbf{Sup}, (\ q \xrightarrow{\sigma^{(e)}} 2 \ ), \boxtimes_q)$ is a monoidal category.

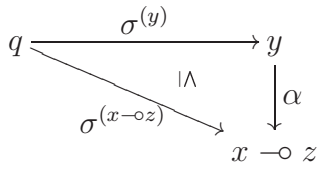
Now we define

$$(q \xrightarrow{\sigma^{(x)}} x) \multimap_q (q \xrightarrow{\sigma^{(z)}} z) := (q \xrightarrow{\sigma^{(x \multimap z)}} x \multimap z)$$

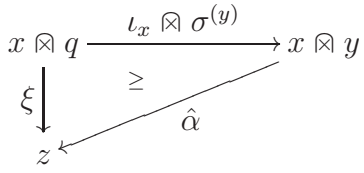
where $\sigma^{(x \multimap z)}$ is the transpose of the largest sup-homomorphism $x \boxtimes q \xrightarrow{\xi} z$ satisfying



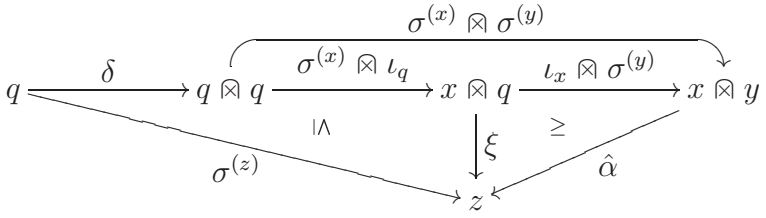
An arrow $(q \xrightarrow{\sigma^{(y)}} y) \xrightarrow{\alpha} (q \xrightarrow{\sigma^{(x)}} x) \multimap_q (q \xrightarrow{\sigma^{(z)}} z)$ is a lax triangle



in **Sup**. But this is equivalent to



—thus we obtain



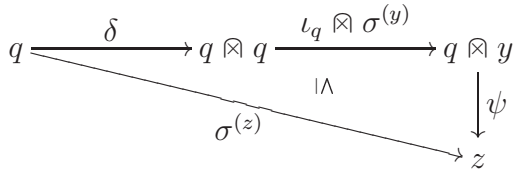
which defines an arrow $(q \xrightarrow{\sigma^{(x)}} x) \boxtimes_q (q \xrightarrow{\sigma^{(y)}} y) \xrightarrow{\hat{\alpha}} (q \xrightarrow{\sigma^{(z)}} z)$ in $q \backslash\backslash \mathbf{Sup}$.

The latter step is reversible by virtue of the maximality criterion in the definition of ξ.

Similarly, we define

$$(q \xrightarrow{\sigma^{(z)}} z) \circ\!\!-_q (q \xrightarrow{\sigma^{(y)}} y) := (q \xrightarrow{\sigma^{(z \circ\!- y)}} z \circ\!\!- y)$$

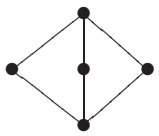
where $\sigma^{(z \circ\!- y)}$ is the transpose of the largest sup-homomorphism $q \otimes y \xrightarrow{\psi} z$ satisfying



Q.E.D.

Remark 5.3.2

$\otimes$-comonoid structures are, in general, rarer than one might expect. For example, the five-element modular but non-distributive lattice, ℓ (displayed below), admits no $\otimes$-comonoid structure.

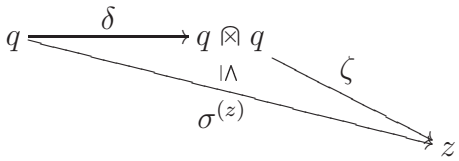


By contrast, $\bowtie$-comonoid structures on ℓ are in bijective correspondence with $\otimes$-monoid structures—i.e., quantale [39] structures—on ℓ^*, and there are plenty of these.

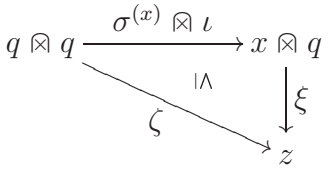
Of course, in the case $q = D(j)$, there is no distinction between the two concepts, by Theorem 5.1.6.

Remark 5.3.3

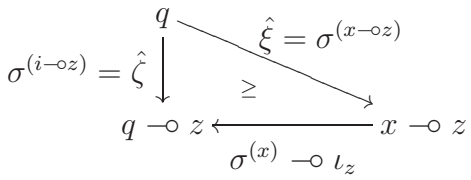
$\circ\!\!-_q$ can always be calculated in two stages: first, find the largest map



then, find the largest map



—but the latter diagram is equivalent to



(where $\sigma^{(i \multimap z)}$ is the structure map of $(q \xrightarrow{\sigma^{(i)}} q) \multimap_q (q \xrightarrow{\sigma^{(z)}} z)$—cf. Remark 5.2.5).

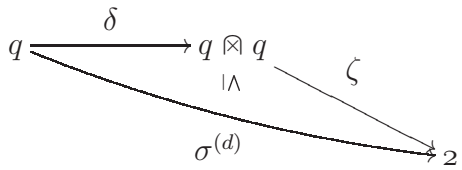
So, by Lemma 5.1.7, in the case $q = D(j)$, we can calculate $\sigma^{(x \multimap z)}$ as the extension of

$$j \rightarrowtail UD(j) \xrightarrow{U(\sigma^{(i \multimap z)})} U(q \multimap z) \xrightarrow{U(\sigma^{(x)} \multimap \iota_z)^\sharp} U(x \multimap z)$$

to q.

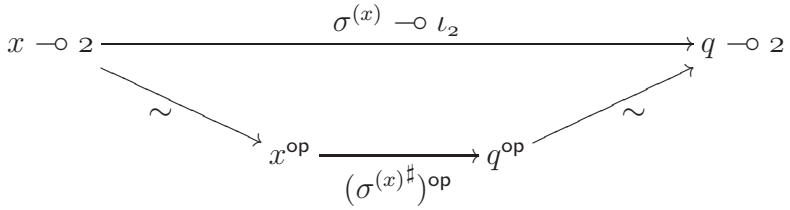
This remark is particularly useful in the case of a potential dualising object $(q \xrightarrow{\sigma^{(d)}} 2)$.

Firstly, note that $\sigma^{(d)}$ is necessarily of the form $\ulcorner \ni \urcorner$ for some $\ni \in q$. Then ζ satisfies the lax triangle



if and only if $\zeta(\delta(\ni)) = \phi$. Hence the largest such ζ is simply $\ulcorner \delta(\ni) \urcorner$.

Moreover, the right adjoint of

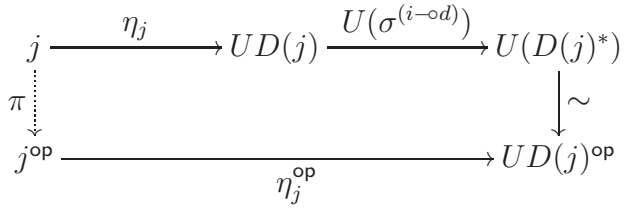


is simply $q \multimap 2 \xrightarrow{\sim} q^{\mathsf{op}} \xrightarrow{(\sigma^{(x)})^{\mathsf{op}}} x^{\mathsf{op}} \xrightarrow{\sim} x \multimap 2$. [$\sigma^{(x)}$ is the left adjoint of $\sigma^{(x)\sharp}$, but $(-)^{\mathsf{op}}$ reverses adjoints!]

So, in the case $q = D(j)$, $(\ q \xrightarrow{\sigma^{(x)}} x\ ) \multimap_q (\ q \xrightarrow{\sigma^{(d)}} 2\ )$ can be calculated as the transpose of

$$j \rightarrowtail UD(j) \xrightarrow{U(\sigma^{(i \multimap d)})} U(q^*) \cong U(q)^{\mathsf{op}} \xrightarrow{(U(\sigma^{(x)}))^{\mathsf{op}}} U(x)^{\mathsf{op}} \cong U(x^*)$$

This brings us preciously close to our original duality of Corollary 5.2.4. What we still need to know is that the composite $j \rightarrowtail UD(j) \xrightarrow{U(\sigma^{(i \multimap d)})} U(D(j)^*) \cong UD(j)^{\mathsf{op}}$ factors through η_j^{op}.



If this occurs, then $\sigma^{(x \multimap d)}$ can indeed be calculated as the extension of

$$j \xrightarrow{\pi} j^{\mathsf{op}} \xrightarrow{\eta_j^{\mathsf{op}}} UD(j)^{\mathsf{op}} \xrightarrow{(\sigma^{(x)})^{\mathsf{op}}} U(x)^{\mathsf{op}} \xrightarrow{\sim} U(x^*)$$

along η_j.

Theorem 5.3.4

For $(\ q \xrightarrow{\sigma^{(d)}} 2\ )$ to be a dualising object for $(q\backslash\backslash\mathbf{Sup}, (\ q \xrightarrow{\sigma^{(e)}} 2\ ), \otimes_q, \multimap_q, {}_q\multimap)$ in the case where $q = D(j)$, it suffices to show that we can find π, ϱ such that

1. $$\begin{array}{ccccc} j & \xrightarrow{\eta_j} & UD(j) & \xrightarrow{U(\sigma^{(i \multimap d)})} & U(D(j)^*) \\ {\scriptstyle \pi}\,\vdots & & & & \downarrow \sim \\ j^{\mathsf{op}} & & \xrightarrow[\eta_j^{\mathsf{op}}]{} & & UD(j)^{\mathsf{op}} \end{array} ,$$

2. $$\begin{array}{ccccc} j & \xrightarrow{\eta_j} & UD(j) & \xrightarrow{U(\sigma^{(d \multimap i)})} & U(D(j)^*) \\ {\scriptstyle \varrho}\,\vdots & & & & \downarrow \sim \\ j^{\mathsf{op}} & & \xrightarrow[\eta_j^{\mathsf{op}}]{} & & UD(j)^{\mathsf{op}} \end{array} , \text{ and}$$

3. π and ϱ^{op} are inverse.

Proof

In this case, the functors $(-) \multimap_q (q \xrightarrow{\sigma^{(d)}} 2)$ and $(q \xrightarrow{\sigma^{(d)}} 2) \circ\!\!- _q (-)$ coincide with the two halves of the duality given by $j \xrightarrow{\pi} j^{\mathsf{op}}$ according to Corollary 5.2.4. Q.E.D.

5.4 Comparison with poset-valued sets

In this section, we compare our notion of adherence space with the related notion of q-valued set, due to Schalk and de Paiva [40].

Definition 5.4.1

Let q be an arbitrary poset. Then a *q-valued set* consists of a set x together with a function $x \xrightarrow{\varsigma^{(x)}} q$. A morphism of q-valued sets

$$(x \xrightarrow{\varsigma^{(x)}} q) \xrightarrow{r} (y \xrightarrow{\varsigma^{(y)}} q)$$

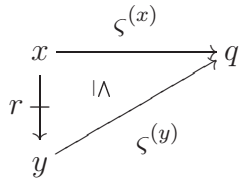
is a relation $r \subseteq x \times y$ such that $\alpha r \beta$ implies $\varsigma^{(x)}(\alpha) \leq \varsigma^{(y)}(\beta)$ for all $\alpha \in x$ and $\beta \in y$; or, equivalently,

$$\sup_{\alpha \in x, \alpha r \beta} \varsigma^{(x)}(\alpha) \quad \leq \quad \varsigma^{(y)}(\beta)$$

for all $\beta \in y$.

