On the definition of C^*-algebra

(a summary of a talk given at PSSL#85 in Nice)

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Abstract

One of the recent advances in Functional Analysis has been the introduction of the notion of an *(abstract) operator space*. Proponents of this theory have, since its inception, maintained that it provides a much more solid "foundations" for Operator Algebra than the more traditional theory of Banach spaces.

In this paper, we show that there exists a (non-trivial) involutive monoidal structure on the category of operator spaces and linear complete contractions. Every C^*-algebra gives rise to an involutive monoid in this involutive monoidal category, but (unfortunately) not *vice versa*.

1 Preliminaries

It is well known that there exists a symmetric monoidal closed category of Banach spaces and linear contractions, here denoted $(\mathbf{Ban}, \boxtimes, \mathbb{C}, \multimap)$—*i.e.*, we use the symbol $\boxtimes$ to denote the *projective tensor product* of Banach spaces, which is more commonly denoted $\otimes_\gamma$, and the symbol $\multimap$ to denote the internal hom functor, which is more commonly denoted $\mathcal{B}(-,-)$.

Linear contractions of the form $x \boxtimes y \longrightarrow z$ are in bijective correspondence with what we shall call *naïvely contractive* bilinear transformations; these are bilinear transformations which restrict to the unit balls of x, y, and z,

$$\begin{array}{ccc} \mathcal{U}(x) \times \mathcal{U}(y) & \cdots\cdots\to & \mathcal{U}(z) \\ \downarrow & & \downarrow \\ x \times y & \xrightarrow{\vartheta} & z \end{array}$$

—or, equivalently, those which satisfy $\|\vartheta(\alpha,\beta)\|_z \leq \|\alpha\|_x \cdot \|\beta\|_y$. We follow the common convention of writing x^* for $x \multimap \mathbb{C}$ despite the fact that $\mathbb{C}$ is not a dualising object for $(\mathbf{Ban}, \boxtimes, \mathbb{C}, \multimap)$.

An important class of Banach spaces are the Hilbert spaces; these are self-dual in the sense that there are (unnatural!) isomorphisms $h \xrightarrow{\sim} \overline{h} \xrightarrow{\sim} h^*$. [The first of these two

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isomorphisms is extremely unnatural, in that its construction depends upon a choice of basis, whereas the second is simply the transpose of the inner product on h, which—thanks to the Cauchy-Schwartz inequality—can be regarded as an arrow of the form $\overline{h} \boxtimes h \longrightarrow \mathbb{C}$ in **Ban**.] From this, it can be derived that Hilbert spaces are among the reflexive Banach spaces—those such that the canonical map $x \longrightarrow x^{**}$ is invertible.

A monoid in $(\mathbf{Ban}, \boxtimes, \mathbb{C})$ is called a *Banach algebra.* [Throughout this paper, all algebras are assumed unital; further, the unit of a Banach algebra is assumed to lie in its unit ball (equivalently, sphere).] A very important class of Banach algebras are those of the form $(h \multimap h; \circ, \mathrm{id}_h)$, where h is a Hilbert space; closed subalgebras of these are called *concrete operator algebras.*

But since h is self-dual, (the inverse of) the “contrapositive map”

$$h \multimap h \xrightarrow{\quad (\ )^* \quad} h^* \multimap h^*$$

can be viewed as an extra piece of structure on $h \multimap h$. [More precisely, one can avoid basis-dependent constructions by noting the existence of a natural isomorphism $\overline{x \multimap y} \xrightarrow{\sim} \overline{x} \multimap \overline{y}$. Thus, the map we are really interested in is the composite

$$\overline{h \multimap h} \longrightarrow \overline{h} \multimap \overline{h} \longrightarrow h^* \multimap h^* \longrightarrow h \multimap h$$

which, by abuse of language, is also denoted $(\ )^*$.] This observation motivates the definition of Banach $*$-algebra.

The algebraic theory of *involutive monoids*, that is, of monoids equipped with a unary operation, denoted either $(\ )^*$ or $\overline{(\ )}$, satisfying the identities

$$\begin{aligned} (\alpha \cdot \beta)^* &= \beta^* \cdot \alpha^* \\ \alpha^{**} &= \alpha \end{aligned}$$

can clearly be modelled in any symmetric monoidal category; and it is clear that the contraposition operation on $h \multimap h$ defined above must satisfy these identities.