We write q-**Val** for the category of q-valued sets and their morphisms.

Schalk and de Paiva write, suggestively,



to mean that r is a morphism of q-valued sets; we shall follow this convention despite the fact that it jars slightly with our previous use of such diagrams, as demonstrated below.

Remark 5.4.2

In the case where q is complete, $\varsigma^{(x)}$ extends to a sup-homomorphism $P(x) \xrightarrow{\hat{\varsigma}^{(x)}} q$. Now the left-hand side of the inequality displayed in Definition 5.4.1 can be re-written as follows:

$$\begin{aligned} \sup_{\alpha \in x, \alpha r \beta} \varsigma^{(x)}(\alpha) &= \hat{\varsigma}^{(x)}(\{\alpha \in x \mid \alpha r \beta\}) \\ &= \hat{\varsigma}^{(x)}(\overleftarrow{r}\ (\{\beta\})) \end{aligned}$$

—where $\overleftarrow{r}$ denotes the *inverse* image map associated to r. So, for r to be a morphism of q-valued sets, one must have

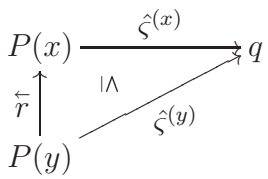
$$(\hat{\varsigma}^{(x)} \circ \overleftarrow{r})(\{\beta\}) \leq \varsigma^{(y)}(\beta) = \hat{\varsigma}^{(y)}(\{\beta\})$$

for all $\beta \in y$, and therefore

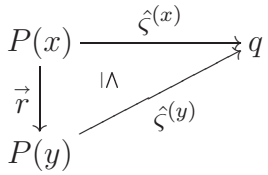
$$(\hat{\varsigma}^{(x)} \circ \overleftarrow{r})(\psi) \leq \hat{\varsigma}^{(y)}(\psi)$$

for all $\psi \in P(y)$.

Note that the diagrammatic expression of the latter statement is



and not, as one might have expected,



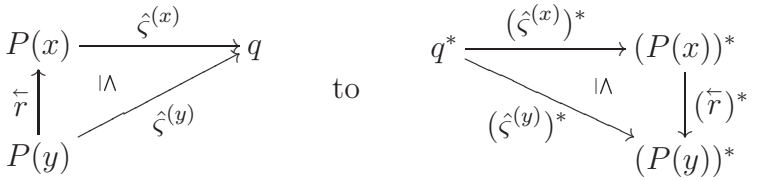
—where $\vec{r}$ denotes the *direct* image map associated to r.

Theorem 5.4.3

If q is complete, then q-**Val** is isomorphic to a full subcategory of $q^* \backslash\backslash \mathbf{Sup}$.

Proof

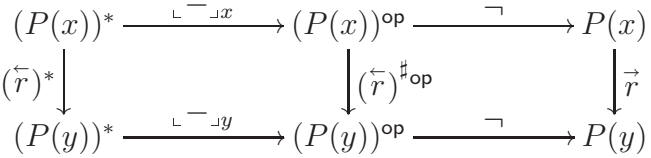
What Remark 5.4.2 demonstrates is that q-**Val** is isomorphic to a full subcategory of $(\mathbf{Sup}//q)^{\mathsf{op}}$—namely, the full subcategory consisting of those objects of the form $(P(x) \longrightarrow q)$. But $(\mathbf{Sup}//q)^{\mathsf{op}}$ is clearly isomorphic to $q^* \backslash\backslash \mathbf{Sup}$ by taking duals. Thus, in particular, we map



Q.E.D.

Remark 5.4.4

The isomorphism described in the proof above can be refined by the observation that the diagram



commutes.

I.e., instead of mapping the q-valued set $x \xrightarrow{\varsigma^{(x)}} q$ to $q^* \xrightarrow{(\hat{\varsigma}^{(x)})^*} (P(x))^*$, we may map it to the composite

$$q^* \xrightarrow{(\hat{\varsigma}^{(x)})^*} (P(x))^* \xrightarrow{\llcorner - \lrcorner x} (P(x))^{\mathsf{op}} \xrightarrow{\neg} P(x)$$

—then a morphism of q-valued sets $(x \xrightarrow{\varsigma^{(x)}} q) \xrightarrow{r} (y \xrightarrow{\varsigma^{(y)}} q)$ is mapped to $\vec{r}$.

Moreover, if q is of the form $P(j)$, then the isomorphism $q^*\backslash\backslash\mathbf{Sup} \cong q\backslash\backslash\mathbf{Sup}$ induced by the map $q \xrightarrow{\neg} q^{\mathsf{op}} \xrightarrow{\llcorner - \lrcorner q^{-1}} q^*$ maps said composite

$$q^* \xrightarrow{(\hat{\varsigma}^{(x)})^*} (P(x))^* \xrightarrow{\llcorner - \lrcorner x} (P(x))^{\mathsf{op}} \xrightarrow{\neg} P(x)$$

to the composite

$$\begin{array}{ccccc} q & \xrightarrow{\neg} & q^{\mathsf{op}} & \xrightarrow{\llcorner - \lrcorner q^{-1}} & q^* \\ & & & & \downarrow{\scriptstyle (\hat{\varsigma}^{(x)})^*} \\ P(x) & \xleftarrow[\neg]{} & (P(x))^{\mathsf{op}} & \xleftarrow[\llcorner - \lrcorner x]{} & (P(x))^* \end{array}$$

—which can be more easily expressed as $\overleftarrow{s}$, where $s \subseteq x \times j$ denotes the relation corresponding to the original q-valued set, $x \xrightarrow{\varsigma^{(x)}} P(j)$.

The advantage of q-valued sets, as compared to q-adherence spaces, is that their monoidal-closed structures may be described in a convenient "pointwise" fashion.

Definitions 5.4.5

Suppose $(q, \eta, \&, \rightarrow, \leftarrow)$ is a monoidal-closed poset. Then we define

$$(x \xrightarrow{\varsigma^{(x)}} q) \otimes (y \xrightarrow{\varsigma^{(y)}} q) := (x \times y \xrightarrow{\varsigma^{(x)} \times \varsigma^{(y)}} q \times q \xrightarrow{\&} q)$$

$$(x \xrightarrow{\varsigma^{(x)}} q) \multimap (z \xrightarrow{\varsigma^{(z)}} q) := (x \times z \xrightarrow{\varsigma^{(x)} \times \varsigma^{(z)}} q \times q \xrightarrow{\rightarrow} q)$$

$$(z \xrightarrow{\varsigma^{(z)}} q) \circ\!\!- (y \xrightarrow{\varsigma^{(y)}} q) := (z \times y \xrightarrow{\varsigma^{(z)} \times \varsigma^{(y)}} q \times q \xrightarrow{\leftarrow} q)$$

Theorem 5.4.6 ([40])

$(q\text{-}\mathbf{Val}, (1 \xrightarrow{\eta} q), \boxtimes, \multimap, \circ\!\!-)$ is a monoidal-closed category.

Moreover, if $\ni$ is a dualising element for $(q, \eta, \&, \rightarrow, \leftarrow)$ then $(1 \xrightarrow{\ni} q)$ is a dualising object for $(q\text{-}\mathbf{Val}, (1 \xrightarrow{\eta} q), \boxtimes, \multimap, \circ\!\!-)$.

We now return to the case where q is of the form $P(j)$.

Theorem 5.4.7

If q is of the form $P(j)$, then the monoidal structures on $q\text{-}\mathbf{Val}$ and $q\backslash\backslash\mathbf{Sup}$ are the same.

Proof

Suppose that

$$P(j) \times P(j) \xrightarrow{\quad\&\quad} P(j) \xleftarrow{\quad\eta\quad} 1$$

is the monoidal closed structure corresponding to

$$j \times j \xrightarrow{\quad m \quad} j \xleftarrow{\quad e \quad} 1$$

and that $(\, x \xrightarrow{\varsigma^{(x)}} q \,)$, $(\, y \xrightarrow{\varsigma^{(y)}} q \,)$ correspond to $(\, j \xrightarrow{s^{(x)}} x \,)$, $(\, j \xrightarrow{s^{(y)}} y \,)$, respectively. I.e., $\gamma s^{(x)} \alpha \iff \gamma \in \varsigma^{(x)}(\alpha)$ and $\gamma s^{(y)} \beta \iff \gamma \in \varsigma^{(y)}(\beta)$.

Then we have

$$\begin{aligned}
\gamma s^{(x \boxtimes y)}(\alpha, \beta) &\iff (\exists \kappa, \xi \in j)(\gamma m^{\perp}(\kappa, \xi) \wedge \kappa s^{(x)} \alpha \wedge \xi s^{(y)} \beta) \\
&\iff (\exists \kappa, \xi \in j)((\gamma \in \{\kappa\} \,\&\, \{\xi\}) \wedge (\kappa \in \varsigma^{(x)}(\alpha)) \wedge (\xi \in \varsigma^{(y)}(\beta))) \\
&\iff \gamma \in \varsigma^{(x)}(\alpha) \,\&\, \varsigma^{(y)}(\beta)
\end{aligned}$$

—where the latter step is justified by:

$$\begin{aligned}
\varsigma^{(x)}(\alpha) \,\&\, \varsigma^{(y)}(\beta) &= \left[\bigcup_{\kappa \in \varsigma^{(x)}(\alpha)} \{\kappa\} \right] \,\&\, \left[\bigcup_{\xi \in \varsigma^{(y)}(\beta)} \{\xi\} \right] \\
&= \bigcup_{\kappa \in \varsigma^{(x)}(\alpha)} \left[\{\kappa\} \,\&\, \bigcup_{\xi \in \varsigma^{(y)}(\beta)} \{\xi\} \right] \\
&= \bigcup_{\kappa \in \varsigma^{(x)}(\alpha), \xi \in \varsigma^{(y)}(\beta)} \{\kappa\} \,\&\, \{\xi\}
\end{aligned}$$

So the adherence space

$$j \xrightarrow{\ m^{\perp}\ \mid} j \times j \xrightarrow{\ s^{(x)} \times s^{(y)}\ \mid} x \times y$$

corresponds to the q-valued set

$$x \times y \xrightarrow{\ \varsigma^{(x)} \times \varsigma^{(y)}\ } q \times q \xrightarrow{\quad \& \quad} q$$

Q.E.D.

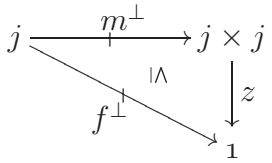
Theorem 5.4.8

If $\ni$ is a dualising element for $q = P(j)$, then $q \xrightarrow{\ulcorner \ni \urcorner} 2$ is a dualising object for $q\backslash\backslash\mathbf{Sup}$.

Proof

Essentially this amounts to the fact that in order to be invertible, $(-) \to \ni$ and $\ni \leftarrow (-)$ must map atoms—i.e., singletons—to co-atoms, i.e., complements of singletons.

To compute $s^{(i \multimap d)}$, we need to find the largest z such that



where f denotes the relation $1 \rightarrowtail j$ which relates $*$ to $\ni$.

By an easy direct computation

$$\begin{aligned}
(\gamma,\kappa)z* &\iff (\forall \xi \in j)(\xi m^{\perp}(\gamma,\kappa) \Rightarrow \xi f^{\perp} *) \\
&\iff (\forall \xi \in j)((\gamma,\kappa)m\xi \Rightarrow *f\xi) \\
&\iff \{\gamma\} \,\&\, \{\kappa\} \subseteq \ni \\
&\iff \{\kappa\} \subseteq \{\gamma\} \to \ni \\
&\iff \kappa \in \{\gamma\} \to \ni
\end{aligned}$$

Now the transpose of z in **Rel** is a relation $j \rightarrowtail j$ which extends to a function $j \longrightarrow P(j)$; namely,

$$\gamma \mapsto \{\kappa \in j \mid (\gamma,\kappa)z*\} = \{\gamma\} \to \ni$$

But the transpose in **Rel** of the sup-homomorphism $P(j) \otimes P(j) \cong P(j \times j) \xrightarrow{\zeta} P(1) \cong 2$ corresponding to z ought to be of the form $P(j) \longrightarrow P(j)^*$, corresponding to a function $j \longrightarrow P(j)^{\mathsf{op}}$.

This indicates the perils of too closely identifying the monoidal-closed structure of **Rel** with that of **Sup**. In order to obtain the correct transpose, we must follow

$$\gamma \mapsto \{\kappa \in j \mid (\gamma, \kappa) z *\} = \{\gamma\} \to \mathfrak{d}$$

by the complementation map $P(j) \longrightarrow P(j)^{\mathsf{op}}$ (which is a sup-homomorphism).

Thus we obtain

$$\gamma \mapsto j \setminus (\{\gamma\} \to \mathfrak{d})$$

whose range does lie in that of η_j^{op}—i.e., among the singletons. Q.E.D.

5.5 $P(2)$-adherence spaces

We devote the rest of this chapter to a single example.

Let 2 denote the unordered set $\{0, 1\}$. [Note: $2 \neq 2$!] Then each monoid structure on 2 in **Set** determines a comonoid structure on 2 in **Rel**. [Also, the unique comonoid structure on 2 in **Set** determines a monoid structure on 2 in **Rel**. In fact, there are a total of five (co)monoid structures on 2 in **Rel**.]

We shall consider the structure determined in this way by addition—i.e.,

$$1 \xleftarrow{0^{\perp}} 2 \xrightarrow{+^{\perp}} 2 \times 2.$$

Viewed in **Sup**, this comonoid may be described as follows:

$$2 \xleftarrow{\varepsilon} P(2) \xrightarrow{\delta} P(2) \otimes P(2)$$

$$\top \longleftarrow \{0\} \longmapsto (\{0\} \otimes \{0\}) \vee (\{1\} \otimes \{1\})$$

$$\perp \longleftarrow \{1\} \longmapsto (\{0\} \otimes \{1\}) \vee (\{1\} \otimes \{0\})$$

As discussed previously, a $P(2)$-adherence space may be thought of simply as a sup-lattice x together with two distinguished points, α_0, α_1, which we write as a triple

(x, α_0, α_1). To be precise, $\alpha_0 = \sigma^{(x)}(\{0\})$ and $\alpha_1 = \sigma^{(x)}(\{1\})$. Thus the tensor unit $P(2) \xrightarrow{\varepsilon} 2$ will be written as $(2, \top, \bot)$.

Chasing through the definition of $\circledR_q$ (Proof of Theorem 5.3.1), we see that the tensor product of two $P(2)$-adherence spaces may be conveniently described as follows:

$$(x, \alpha_0, \alpha_1) \circledR (y, \beta_0, \beta_1) = (x \circledR y, (\alpha_0 \circledR \beta_0) \vee (\alpha_1 \circledR \beta_1), (\alpha_0 \circledR \beta_1) \vee (\alpha_1 \circledR \beta_0))$$

A curious feature of this category is that it possesses two non-isomorphic dualising objects: $(2, \top, \bot)$ and $(2, \bot, \top)$.

In the first case, we obtain the following 'negation' operation:

$$(x, \alpha_0, \alpha_1) \mapsto (x^*, \llcorner\alpha_1\lrcorner, \llcorner\alpha_0\lrcorner) \cong (x^{\mathsf{op}}, \alpha_1, \alpha_0)$$

—whereas in the second, we have

$$(x, \alpha_0, \alpha_1) \mapsto (x^*, \llcorner\alpha_0\lrcorner, \llcorner\alpha_1\lrcorner) \cong (x^{\mathsf{op}}, \alpha_0, \alpha_1)$$

Henceforth we shall consider only the former structure.

An easy calculation shows that we then have

$$(x, \alpha_0, \alpha_1) \barwedge (y, \beta_0, \beta_1) = (x \barwedge y, (\alpha_0 \barwedge \beta_1) \wedge (\alpha_1 \barwedge \beta_0), (\alpha_0 \barwedge \beta_0) \wedge (\alpha_1 \barwedge \beta_1))$$

where, for any $\alpha \in x$ and $\beta \in y$, $\alpha \barwedge \beta$ denotes $\llcorner(\llcorner\alpha\lrcorner \circledR \llcorner\beta\lrcorner)\lrcorner \in (x^* \circledR y^*)^* = x \barwedge y$. [See Lemma 5.5.2 below for a more concrete understanding of $\alpha \barwedge \beta$.]

We shall now attempt to characterise the core of $P(2)\backslash\backslash\mathbf{Sup}$. Clearly, x must belong to the core of $\mathbf{Sup}$, in order for (x, α_0, α_1) to belong to the core of $P(2)\backslash\backslash\mathbf{Sup}$; such x are called completely distributive [38].

Now consider the adherence spaces $(2, \phi, \phi)$ and $(2, \tau, \tau)$. Using the formulas presented above we see that

$$\begin{aligned}(x, \alpha_0, \alpha_1) \circledR (2, \phi, \phi) &= (x \circledR 2, (\alpha_0 \circledR \phi) \vee (\alpha_1 \circledR \phi), (\alpha_0 \circledR \phi) \vee (\alpha_1 \circledR \phi)) \\ &\cong (x, \phi, \phi)\end{aligned}$$

whereas

$$\begin{aligned}(x, \alpha_0, \alpha_1) \barwedge (2, \phi, \phi) &= (x \barwedge 2, (\alpha_0 \barwedge \phi) \wedge (\alpha_1 \barwedge \phi), (\alpha_0 \barwedge \phi) \wedge (\alpha_1 \barwedge \phi)) \\ &\cong (x, \alpha_0 \wedge \alpha_1, \alpha_0 \wedge \alpha_1)\end{aligned}$$

—so, in order for (x, α_0, α_1) to be core, we must have $\alpha_0 \wedge \alpha_1 = \phi$.

Similarly,

$$\begin{aligned}(x, \alpha_0, \alpha_1) \circledR (2, \tau, \tau) &= (x \circledR 2, (\alpha_0 \circledR \tau) \vee (\alpha_1 \circledR \tau), (\alpha_0 \circledR \tau) \vee (\alpha_1 \circledR \tau)) \\ &\cong (x, \alpha_0 \vee \alpha_1, \alpha_0 \vee \alpha_1)\end{aligned}$$

whereas

$$\begin{aligned}(x, \alpha_0, \alpha_1) \circledL (2, \tau, \tau) &= (x \circledL 2, (\alpha_0 \circledL \tau) \wedge (\alpha_1 \circledL \tau), (\alpha_0 \circledL \tau) \wedge (\alpha_1 \circledL \tau)) \\ &\cong (x, \tau, \tau)\end{aligned}$$

—so, in order for (x, α_0, α_1) to be core, we must also have $\alpha_0 \vee \alpha_1 = \tau$.

Theorem 5.5.1

The necessary conditions enumerated above—namely, that x be completely distributive and that α_0 and α_1 are complementary in x—also suffice for an adherence space (x, α_0, α_1) to be core.

Our proof hinges on the following description of invertible mix maps in **Sup**.

Lemma 5.5.2

For a completely distributive sup-lattice x and an arbitrary sup-lattice y, the mix isomorphism $x \circledR y \xrightarrow{\sim} x \circledL y$ is given by the formula

$$\alpha \circledR \beta \mapsto (\alpha \circledL \bot) \wedge (\bot \circledL \beta)$$

Similarly, the inverse map $x \circledL y \xrightarrow{\sim} x \circledR y$ is given by the formula

$$\alpha \circledL \beta \mapsto (\alpha \circledR \top) \wedge (\top \circledR \beta)$$

Proof

In general, the mix map $x \circledR y \longrightarrow x \circledL y$ is given by the formula

$$\alpha \circledR \beta \mapsto [\omega \circledR \psi \Rrightarrow \omega(\alpha) \wedge \psi(\beta)]$$

But recall that $\alpha \bowtie \beta$ can be described as follows:

$$\omega \boxtimes \psi \mapsto \begin{cases} \omega(\alpha) \vee \psi(\beta) & \text{if } \omega \neq \bot \text{ and } \psi \neq \bot \\ \bot & \text{if } \omega = \bot \text{ or } \psi = \bot \end{cases}$$

Thus $\alpha \bowtie \bot$ has the following action on non-trivial pure tensors

$$\omega \boxtimes \psi \mapsto \omega(\alpha) \vee \psi(\bot) = \omega(\alpha)$$

But, $\psi(\top) = \bot$ if and only if $\psi = \bot$—in all other cases we have $\psi(\top) = \top$. Hence the action of $\mathsf{mix}(\alpha \boxtimes \top)$ on non-trivial pure tensors is also

$$\omega \boxtimes \psi \mapsto \omega(\alpha) \wedge \psi(\top) = \omega(\alpha)$$

Hence the formula

$$\mathsf{mix}(\alpha \boxtimes \top) = \alpha \bowtie \bot$$

hold for all sup-lattices x and y; by symmetry, $\mathsf{mix}(\top \boxtimes \beta) = \bot \bowtie \beta$ also.

Now, if mix is invertible, then it preserves meets as well as joins! Hence

$$\begin{aligned} \mathsf{mix}(\alpha \boxtimes \beta) &= \mathsf{mix}((\alpha \boxtimes \top) \wedge (\top \boxtimes \beta)) \\ &= (\alpha \bowtie \bot) \wedge (\bot \bowtie \beta) \end{aligned}$$

[In general one can only be sure that $\mathsf{mix}(\alpha \boxtimes \beta) \leq (\alpha \bowtie \bot) \wedge (\bot \bowtie \beta)$.] Q.E.D.

Returning to our main calculation ...