Were we working over the reals instead of $\mathbb{C}$, and thus able to identify $\overline{x}$ with x, we would be able to define a Banach $*$-algebra as an involutive monoid in the symmetric monoidal closed category of real Banach spaces. To arrive at the correct notion of (complex) Banach $*$-algebra, however, we must tweak the definition of involutive monoid, by using (yet again) the existence of a natural isomorphism $\overline{x} \boxtimes \overline{y} \xrightarrow{\quad \sim \quad} \overline{x \boxtimes y}$.

Definitions 1.1

1. A *Banach $*$-algebra* is a Banach algebra $(a; \mu, \eta)$ together with a map $\overline{a} \xrightarrow{\nu} a$ such that the diagrams

$$\begin{array}{ccccc} \overline{a} \boxtimes \overline{a} & \xrightarrow{\quad \chi_{\overline{a},\overline{a}} \quad} & \overline{a} \boxtimes \overline{a} & \xrightarrow{\quad \sim \quad} & \overline{a \boxtimes a} \\ {\scriptstyle \nu \boxtimes \nu}\downarrow & & & & \downarrow{\scriptstyle \overline{\mu}} \\ a \boxtimes a & \xrightarrow[\quad \mu \quad]{} & a & \xleftarrow[\quad \nu \quad]{} & \overline{a} \end{array}$$

(where χ denotes the symmetry of $\boxtimes$) and

$$\overline{\overline{a}} \xrightarrow{\;\overline{\nu}\;} \overline{a} \xrightarrow{\;\nu\;} a, \qquad \overline{\overline{a}} \xrightarrow{\;=\;} a$$

commute.

2. A C^**-algebra* is a Banach $*$-algebra which can be regularly (=isometrically) embedded into one of the form $(h \multimap h; \circ, \mathrm{id}_h, (\ )^*)$.

Unlike concrete operator algebras (which are notoriously difficult to classify (up to isomorphism) among Banach algebras—see, for instance, [1]), there exists a simple characterisation of C^*-algebras.

Theorem 1.2 (Gelfand-Naimark-Segal,1943)
A Banach $*$-algebra is a C^*-algebra if and only if it satisfies the so-called C^**-identity*:

$$\|\alpha\| = \sqrt{\|\alpha^*\alpha\|}.$$

But the C^*-identity, unlike anything else we have encountered so far, can clearly not be expressed in terms of a commutative diagram in **Ban**.

2 Operator spaces

In this section we quickly sketch as much of the basic theory of operator spaces as the reader shall need to understand and appreciate the statement and proof our main results.

Definitions 2.1

1. An *(abstract) operator space* $\mathbf{x}$ consists of a vector space x together with a suite of (Cauchy-)complete norms

$$x^{n\times n} \xrightarrow{\;\|-\|_{(\mathbf{x},n)}\;} [0,\infty)$$

satisfying the following axioms:

(a) for every $\zeta \in \mathbb{C}^{n\times m}$, $\alpha \in x^{n\times n}$ and $\omega \in \mathbb{C}^{m\times n}$,

$$\|\zeta \cdot \alpha \cdot \omega\|_{(\mathbf{x},m)} \leq \|\zeta\|_\infty \cdot \|\alpha\|_{(\mathbf{x},n)} \cdot \|\omega\|_\infty$$

(where $\|-\|_\infty$ denotes the usual "operator norm" on $\mathbb{C}^{p\times q}$ for all $p, q \in \mathbb{N}$); and

(b) for every $\alpha \in x^{n\times n}$ and every $\beta \in x^{m\times m}$,

$$\left\| \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right\|_{(\mathbf{x},n+m)} \leq \max\left\{ \|\alpha\|_{(\mathbf{x},n)}, \|\beta\|_{(\mathbf{x},m)} \right\}.$$

[$\geq$ follows from axiom (a) so it would be equivalent to write $=$ in place of $\leq$.]

Note that we shall usually denote operator spaces by bold letters, and that in this case their underlying vector spaces will always be denoted by the same letter in italic font.

2. A linear *complete contraction* $\mathbf{x} \longrightarrow \mathbf{y}$ is a linear transformation $x \xrightarrow{\vartheta} y$ such that, for every $n \in \mathbb{N}$, the linear transformation $x^{n\times n} \xrightarrow{\vartheta^{n\times n}} y^{n\times n}$ defined by

$$\vartheta^{n\times n}\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} := \begin{pmatrix} \vartheta(\alpha_{11}) & \cdots & \vartheta(\alpha_{1n}) \\ \vdots & \ddots & \vdots \\ \vartheta(\alpha_{n1}) & \cdots & \vartheta(\alpha_{nn}) \end{pmatrix}$$

is norm-non-increasing (with respect to $\|-\|_{(\mathbf{x},n)}$ and $\|-\|_{(\mathbf{y},n)}$).