Proof of Theorem

In a distributive lattice, we have

$$\begin{aligned} (\alpha_0 \bowtie \beta_1) \wedge (\alpha_1 \bowtie \beta_0) &= [(\alpha_0 \bowtie \bot) \vee (\bot \bowtie \beta_1)] \wedge [(\alpha_1 \bowtie \bot) \vee (\bot \bowtie \beta_0)] \\ &= [(\alpha_0 \bowtie \bot) \wedge (\alpha_1 \bowtie \bot)] \vee [(\alpha_0 \bowtie \bot) \wedge (\bot \bowtie \beta_0)] \\ &\quad \vee [(\alpha_1 \bowtie \bot) \wedge (\bot \bowtie \beta_1)] \vee [(\bot \bowtie \beta_0) \wedge (\bot \bowtie \beta_1)] \\ &= [(\alpha_0 \wedge \alpha_1) \bowtie \bot] \vee \mathsf{mix}(\alpha_0 \boxtimes \beta_0) \vee \mathsf{mix}(\alpha_1 \boxtimes \beta_1) \\ &\quad \vee [\bot \bowtie (\beta_0 \wedge \beta_1)] \end{aligned}$$

Now it is clear that if $(\alpha_0 \wedge \alpha_1) = \bot$, then the first term $\bot \bowtie \bot = \bot$ disappears. What is less clear, but equally true, is that if $(\alpha_0 \vee \alpha_1) = \top$, then the last term is absorbed into the others.

$$\begin{aligned}
\bot \bowtie (\beta_0 \wedge \beta_1) &= \mathsf{mix}(\top \boxtimes (\beta_0 \wedge \beta_1)) \\
&= \mathsf{mix}((\alpha_0 \vee \alpha_1) \boxtimes (\beta_0 \wedge \beta_1)) \\
&= \mathsf{mix}([\alpha_0 \boxtimes (\beta_0 \wedge \beta_1)] \vee [\alpha_1 \boxtimes (\beta_0 \wedge \beta_1)]) \\
&\leq \mathsf{mix}((\alpha_0 \boxtimes \beta_0) \vee (\alpha_1 \boxtimes \beta_1))
\end{aligned}$$

Mutatis mutandis,

$$\begin{aligned}
&(\alpha_0 \bowtie \beta_0) \wedge (\alpha_1 \bowtie \beta_1) \\
&\quad = [(\alpha_0 \wedge \alpha_1) \bowtie \bot] \vee \mathsf{mix}(\alpha_0 \boxtimes \beta_1) \vee \mathsf{mix}(\alpha_1 \boxtimes \beta_0) \vee [\bot \bowtie (\beta_0 \wedge \beta_1)] \\
&\quad = \mathsf{mix}((\alpha_0 \boxtimes \beta_1) \vee (\alpha_1 \boxtimes \beta_0))
\end{aligned}$$

if $(\alpha_0 \wedge \alpha_1) = \bot$ and $(\alpha_0 \vee \alpha_1) = \top$. Q.E.D.

Chapter 6

Homotopy of Adherence spaces

We finally return to the project, described in the introduction, of finding a Quillen model structure on a $*$-autonomous category with finite products and coproducts in such a way that the resultant homotopy category is compact closed.

The $*$-autonomous category we shall use is that of the preceding section: $P(2)\backslash\backslash\mathbf{Sup}$, with the $*$-autonomous structure described therein. Since this $*$-autonomous category satisfies the mix rule, it suffices to ensure that the class of weak equivalences contains the class of mix maps, and that the homotopy category preserves the $*$-autonomous structure of $P(2)\backslash\backslash\mathbf{Sup}$; for the latter condition ensures that we will have mix maps in $\mathsf{Ho}[P(2)\backslash\backslash\mathbf{Sup}]$ which are the image of those in $P(2)\backslash\backslash\mathbf{Sup}$, while the former condition ensures that these very maps are invertible.

6.1 Definitions

In light of Theorem 5.5.1, we make the following definitions.

Definitions 6.1.1

We say that a $P(2)$-adherence space (x, α_0, α_1) is:

1. *wholly coloured* if $\alpha_0 \vee \alpha_1 = \top$;

2. *distinctly coloured* if $\alpha_0 \wedge \alpha_1 = \bot$;

3. *perfectly coloured* if it is both wholly and distinctly coloured.

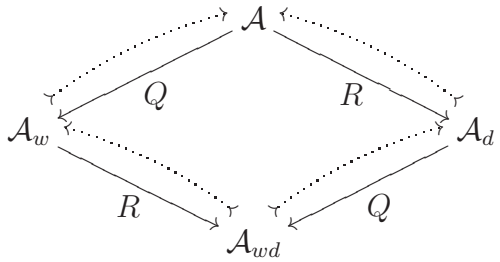
We write $\mathcal{A}_w$, $\mathcal{A}_d$ and $\mathcal{A}_{wd}$ for the full subcategories of $\mathcal{A} = P(2)\backslash\backslash\mathbf{Sup}$ consisting of wholly, distinctly, and perfectly coloured spaces, respectively.

To state that (x, α_0, α_1) is perfectly coloured is equivalent to the assertion that the structure map $P(2) \xrightarrow{\sigma^{(x)}} x$ is a lattice homomorphism. But the advantage of dividing this property into two pieces is demonstrated by the following theorem.

Theorem 6.1.2

$\mathcal{A}_d$ is a reflective subcategory of $\mathcal{A}$; dually, $\mathcal{A}_w$ is co-reflective. Moreover, the reflection $\mathcal{A} \longrightarrow \mathcal{A}_d$, which we call R, restricts to a reflection $\mathcal{A}_w \longrightarrow \mathcal{A}_{wd}$, which we also call R. Dually, the co-reflection $\mathcal{A} \longrightarrow \mathcal{A}_w$, which we call Q, restricts to a co-reflection $\mathcal{A}_d \longrightarrow \mathcal{A}_{wd}$, which we also call Q.

The diagram



commutes, and we have natural isomorphisms $(Q(x, \alpha_0, \alpha_1))^* \cong R((x, \alpha_0, \alpha_1)^*)$ and $(R(x, \alpha_0, \alpha_1))^* \cong Q((x, \alpha_0, \alpha_1)^*)$. In particular, x is wholly coloured if and only if x^* is distinctly coloured.

Proof

Suppose that (x, α_0, α_1) is wholly coloured, and that $(x, \alpha_0, \alpha_1) \xrightarrow{\omega} (y, \beta_0, \beta_1)$ is an arbitrary morphism of adherence spaces. Then

$$\begin{aligned} \omega(\top_x) &= \omega(\alpha_0 \vee \alpha_1) \\ &= \omega(\alpha_0) \vee \omega(\alpha_1) \\ &\leq \beta_0 \vee \beta_1 \end{aligned}$$

—hence the range of ω is contained in $\downarrow(\beta_0 \vee \beta_1) \subseteq y$.

Equivalently, ω factors through $(\downarrow(\beta_0 \vee \beta_1), \beta_0, \beta_1)$—which is a wholly coloured adherence space. I.e., the co-reflection $\mathcal{A} \xrightarrow{Q} \mathcal{A}_w$ is given by the formula

$$Q(y, \beta_0, \beta_1) = (\downarrow(\beta_0 \vee \beta_1), \beta_0, \beta_1)$$

—the co-unit $Q(y, \beta_0, \beta_1) \longrightarrow (y, \beta_0, \beta_1)$ is just the inclusion.

Now it is tautologous that $(\downarrow(\beta_0 \vee \beta_1), \beta_0, \beta_1)$ is disjointly coloured if (and only if) (y, β_0, β_1) is.

The dual theorem holds by general nonsense, that is, we could define the reflection $\mathcal{A} \longrightarrow \mathcal{A}_d$ by the formula

$$R(x, \alpha_0, \alpha_1) = (x/\downarrow(\alpha_0 \wedge \alpha_1), [\alpha_0], [\alpha_1])$$

—where $x/\downarrow(\alpha_0 \wedge \alpha_1)$, of course, denotes the coequaliser of $\downarrow(\alpha_0 \wedge \alpha_1) \underset{\top}{\rightrightarrows} x$. But it will be more convenient to provide an alternate definition of R.

Note that the duality $\mathbf{Sup}^{\mathsf{op}} \simeq \mathbf{Sup}$ allows one to calculate colimits more efficiently than is usual. In particular, we have a sup-isomorphism

$$x/\downarrow(\alpha_0 \wedge \alpha_1) \xrightarrow{\sim} \uparrow(\alpha_0 \wedge \alpha_1) := \{\xi \in x \mid \xi \geq (\alpha_0 \wedge \alpha_1)\}$$

given by $[\xi] \mapsto \xi \vee (\alpha_0 \wedge \alpha_1)$. [In effect, we pick out the largest element of each equivalence class.] Hence, using the absorption rule $\alpha \vee (\alpha \wedge \beta) = \alpha$, we see that the adherence space $(x/\downarrow(\alpha_0 \wedge \alpha_1), [\alpha_0], [\alpha_1])$ is isomorphic to

$$(\uparrow(\alpha_0 \wedge \alpha_1), \alpha_0, \alpha_1)$$

and it is convenient to take the latter as the definition of $R(x, \alpha_0, \alpha_1)$. Under this description of R, the unit map $(x, \alpha_0, \alpha_1) \longrightarrow R(x, \alpha_0, \alpha_1)$ is given by $\xi \mapsto \xi \vee (\alpha_0 \wedge \alpha_1)$.

Now, according to the definitions given above, both $QR(x, \alpha_0, \alpha_1)$ and $RQ(x, \alpha_0, \alpha_1)$ equal the adherence space

$$(\{\xi \in x \mid \alpha_0 \wedge \alpha_1 \leq \xi \leq \alpha_0 \vee \alpha_1\}, \alpha_0, \alpha_1)$$

so we have $QR = RQ$.

Finally, recall that we have a natural isomorphism

$$(x, \alpha_0, \alpha_1)^* \cong (x^{\mathsf{op}}, \alpha_0, \alpha_1).$$

Now the underlying sup-lattice of $R(x^{\mathsf{op}}, \alpha_0, \alpha_1)$ consists of those $\xi \in x^{\mathsf{op}}$ which are greater or equal to, in the sense of x^{op}, the meet of α_0 and α_1, calculated in x^{op}. Not wishing to belabour the point, these are evidently the same ξ which belong to the underlying sup-lattice of $Q(x, \alpha_0, \alpha_1)$, but ordered in the opposite fashion. I.e., we have

$$\uparrow^{x^{\mathsf{op}}}(\alpha_0 \wedge \alpha_1) \cong (\downarrow^x (\alpha_0 \vee \alpha_1))^{\mathsf{op}}$$

Hence $R((x, \alpha_0, \alpha_1)^*) \cong (Q(x, \alpha_0, \alpha_1))^*$. $Q((x, \alpha_0, \alpha_1)^*) \cong (R(x, \alpha_0, \alpha_1))^*$ is similar.

Q.E.D.

Of course, the other part of of Theorem 5.5.1, made reference to the core of **Sup**. Since we know that sup-lattices of the form $P(x)$ belong to the core of **Sup** (Lemma 5.1.6), we investigate the action of Q and R on adherence spaces whose underlying sup-lattice is of this form.

Example 6.1.3

Consider an adherence space of the form $(P(x), a_0, a_1)$. Then an element of $P(x)$ belongs to $\downarrow(a_0 \cup a_1)$ if and only if it is contained in $a_0 \cup a_1$. I.e., we have $Q(P(x), a_0, a_1) \cong (P(a_0 \cup a_1), a_0, a_1)$.

Dually, we have a sup-isomorphism

$$\uparrow(a_0 \cap a_1) \xrightarrow{\quad\sim\quad} P(x \setminus (a_0 \cap a_1))$$

given by $b \mapsto b \setminus (a_0 \cap a_1)$. Hence $R(P(x), a_0, a_1) \cong (P(x \setminus (a_0 \cap a_1)), a_0 \setminus a_1, a_1 \setminus a_0)$.

We are now ready to define the component parts of a Quillen model structure on $\mathcal{A}$. The proof that they do indeed a Quillen model structure will be postponed to the next section.

Definitions 6.1.4

$\mathcal{C}$ denotes the class of all $\mathcal{A}$-arrows $(x, \alpha_0, \alpha_1) \xrightarrow{\kappa} (y, \beta_0, \beta_1)$ such that the naturality square

$$\begin{array}{ccc} Q(x, \alpha_0, \alpha_1) & \rightarrowtail & (x, \alpha_0, \alpha_1) \\ {\scriptstyle Q\kappa}\downarrow & & \downarrow{\scriptstyle \kappa} \\ Q(y, \beta_0, \beta_1) & \rightarrowtail & (y, \beta_0, \beta_1) \end{array}$$

is a pushout.

$\mathcal{F}$ denotes the class of all $\mathcal{A}$-arrows $(x, \alpha_0, \alpha_1) \xrightarrow{\varphi} (y, \beta_0, \beta_1)$ such that the naturality square

$$\begin{array}{ccc} Q(x, \alpha_0, \alpha_1) & \twoheadrightarrow & RQ(x, \alpha_0, \alpha_1) \\ {\scriptstyle Q\varphi}\downarrow & & \downarrow{\scriptstyle RQ\varphi} \\ Q(y, \beta_0, \beta_1) & \twoheadrightarrow & RQ(y, \beta_0, \beta_1) \end{array}$$

is a pullback.

$\mathcal{W}$ denotes the class of $\omega \in \mathsf{mor}[\mathcal{A}]$ such that $RQ\omega$ is invertible.

As usual, we will call the elements of $\mathcal{C}$ *cofibrations*, those of $\mathcal{F}$, *fibrations*, and those of $\mathcal{W}$, *weak equivalences*.

Lemma 6.1.5

An adherence space (x, α_0, α_1) is cofibrant if and only if it is wholly-coloured and fibrant if and only if it is distinctly coloured.

Proof

Recall that the terminal object of $\mathcal{A}$ is $(1, *, *)$. Since $1^{\mathsf{op}} \cong 1$, it is also the initial object of $\mathcal{A}$.

Now to ask that $(1, *, *) \longrightarrow (x, \alpha_0, \alpha_1)$ be a cofibration means that

$$\begin{array}{ccc} (1, *, *) & = & (1, *, *) \\ \downarrow & & \downarrow \\ Q(x, \alpha_0, \alpha_1) & \rightarrowtail & (x, \alpha_0, \alpha_1) \end{array}$$

is a pushout; but the pushout along any isomorphism is also an isomorphism, therefore we must have $Q(x, \alpha_0, \alpha_1) = (x, \alpha_0, \alpha_1)$—i.e., (x, α_0, α_1) is wholly coloured.

Similarly, to ask that $(x, \alpha_0, \alpha_1) \longrightarrow (1, *, *)$ be a fibration means that

$$Q(x, \alpha_0, \alpha_1) \twoheadrightarrow RQ(x, \alpha_0, \alpha_1)$$

should be the pullback of an isomorphism and therefore invertible itself. In light of the descriptions of Q and R contained in the proof of Theorem 6.1.2, this means that $\alpha_0 \wedge \alpha_1 = \perp$—i.e., that (x, α_0, α_1) is distinctly coloured. Q.E.D.

Now while we have shown, in Theorem 6.1.2, that cofibrant objects are the duals of fibrant objects (and vice versa), the notions of fibration and cofibration are not generally dual; but we do at least have the following.

Lemma 6.1.6

If $\kappa \in \mathcal{C}$, then $\kappa^* \in \mathcal{F}$. Also, if $\omega \in \mathcal{W}$, then $\omega^* \in \mathcal{W}$ too.

Proof

If

$$\begin{array}{ccc} Q(y, \beta_0, \beta_1) & \rightarrowtail & (y, \beta_0, \beta_1) \\ {\scriptstyle Q\kappa}\downarrow & & \downarrow{\scriptstyle \kappa} \\ Q(z, \gamma_0, \gamma_1) & \rightarrowtail & (z, \gamma_0, \gamma_1) \end{array}$$

is a pushout, then, by duality,

$$\begin{array}{ccccc} (z, \gamma_0, \gamma_1)^* & \twoheadrightarrow & (Q(z, \gamma_0, \gamma_1))^* & \xrightarrow{\sim} & R((z, \gamma_0, \gamma_1)^*) \\ {\scriptstyle \kappa^*}\downarrow & & \downarrow{\scriptstyle (Q\kappa)^*} & & \downarrow{\scriptstyle R(\kappa^*)} \\ (y, \beta_0, \beta_1)^* & \twoheadrightarrow & (Q(y, \beta_0, \beta_1))^* & \xrightarrow{\sim} & R((y, \beta_0, \beta_1)^*) \end{array}$$

is a pullback.

But Q is right adjoint to the inclusion, and right adjoints always preserve pullbacks. Hence

$$\begin{array}{ccc} Q(z, \gamma_0, \gamma_1)^* & \twoheadrightarrow & QR((z, \gamma_0, \gamma_1)^*) \\ {\scriptstyle Q\kappa^*}\downarrow & & \downarrow{\scriptstyle QR\kappa^*} \\ Q(y, \beta_0, \beta_1)^* & \twoheadrightarrow & QR((y, \beta_0, \beta_1)^*) \end{array}$$

is a pullback; i.e., κ^* is a fibration.

The latter statement follows from the last part of Theorem 6.1.2. Since we have natural isomorphisms

$$\begin{aligned}(RQ(x,\alpha_0,\alpha_1))^* &\cong Q((Q(x,\alpha_0,\alpha_1))^*)\\ &\cong QR((x,\alpha_0,\alpha_1)^*)\\ &= RQ((x,\alpha_0,\alpha_1)^*)\end{aligned}$$

it follows that $RQ(\omega^*) = (RQ\omega)^*$. Q.E.D.

Before proceeding to the main theorem, let us attempt to gain some intuition for the above notions of fibration and cofibration by returning to the case of adherence spaces whose underlying sup-lattice is a power-set.

Example 6.1.7

Let $(P(x), a_0, a_1) \xrightarrow{\kappa} (P(y), b_0, b_1)$, and k be the corresponding relation $x \longmapsto y$.

Then the pushout of

$$\begin{array}{ccc}(P(a_0 \cup a_1), a_0, a_1) & \rightarrowtail & (P(x), a_0, a_1)\\ {\scriptstyle Q\kappa}\downarrow & & \\ (P(b_0 \cup b_1), b_0, b_1) & & \end{array}$$

is $(P(b_0 \cup b_1 \cup (x \setminus (a_0 \cup a_1))), b_0, b_1)$, because the pushout of

$$\begin{array}{ccc}a_0 \cup a_1 & \rightarrowtail & x\\ {\scriptstyle k\restriction_{a_0\cup a_1}}\downarrow & & \\ b_0 \cup b_1 & & \end{array}$$

in **Rel** is (isomorphic to) $b_0 \cup b_1 \cup (x \setminus (a_0 \cup a_1))$ irrespective of k, and the inclusion $\mathbf{Rel} \xrightarrow{P} \mathbf{Sup}$ preserves whatever pushouts happen to exist in **Rel**. The coprojection $x \longrightarrow b_0 \cup b_1 \cup (x \setminus (a_0 \cup a_1))$ is the (disjoint) union of $(k\restriction_{a_0\cup a_1})$ and the identity relation on $(x \setminus (a_0 \cup a_1))$.

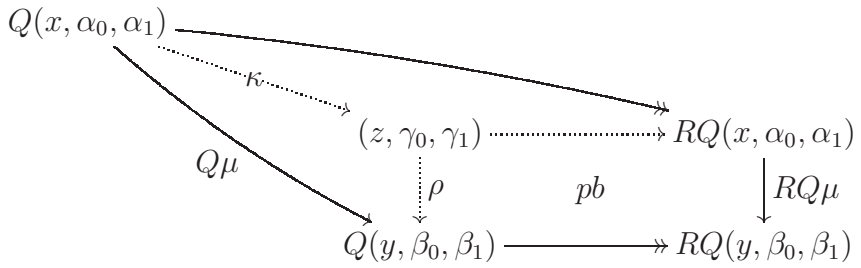
So to say that κ is a cofibration means, not only that $y \setminus (b_0 \cup b_1) \cong x \setminus (a_0 \cup a_1)$, but also that k is the disjoint union of said isomorphism with $k\restriction_{a_0\cup a_1}$. In particular, k may not relate any element of $x \setminus (a_0 \cup a_1)$ to any element of $b_0 \cup b_1$.

Similarly, if $(P(x), a_0, a_1) \xrightarrow{\varphi} (P(y), b_0, b_1)$ is a fibration, and f is the corresponding relation $x \rightarrowtail\!\!\!\!\!\rightarrow y$, then the restriction of f to $(a_0 \cap a_1) \times (b_0 \cap b_1)$ must be a bijection, and f may not relate any element of $(a_0 \cup a_1) \setminus (a_0 \cap a_1)$ to any element of $b_0 \cap b_1$. [But it may relate elements of $x \setminus (a_0 \cup a_1)$ to elements of $b_0 \cap b_1$—so the notion of fibration is more general than the notion of "dual of a cofibration".]

6.2 Theorems

The most difficult part of our main theorem will be showing that any arrow in $\mathcal{A}$ can be factorised as an element of $\mathcal{C} \cap \mathcal{W}$ followed by one in $\mathcal{F}$. It is, unfortuately, a two-stage process with many lemmas.

The first stage is as follows: given an arbitrary $\mathcal{A}$-arrow $(x, \alpha_0, \alpha_1) \xrightarrow{\mu} (y, \beta_0, \beta_1)$, we define (z, γ_0, γ_1), κ and ρ as follows:



Lemma 6.2.1

(z, γ_0, γ_1) is wholly coloured, $R\kappa$ is invertible, and the naturality square

$$\begin{array}{ccc} (z, \gamma_0, \gamma_1) & \twoheadrightarrow & R(z, \gamma_0, \gamma_1) \\ \downarrow \rho & & \downarrow R\rho \\ Q(y, \beta_0, \beta_1) & \twoheadrightarrow & RQ(y, \beta_0, \beta_1) \end{array}$$

is a pullback.

Proof

First, we note that, following the description of limits in $\mathcal{A}$ given in Lemma 5.2.6, and the descriptions of Q and R furnished in Theorem 6.1.2,

$$z = \{(\xi, \psi) \in x \times y \mid \alpha_0 \wedge \alpha_1 \leq \xi \leq \alpha_0 \vee \alpha_1, \psi \leq \beta_0 \vee \beta_1, \mu(\xi) \vee (\beta_0 \wedge \beta_1) = \psi \vee (\beta_0 \wedge \beta_1)\}$$

equipped with the pointwise ordering, and γ_k is the largest $(\xi_k, \psi_k) \in z$ satisfying $\xi_k \leq \alpha_k$ and $\psi_k \leq \beta_k$, for $k = 0, 1$.