3. The category of operator spaces and linear complete contractions is denoted **Oper**.

4. The forgetful functor $\mathbf{Oper} \longrightarrow \mathbf{Vec}$, defined by $\mathbf{x} \mapsto x$, will be denoted $\mathcal{V}$ (when needs be).

Standard references for operator spaces include [2, 7] while a more categorical approach will be appearing in [4]; there is also an "on-line dictionary" of operator spaces [9]. All of the standard references give detailed proofs of all the facts listed in this section.

Examples 2.2

1. Let $A = (a; \mu, \eta, \nu)$ be a C^*-algebra. Then the matrix algebra $A^{n\times n}$ equipped with the (Hermitianesque) involution

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix}^* = \begin{pmatrix} \alpha_{11}^* & \cdots & \alpha_{n1}^* \\ \vdots & \ddots & \vdots \\ \alpha_{1n}^* & \cdots & \alpha_{nn}^* \end{pmatrix}$$

is again a C^*-algebra —and so $a^{n\times n}$ carries a canonical norm, its so-called C^*-norm.

These C^*-norms satisfy axioms (a) and (b), and so there exists an *underlying operator space of* A, denoted $\mathbf{a}$, consisting of a together with the C^*-norms.

Moreover, every C^*-homomorphism $A \longrightarrow B$ defines a linear complete contraction $\mathbf{a} \longrightarrow \mathbf{b}$, and so we obtain a forgetful functor $\mathbf{C^*Alg} \longrightarrow \mathbf{Oper}$.

2. In particular, if A is a C^*-algebra of the form $(h \multimap h; \circ, \mathrm{id}_h, (\ )^*)$ where h is a Hilbert space, then the C^*-norm on $a^{n\times n}$ is that of $h^{\boxplus n} \multimap h^{\boxplus n}$, where $\boxplus$ denotes the sum of Hilbert spaces (*i.e.*, the ℓ_2-direct sum). [Note that $A^{n\times n}$ is canonically isomorphic to the underlying algebra of $h^{\boxplus n} \multimap h^{\boxplus n}$, so the former statement does make sense.]

The underlying operator space of $(h \multimap h; \circ, \mathrm{id}_h, (\ )^*)$ will be denoted $h \multimap\!\!\bullet\, h$.

3. In particular, the vector space $\mathbb{C}^{n\times n}$ carries a canonical operator space structure, that of $\mathbb{C}^n \multimap\!\!\bullet\, \mathbb{C}^n$. Unless otherwise specified, if we refer to $\mathbb{C}^{n\times n}$ as an operator space, then we mean the latter operator space. [We shall later provide an example of a different operator space structure overlying $\mathbb{C}^{n\times n}$, for $n > 1$.]

4. A celebrated example of a linear contraction which fails to be a linear complete contraction is $\mathbb{C}^{n\times n} \xrightarrow{(\ )^T} \mathbb{C}^{n\times n}$ for $n > 1$.

Theorem 2.3 (Ruan,[8])

Every operator space can be regularly (=completely isometrically) embedded into the underlying operator space of some C^*-algebra.

Equivalently, every operator space can be regularly embedded into some $h \multimap h$.

Definition 2.4

Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be operator spaces, and let $\odot$ denote the obvious notion of "matrix multiplication"

$$x^{n\times n} \times y^{n\times n} \longrightarrow (x \otimes y)^{n\times n}$$

with the symbol $\otimes$ replacing $\cdot$ (ordinary multiplication).

Then a bilinear transformation $x \times y \xrightarrow{\vartheta} z$ is called *completely contractive* if, for every $n \in \mathbb{N}$, the map

$$x^{n\times n} \times y^{n\times n} \xrightarrow{\quad\odot\quad} (x \otimes y)^{n\times n} \xrightarrow{\quad\hat{\vartheta}^{n\times n}\quad} z^{n\times n}$$

is naïvely contractive (with respect to the norms $\|-\|_{(\mathbf{x},n)}$, $\|-\|_{(\mathbf{y},n)}$ and $\|-\|_{(\mathbf{z},n)}$).

[$\hat{\vartheta}$ denotes the "linearisation" of ϑ.]

Theorem 2.5

For every pair of operator spaces $\mathbf{x}$ and $\mathbf{y}$, there exists a universal completely contractive bilinear transformation, which we shall denote

$$x \times y \longrightarrow \mathcal{V}\left(\mathbf{x} \boxtimes \mathbf{y}\right).$$

[A more common symbol for $\boxtimes$, which is called the *Haagerup tensor product*, is $\otimes_h$.]