Note that our obvious candidate for γ_0, namely (α_0, β_0), need not be an element of z—for instance, we might have $\beta_0 \wedge \beta_1 < \mu(\alpha_0) < \beta_0$.

But the requirement $\mu(\xi_0) \vee (\beta_0 \wedge \beta_1) = \psi_0 \vee (\beta_0 \wedge \beta_1)$ imposes

$$\psi_0 \leq \mu(\xi_0) \vee (\beta_0 \wedge \beta_1) \leq \mu(\alpha_0) \vee (\beta_0 \wedge \beta_1)$$

—moreover, $(\alpha_0, \mu(\alpha_0) \vee (\beta_0 \wedge \beta_1))$ is indeed an element of z. Hence, we can conclude that the latter tuple is γ_0. Similarly, $\gamma_1 = (\alpha_1, \mu(\alpha_1) \vee (\beta_0 \wedge \beta_1))$.

Joins in z can be calculated pointwise, but meets need not be. So it is easy to see that

$$\begin{aligned} \gamma_0 \vee \gamma_1 &= (\alpha_0 \vee \alpha_1, \mu(\alpha_0) \vee (\beta_0 \wedge \beta_1) \vee \mu(\alpha_1) \vee (\beta_0 \wedge \beta_1)) \\ &= (\alpha_0 \vee \alpha_1, \mu(\alpha_0 \vee \alpha_1) \vee (\beta_0 \wedge \beta_1)) \end{aligned}$$

since μ is a sup-homomorphism. But $(\xi, \psi) \in z$ implies $\xi \leq \alpha_0 \vee \alpha_1$ and

$$\begin{aligned} \psi &\leq \psi \vee (\beta_0 \wedge \beta_1) \\ &= \mu(\xi) \vee (\beta_0 \wedge \beta_1) \\ &\leq \mu(\alpha_0 \vee \alpha_1) \vee (\beta_0 \wedge \beta_1) \end{aligned}$$

hence $\top_z = (\alpha_0 \vee \alpha_1, \mu(\alpha_0 \vee \alpha_1) \vee (\beta_0 \wedge \beta_1))$. This proves that (z, γ_0, γ_1) is wholly coloured.

Similarly, we claim that $\gamma_0 \wedge \gamma_1 = (\alpha_0 \wedge \alpha_1, \beta_0 \wedge \beta_1)$. Certainly, the latter is a lower bound for γ_0 and γ_1; it is also an element of z since

$$\mu(\alpha_0 \wedge \alpha_1) \leq \mu(\alpha_0) \wedge \mu(\alpha_1) \leq (\beta_0 \wedge \beta_1)$$

implies

$$\mu(\alpha_0 \wedge \alpha_1) \vee (\beta_0 \wedge \beta_1) = \beta_0 \wedge \beta_1$$

Thus it remains to show that it is the *greatest* lower bound of γ_0 and γ_1 in z. But if (ξ, ψ) is an element of z satisfying $(\xi, \psi) \leq \gamma_0, \gamma_1$, then we have $\xi \leq \alpha_0, \alpha_1$ and

therefore $\xi \leq \alpha_0 \wedge \alpha_1$. Hence,

$$\begin{aligned} \psi &\leq \psi \vee (\beta_0 \wedge \beta_1) \\ &= \mu(\xi) \vee (\beta_0 \wedge \beta_1) \\ &\leq \mu(\alpha_0 \wedge \alpha_1) \vee (\beta_0 \wedge \beta_1) \\ &= \beta_0 \wedge \beta_1. \end{aligned}$$

So $R(z, \gamma_0, \gamma_1)$ consists of those $(\xi, \psi) \in z$ satisfying, $\xi \geq \alpha_0 \wedge \alpha_1$ and $\psi \geq \beta_0 \wedge \beta_1$. In this case, the key equation in the definition of z:

$$\mu(\xi) \vee (\beta_0 \wedge \beta_1) = \psi \vee (\beta_0 \wedge \beta_1)$$

becomes just

$$\mu(\xi) \vee (\beta_0 \wedge \beta_1) = \psi$$

so the projection $(\xi, \psi) \mapsto \xi$ restricts to a bijection $R(z, \gamma_0, \gamma_1) \longrightarrow RQ(x, \alpha_0, \alpha_1)$ and is therefore invertible. I.e., we have

$$\begin{array}{ccc} RQ(x, \alpha_0, \alpha_1) & & \\ & \searrow^{R\kappa} & \\ & R(z, \gamma_0, \gamma_1) & \xrightarrow{\sim} RQ(x, \alpha_0, \alpha_1) \end{array}$$

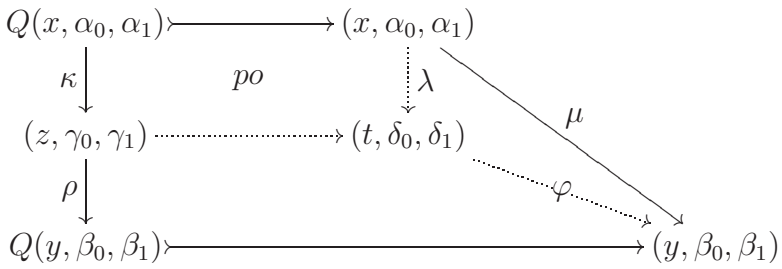
and hence $R\kappa$ is invertible.

Finally,

$$\begin{array}{ccc} (z, \gamma_0, \gamma_1) & \twoheadrightarrow & R(z, \gamma_0, \gamma_1) \\ \downarrow \rho & & \downarrow R\rho \\ Q(y, \beta_0, \beta_1) & \twoheadrightarrow & RQ(y, \beta_0, \beta_1) \end{array}$$

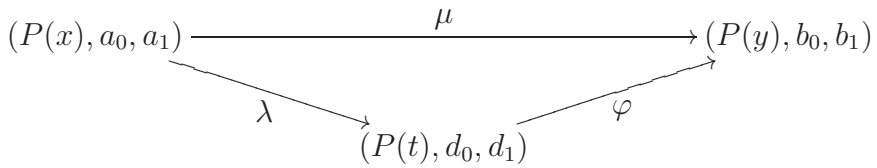
is a pullback, precisely because $R(z, \gamma_0, \gamma_1) \cong RQ(x, \alpha_0, \alpha_1)$, and because that's how ρ was defined. Q.E.D.

For the second stage of our construction, we define (t, δ_0, δ_1), λ, φ as follows:



Example 6.2.2

The result of applying this construction to an $\mathcal{A}$-arrow $(P(x), a_0, a_1) \xrightarrow{\mu} (P(y), b_0, b_1)$ with corresponding relation $m \subseteq x \times y$ is



where

$$\begin{aligned} t &= (x \setminus (a_0 \cap a_1)) \cup (b_0 \cap b_1) \\ d_0 &= (a_0 \setminus a_1) \cup (b_0 \cap b_1) \\ d_1 &= (a_1 \setminus a_0) \cup (b_0 \cap b_1) \end{aligned}$$

and λ corresponds to the relation $r \subseteq x \times t$ given by the union of the identity on $x \setminus (a_0 \cap a_1)$ and the intersection of m with $[(a_0 \cup a_1) \setminus (a_0 \cap a_1)] \times (b_0 \cap b_1)$, and φ corresponds to the relation $f \subseteq t \times y$ given by the union of the identity on $b_0 \cap b_1$ with $m \cap (t \times y)$.

Lemma 6.2.3

$\lambda \in \mathcal{C} \cap \mathcal{W}$ and $\varphi \in \mathcal{F}$.

Proof

By similar arguments to those above, we have

$$t = \{(\zeta, \chi) \in z \times x \mid \kappa^\sharp(\zeta) = \chi \wedge (\alpha_0 \vee \alpha_1)\}$$

(where $\kappa^\sharp$ denotes the right adjoint of κ) equipped with the pointwise order, and $\delta_k = (\gamma_k, \kappa^\sharp(\gamma_k))$, for $k = 0, 1$.

Specifically, pushouts in **Sup** can be calculated by taking the pullback of the dual diagram. [So meets in t may be calculated pointwise, but joins can not.] Moreover, δ_0 should (by the duality on $\mathcal{A}$) equal the *smallest* $(\zeta_0, \chi_0) \in t$ satisfying $\gamma_0 \leq \zeta_0$ and $\alpha_0 \leq \chi_0$. As before, the equation

$$\kappa^\sharp(\zeta_0) = \chi_0 \wedge (\alpha_0 \vee \alpha_1) \leq \chi_0$$

guarantees that these are γ_0 and $\kappa^\sharp(\gamma_0)$ respectively.

Similarly also, $\delta_0 \vee \delta_1 = (\gamma_0 \vee \gamma_1, \alpha_0 \vee \alpha_1)$ and $Q(t, \delta_0, \delta_1) \cong (z, \gamma_0, \gamma_1)$.

So the naturality square

$$\begin{array}{ccccc} & & Q(x,\alpha_0,\alpha_1) & \rightarrowtail & (x,\alpha_0,\alpha_1) \\ & \overset{\kappa}{\swarrow} & \downarrow{\scriptstyle Q\lambda} & & \downarrow{\scriptstyle \lambda} \\ (z,\gamma_0,\gamma_1) & \xrightarrow{\sim} & Q(t,\delta_0,\delta_1) & \rightarrowtail & (t,\delta_0,\delta_1) \end{array}$$

is a pushout, and $RQ\lambda \cong R\kappa$ is invertible. So $\lambda \in \mathcal{C} \cap \mathcal{W}$.

Also,

$$\begin{array}{ccccc} (z,\gamma_0,\gamma_1) & \xrightarrow{\sim} & Q(t,\delta_0,\delta_1) & \rightarrowtail & (t,\delta_0,\delta_1) \\ \downarrow{\scriptstyle \rho} & \swarrow{\scriptstyle Q\varphi} & & & \downarrow{\scriptstyle \varphi} \\ Q(y,\beta_0,\beta_1) & & \rightarrowtail & & (y,\beta_0,\beta_1) \end{array}$$

so the naturality square

$$\begin{array}{ccc} Q(t,\delta_0,\delta_1) & \twoheadrightarrow & RQ(x,\delta_0,\delta_1) \\ \downarrow{\scriptstyle Q\varphi} & & \downarrow{\scriptstyle RQ\varphi} \\ Q(y,\beta_0,\beta_1) & \twoheadrightarrow & RQ(y,\beta_0,\beta_1) \end{array}$$

is "essentially the same"—that is the same up to natural isomorphism—as

$$\begin{array}{ccc} (z,\gamma_0,\gamma_1) & \twoheadrightarrow & R(z,\gamma_0,\gamma_1) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle R\rho} \\ Q(y,\beta_0,\beta_1) & \twoheadrightarrow & RQ(y,\beta_0,\beta_1) \end{array}$$

which is a pullback. Hence $\varphi \in \mathcal{F}$.

Q.E.D.

Lemma 6.2.4

$\mathcal{W} \cap \mathcal{F}$ consists of those $\rho \in \mathsf{mor}[\mathcal{A}]$ such that $Q(\rho)$ is invertible.

Proof

If $\lambda \in \mathcal{W} \cap \mathcal{F}$ then $RQ\lambda$ is invertible and

$$\begin{array}{ccc} Qx & \twoheadrightarrow & RQx \\ {\scriptstyle Q\lambda}\downarrow & & \downarrow{\scriptstyle RQ\lambda} \\ Qy & \twoheadrightarrow & RQy \end{array}$$

is a pullback. But the pullback of an iso is an iso.

Conversely, if $Q\lambda$ is invertible, then so is $RQ\lambda$; hence, the square above is a pullback. Q.E.D.

Theorem 6.2.5

$\mathcal{C}$, $\mathcal{W}$ and $\mathcal{F}$, as defined above, form a Quillen model structure for $\mathcal{A}$.

Proof

We have already (exhaustively) established that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ satisfy the factoring axiom.

By contrast, verifying the lifting axiom is comparatively routine: let $(x, \alpha_0, \alpha_1) \xrightarrow{\lambda} (y, \beta_0, \beta_1)$ be an element of $\mathcal{C} \cap \mathcal{W}$, $(z, \gamma_0, \gamma_1) \xrightarrow{\varphi} (t, \delta_0, \delta_1)$ be an element of $\mathcal{F}$, and

$$\begin{array}{ccc} (x, \alpha_0, \alpha_1) & \xrightarrow{\mu} & (z, \gamma_0, \gamma_1) \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \varphi} \\ (y, \beta_0, \beta_1) & \xrightarrow[\nu]{} & (t, \delta_0, \delta_1) \end{array}$$

a commutative square in $\mathcal{A}$.

Since $RQ\lambda$ is invertible, we can find a (unique) ϑ_1 satisfying

$$\begin{array}{ccc} RQ(x, \alpha_0, \alpha_1) & \xrightarrow{RQ\mu} & RQ(z, \gamma_0, \gamma_1) \\ {\scriptstyle RQ\lambda}\downarrow & \overset{\vartheta_1}{\nearrow} & \downarrow{\scriptstyle RQ\varphi} \\ RQ(y, \beta_0, \beta_1) & \xrightarrow[RQ\nu]{} & RQ(t, \delta_0, \delta_1) \end{array}$$

—namely, $\vartheta_1 = (RQ\mu) \circ (RQ\lambda)^{-1}$.

Now, the fact that $\varphi \in \mathcal{F}$ implies that we can find a (unique) ϑ_2 satisfying

$$\begin{array}{ccc} Q(y,\beta_0,\beta_1) & \twoheadrightarrow & RQ(y,\beta_0,\beta_1) \\ {\scriptstyle \vartheta_2}\vdots & & \downarrow{\scriptstyle \vartheta_1} \\ Q(z,\gamma_0,\gamma_1) & \twoheadrightarrow & RQ(z,\gamma_0,\gamma_1) \\ {\scriptstyle Q\varphi}\downarrow & & \downarrow{\scriptstyle RQ\varphi} \\ Q(t,\delta_0,\delta_1) & \twoheadrightarrow & RQ(t,\delta_0,\delta_1) \end{array}$$

(with $Q\nu$ on the left and $RQ\nu$ on the right)

We also have that $Q\mu = \vartheta_2 \circ (Q\lambda)$ because both make the diagram

$$\begin{array}{ccc} Q(x,\alpha_0,\alpha_1) & \twoheadrightarrow & RQ(x,\alpha_0,\alpha_1) \\ \vdots & & \downarrow{\scriptstyle RQ\mu} \\ Q(z,\gamma_0,\gamma_1) & \twoheadrightarrow & RQ(z,\gamma_0,\gamma_1) \\ {\scriptstyle Q\varphi}\downarrow & & \downarrow{\scriptstyle RQ\varphi} \\ Q(t,\delta_0,\delta_1) & \twoheadrightarrow & RQ(t,\delta_0,\delta_1) \end{array}$$

(with $Q(\nu \circ \lambda)$ on the left and $RQ(\nu \circ \lambda)$ on the right)

commute, hence

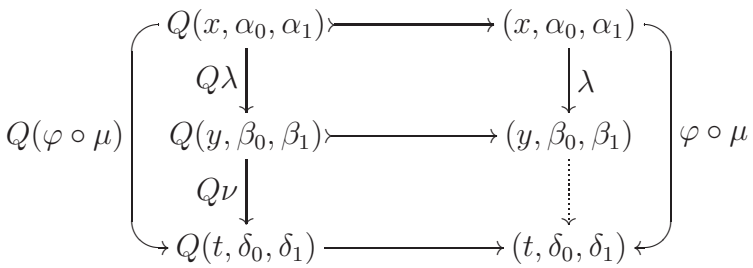
$$\begin{array}{ccc} Q(x,\alpha_0,\alpha_1) & \xrightarrow{Q\mu} & Q(z,\gamma_0,\gamma_1) \\ {\scriptstyle Q\lambda}\downarrow & \overset{\vartheta_2}{\nearrow} & \downarrow{\scriptstyle Q\varphi} \\ Q(y,\beta_0,\beta_1) & \xrightarrow[Q\nu]{} & Q(t,\delta_0,\delta_1) \end{array}$$

Finally, the fact that $\lambda \in \mathcal{C}$ implies that we can find a (unique) ϑ_3 satisfying

$$\begin{array}{ccc} Q(x,\alpha_0,\alpha_1) & \rightarrowtail & (x,\alpha_0,\alpha_1) \\ {\scriptstyle Q\lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\ Q(y,\beta_0,\beta_1) & \rightarrowtail & (y,\beta_0,\beta_1) \\ {\scriptstyle \vartheta_2}\downarrow & & \vdots{\scriptstyle \vartheta_3} \\ Q(z,\gamma_0,\gamma_1) & \rightarrowtail & (z,\gamma_0,\gamma_1) \end{array}$$

(with $Q\mu$ on the left and μ on the right)

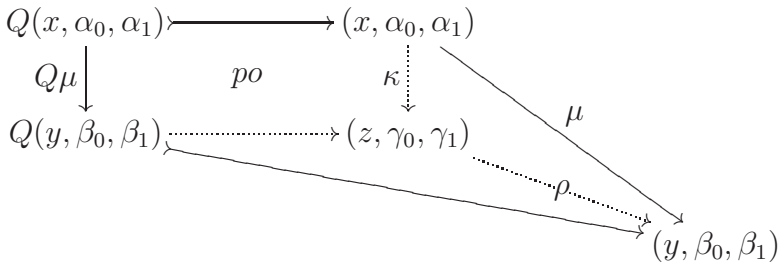
again, we have that $\varphi \circ \vartheta_3 = \nu$, because they both make the diagram



commute.

The proof that $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ is a factorisation scheme uses the essentially the same techniques, but it is considerably shorter in light of Lemma 6.2.4—all proofs are done in one step, instead of two!

To factor an arbitrary $\mathcal{A}$-arrow $(x, \alpha_0, \alpha_1) \xrightarrow{\mu} (y, \beta_0, \beta_1)$, we define (z, γ_0, γ_1), κ and ρ as follows:



So $z = \{(\xi, \psi) \in x \times y \mid \psi \leq \beta_0 \vee \beta_1, \mu^\sharp(\psi) = \xi \wedge (\alpha_0 \vee \alpha_1)\}$, and $\gamma_k = (\alpha_k, \mu^\sharp(\beta_k))$, for $k = 0, 1$. $\gamma_0 \vee \gamma_1 = (\beta_0 \vee \beta_1, \alpha_0 \vee \alpha_1)$, and $Q(z, \gamma_0, \gamma_1) = \downarrow(\gamma_0 \vee \gamma_1) \cong Q(y, \beta_0, \beta_1)$. hence $\kappa \in \mathcal{C}$ and $Q\mu$ is invertible. Q.E.D.

Having established a Quillen model structure, we still need to show that it is a monoidal model structure.

Lemma 6.2.6

For any adherence spaces (x, α_0, α_1) and (y, β_0, β_1), the following isomorphism holds:

$$Q((x, \alpha_0, \alpha_1) \otimes (y, \beta_0, \beta_1)) \cong Q(x, \alpha_0, \alpha_1) \otimes Q(y, \beta_0, \beta_1).$$

Proof

Recall from section 6.5 that

$$(x, \alpha_0, \alpha_1) \boxtimes (y, \beta_0, \beta_1) = (x \boxtimes y, (\alpha_0 \boxtimes \beta_0) \vee (\alpha_1 \boxtimes \beta_1), (\alpha_0 \boxtimes \beta_1) \vee (\alpha_1 \boxtimes \beta_0))$$

but since $\boxtimes$ distributes $\vee$, we have

$$(\alpha_0 \boxtimes \beta_0) \vee (\alpha_1 \boxtimes \beta_1) \vee (\alpha_0 \boxtimes \beta_1) \vee (\alpha_1 \boxtimes \beta_0) = (\alpha_0 \vee \alpha_1) \boxtimes (\beta_0 \vee \beta_1).$$

Hence $Q((x, \alpha_0, \alpha_1) \boxtimes (y, \beta_0, \beta_1))$ equals

$$\left(\downarrow^{x \boxtimes y}(((\alpha_0 \vee \alpha_1) \boxtimes (\beta_0 \vee \beta_1))), (\alpha_0 \boxtimes \beta_0) \vee (\alpha_1 \boxtimes \beta_1), (\alpha_0 \boxtimes \beta_1) \vee (\alpha_1 \boxtimes \beta_0)\right)$$

—whereas $Q(x, \alpha_0, \alpha_1) \boxtimes Q(y, \beta_0, \beta_1)$ equals

$$\left(\downarrow^{x}((\alpha_0 \vee \alpha_1)) \boxtimes \downarrow^{y}((\beta_0 \vee \beta_1)), (\alpha_0 \boxtimes \beta_0) \vee (\alpha_1 \boxtimes \beta_1), (\alpha_0 \boxtimes \beta_1) \vee (\alpha_1 \boxtimes \beta_0)\right).$$

But $\downarrow^{x \boxtimes y}(((\alpha_0 \vee \alpha_1) \boxtimes (\beta_0 \vee \beta_1))) \cong \downarrow^{x}((\alpha_0 \vee \alpha_1)) \boxtimes \downarrow^{y}((\beta_0 \vee \beta_1))$ as sup-lattices. Q.E.D.

Lemma 6.2.7

If (x, α_0, α_1) and (y, β_0, β_1) are cofibrant, then

$$R((x, \alpha_0, \alpha_1) \boxtimes (y, \beta_0, \beta_1)) \cong R(R(x, \alpha_0, \alpha_1) \boxtimes R(y, \beta_0, \beta_1))$$

Proof

The fact that, for cofibrant (x, α_0, α_1) and (y, β_0, β_1),

$$\begin{aligned}
&((\alpha_0 \boxtimes \beta_0) \vee (\alpha_1 \boxtimes \beta_1)) \wedge ((\alpha_0 \boxtimes \beta_1) \vee (\alpha_1 \boxtimes \beta_0)) \\
&\quad \geq \quad ((\alpha_0 \boxtimes \beta_0) \wedge (\alpha_0 \boxtimes \beta_1)) \vee ((\alpha_0 \boxtimes \beta_0) \wedge (\alpha_1 \boxtimes \beta_0)) \vee ((\alpha_1 \boxtimes \beta_1) \wedge (\alpha_0 \boxtimes \beta_1)) \\
&\qquad \vee((\alpha_1 \boxtimes \beta_1) \wedge (\alpha_1 \boxtimes \beta_0)) \\
&\quad \geq \quad (\alpha_0 \boxtimes (\beta_0 \wedge \beta_1)) \vee ((\alpha_0 \wedge \alpha_1) \boxtimes \beta_0) \vee ((\alpha_1 \wedge \alpha_0) \boxtimes \beta_1) \vee (\alpha_1 \boxtimes (\beta_1 \wedge \beta_0)) \\
&\quad = \quad ((\alpha_0 \wedge \alpha_1) \boxtimes (\beta_0 \vee \beta_1)) \vee ((\alpha_0 \vee \alpha_1) \boxtimes (\beta_0 \wedge \beta_1)) \\
&\quad = \quad ((\alpha_0 \wedge \alpha_1) \boxtimes \top_y) \vee (\top_x \boxtimes (\beta_0 \wedge \beta_1))
\end{aligned}$$

implies that the underlying sup-lattice of $R(x, \alpha_0, \alpha_1) \boxtimes R(y, \beta_0, \beta_1)$, namely

$$\uparrow^{x}(\alpha_0 \wedge \alpha_1) \boxtimes \uparrow^{y}(\beta_0 \wedge \beta_1) \cong \uparrow^{x \boxtimes y}(((\alpha_0 \wedge \alpha_1) \boxtimes \top_y) \vee (\top_x \boxtimes (\beta_0 \wedge \beta_1)))$$

contains (an isomorphic copy of) the underlying sup-lattice of

$$R((x,\alpha_0,\alpha_1) \boxtimes (y,\beta_0,\beta_1)) = \uparrow^{x \boxtimes y}(((\alpha_0 \boxtimes \beta_0) \vee (\alpha_1 \boxtimes \beta_1)) \wedge ((\alpha_0 \boxtimes \beta_1) \vee (\alpha_1 \boxtimes \beta_0))).$$

The effect of applying R to $R(x,\alpha_0,\alpha_1) \boxtimes R(y,\beta_0,\beta_1)$ is to restrict to this subspace.

Q.E.D.

Theorem 6.2.8

$(\mathcal{C},\mathcal{W},\mathcal{F})$ form a monoidal almost-Quillen model category in the sense of Definition 3.4.6.

Proof

First, we must show that, if (x,α_0,α_1) is cofibrant, then

$$\mathcal{K} \underset{(x,\alpha_0,\alpha_1) \boxtimes (-)}{\overset{(x,\alpha_0,\alpha_1) \multimap (-)}{\rightleftarrows}} \mathcal{K}$$

is a Quillen adjunction.

Let $(y,\beta_0,\beta_1) \xrightarrow{\kappa} (z,\gamma_0,\gamma_1)$ be a cofibration. Then, since $(x,\alpha_0,\alpha_1) \boxtimes (-)$ is a left adjoint, and left adjoints preserve pushouts, we have that

$$\begin{array}{ccc} (x,\alpha_0,\alpha_1) \boxtimes Q(y,\beta_0,\beta_1) & \rightarrowtail & (x,\alpha_0,\alpha_1) \boxtimes (y,\beta_0,\beta_1) \\ \downarrow & & \downarrow \\ (x,\alpha_0,\alpha_1) \boxtimes Q(z,\gamma_0,\gamma_1) & \rightarrowtail & (x,\alpha_0,\alpha_1) \boxtimes (z,\gamma_0,\gamma_1) \end{array}$$

is a pushout.

But, by Lemma 6.2.6, we have

$$\begin{array}{ccc} Q((x,\alpha_0,\alpha_1) \boxtimes (y,\beta_0,\beta_1)) & \xrightarrow{\sim} & (x,\alpha_0,\alpha_1) \boxtimes Q(y,\beta_0,\beta_1) \\ {\scriptstyle Q(\iota_x \boxtimes \kappa)}\downarrow & & \downarrow{\scriptstyle \iota_x \boxtimes Q\kappa} \\ Q((x,\alpha_0,\alpha_1) \boxtimes (z,\gamma_0,\gamma_1)) & \xrightarrow{\sim} & (x,\alpha_0,\alpha_1) \boxtimes Q(z,\gamma_0,\gamma_1) \end{array}$$

and therefore $\iota_x \boxtimes \kappa$ is a cofibration.

Now if $(y, \beta_0, \beta_1) \xrightarrow{\omega} (z, \gamma_0, \gamma_1)$ is a weak equivalence—i.e., if $RQ\omega$ is invertible, then, using both Lemma 6.2.6 and Lemma 6.2.7, we have

$$\begin{aligned} RQ((x, \alpha_0, \alpha_1) \invamp (y, \beta_0, \beta_1)) &\cong R((x, \alpha_0, \alpha_1) \invamp Q(y, \beta_0, \beta_1)) \\ &\cong R(R(x, \alpha_0, \alpha_1) \invamp RQ(y, \beta_0, \beta_1)) \end{aligned}$$

naturally. So

$$\begin{array}{ccc} RQ((x, \alpha_0, \alpha_1) \invamp (y, \beta_0, \beta_1)) & \xrightarrow{\sim} & R(R(x, \alpha_0, \alpha_1) \invamp RQ(y, \beta_0, \beta_1)) \\ \Big\downarrow{\scriptstyle RQ(\iota_x \invamp \omega)} & & \Big\downarrow{\scriptstyle R(\iota_x \invamp RQ\omega)} \\ RQ((x, \alpha_0, \alpha_1) \invamp (z, \gamma_0, \gamma_1)) & \xrightarrow{\sim} & R(R(x, \alpha_0, \alpha_1) \invamp RQ(z, \gamma_0, \gamma_1)) \end{array}$$

implies $RQ(\iota_x \invamp \omega)$ is invertible too—i.e., $\iota_x \invamp \omega$ is a weak equivalence.

Now suppose (z, γ_0, γ_1) is fibrant. We must show that

$$\mathcal{A}^{\mathsf{op}} \xrightarrow{(-) \multimap (z, \gamma_0, \gamma_1)} \mathcal{A}$$

maps $\mathcal{C}$ to $\mathcal{F}$ and $\mathcal{W} \cap \mathcal{C}$ to $\mathcal{W} \cap \mathcal{F}$. But, since $\mathcal{A}$ is (symmetric) $*$-autonomous, we have

$$(-) \multimap (z, \gamma_0, \gamma_1) \cong ((-) \invamp (z, \gamma_0, \gamma_1)^*)^*.$$

But (z, γ_0, γ_1) fibrant implies $(z, \gamma_0, \gamma_1)^*$ cofibrant, and therefore $\kappa \in \mathcal{C}$ (respectively $\mathcal{W} \cap \mathcal{C}$) implies $\kappa \invamp \iota_{z^*} \in \mathcal{C}$ (respectively $\mathcal{W} \cap \mathcal{C}$). Therefore, by Lemma 6.1.6, $(\kappa \invamp \iota_{z^*})^* \in \mathcal{F}$ (respectively $\mathcal{W} \cap \mathcal{F}$).

Finally, note that the tensor unit, $(\mathbb{2}, \top, \bot)$, is (both fibrant and) cofibrant. Q.E.D.

Theorem 6.2.9

The Quillen model structure described above restricts to the full subcategory of $\mathcal{A}$ consisting of adherence spaces of the form $(P(x), a_0, a_1)$. Moreover, its homotopy category is compact closed.

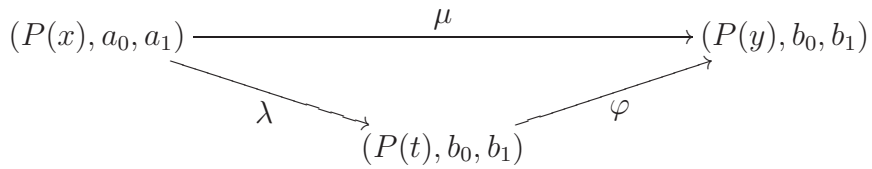
Proof

The only obstruction to restricting a Quillen model structure to a full subcategory is that then the factoring axiom may no longer hold. But we have already shown, in

Example 6.2.2, that the $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$-factorisation of a map between such adherence spaces remains inside the subcategory. As in the proof of the main theorem, the case of the $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$-factorisation is essentially the same only simpler. Specifically, given an arrow $(P(x), a_0, a_1) \xrightarrow{\mu} (P(y), b_0, b_1)$ with corresponding relation $m \subseteq x \times y$, we define

$$t = (x \setminus (a_0 \cup a_1)) \cup (b_0 \cup b_1)$$

then the factorisation described in the proof of Theorem 6.2.5 is just



Now the homotopy category is plainly equivalent to the full subcategory of perfectly coloured adherence spaces of the form $(P(x), a_0, a_1)$—which is compact closed by Theorem 5.5.1. Q.E.D.

Thus we have achieved our objective: a link between the denotational semantics of linear logic and Geometry of Interaction, via a denotational model of linear logic which can also be viewed, through the lens of abstract homotopy theory, as a compact closed category.

6.3 Future Work

As indicated in the Introduction, much work remains to be done.

It would be nice to know whether the Quillen model structure described above restricts to the full subcategory of adherence spaces whose underlying sup-lattice belongs to the core of **Sup** (such lattices are called *completely distributive*). This seems a reasonable conjecture—indeed, the only obstruction is the possibility that the result of factoring an arrow between such adherence spaces (either as a cofibration/acyclic-fibration pair, or as an acyclic-cofibration/fibration pair) might lead us outside the subcategory. The situation is delicate because the factorisations we produced were

phrased in terms of pullbacks and pushouts, and we do not know which pullbacks and pushouts of completely distributive lattices are again completely distributive.

Another question which arises is whether similar model structures may be found on other categories of adherence spaces. In this vein, it is interesting to note that, for any poset j, there exists an idempotent adjunction

$$D(j)\backslash\backslash\mathbf{Sup} \underset{\longleftarrow}{\longrightarrow} \mathbf{Sup}^j$$

inducing an equivalence between the full subcategory of $D(j)\backslash\backslash\mathbf{Sup}$ consisting of lattice homomorphisms $D(j) \xrightarrow{\sigma^{(x)}} x$, and the full subcategory of $\mathbf{Sup}^j$ consisting of functors $j \longrightarrow \mathbf{Sup}_m$—where $\mathbf{Sup}_m$ denotes the category of sup-lattices and *injective* sup-homomorphisms (i.e., sup-monomorphisms).

On a more concrete level, and in a similar vein to our chapter on Chu spaces, Ehrhard's paper on *serial* and *parallel* hypercoherence spaces [16] also reveals arguments similar to those found in abstract homotopy theory; a Quillen model structure might be lurking there.

Finally, an interesting side-project might be interesting to employ Quillen model structures to invert linear distributions, instead of mix maps. The result would then be a shift-monoidal category [14], instead of a compact-closed one.

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