Remarks 2.6

1. Given operator spaces $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, one can also consider bilinear transformations $x \times y \longrightarrow z$ such that, for all $p, q \in \mathbb{N}$, the map

$$x^{p\times p} \times y^{q\times q} \xrightarrow{\quad\otimes\quad} (x \otimes y)^{pq\times pq} \xrightarrow{\quad\vartheta^{pq\times pq}\quad} z^{pq\times pq}$$

   (here, by abuse of language, $\otimes$ also denotes the internalisation of the action

$$(x^p \xrightarrow{\alpha} x^p, y^q \xrightarrow{\beta} y^q) \mapsto \left[(x \otimes y)^{pq} \xrightarrow{\sim} x^p \otimes y^q \xrightarrow{\alpha\otimes\beta} x^p \otimes y^q \xrightarrow{\sim} (x \otimes y)^{pq}\right]$$

   as a map $x^{p\times p} \times y^{q\times q} \longrightarrow (x \otimes y)^{pq\times pq}$) is naïvely contractive (with respect to the norms $\|-\|_{(\mathbf{x},p)}$, $\|-\|_{(\mathbf{y},q)}$ and $\|-\|_{(\mathbf{z},pq)}$).

   This property is strictly weaker than that contained in Definition 2.4; it is therefore potentially misleading that bilinear transformations which satisfy it are called *jointly completely contractive.*

2. There also exists, for every pair of operator spaces $\mathbf{x}$ and $\mathbf{y}$, a universal jointly completely contractive bilinear transformation, here denoted

$$x \times y \longrightarrow \mathcal{V}\left(\mathbf{x} \widehat{\otimes} \mathbf{y}\right).$$

This is also called the *projective tensor product of operator spaces*, and is correctly viewed as the direct analogue of the projective tensor product of Banach spaces. But, be warned, the forgetful functor

$$(\mathbf{Oper}, \widehat{\otimes}, \mathbb{C}) \longrightarrow (\mathbf{Ban}, \widehat{\otimes}, \mathbb{C})$$

is only (lax) monoidal.

3. By contrast, there is no direct analogue of $\boxtimes$ among Banach spaces; it appears to be related to two of Grothendieck's fourteen tensor products on **Ban** [5]: the so-called *Hilbertian* tensor product (often denoted $\otimes_H$ or $\otimes_{\gamma_2}$) and its "dual" ($\otimes_{H'}$ or $\otimes_{\lambda_2}$).

4. The presence of matrix multiplication in the definition of completely contractive bilinear transformation renders $\boxtimes$ highly non-symmetric.

   For instance, given Hilbert spaces h and k, we can construct operator spaces $\mathbf{h}$ and $\mathbf{k}$ (overlying h and k respectively), such that $\mathbf{h} \boxtimes \mathbf{k}$ overlies the Banach space of all compact operators $k \longrightarrow h$, and such that $\mathbf{k} \boxtimes \mathbf{h}$ overlies that of all trace-class operators $k \longrightarrow h$. [Even in the finite-dimensional case, these spaces carry different norms, and are therefore not isomorphic as Banach spaces.]

   By contrast, $\widehat{\otimes}$ is indeed symmetric.

5. Both $\widehat{\otimes}$ and $\boxtimes$ are known to be (left- and right-) closed; but we shall not be needing this extra structure in the current paper.

6. The fact that joint complete contractivity is weaker than complete contractivity induces a natural transformation $\widehat{\otimes} \longrightarrow \boxtimes$.

   This natural transformation is the "multiplication" part of a (lax) monoidal functor

$$(\mathbf{Oper}, \boxtimes, \mathbb{C}) \longrightarrow (\mathbf{Oper}, \widehat{\otimes}, \mathbb{C})$$

   overlying the identity functor on **Oper**.

Theorem 2.7
A Banach algebra is isomorphic to a concrete operator algebra if and only if it lies in the range of the functor

$$\mathbf{Mon}(\mathbf{Oper}, \boxtimes, \mathbb{C}) \to \mathbf{Mon}(\mathbf{Oper}, \widehat{\otimes}, \mathbb{C}) \to \mathbf{Mon}(\mathbf{Ban}, \widehat{\otimes}, \mathbb{C})$$

—*i.e.*, if and only if its underlying Banach space can be endowed with an operator space structure, in such a way that its multiplication becomes a completely contractive bilinear map.

At this point, one should like to state and prove an analogous theorem for Banach $*$-algebras and C^*-algebras. But there are, at first glance, two monstrous obstacles to such a theorem:

1. the non-commutativity of $\boxtimes$ means that we cannot model the concept of involutive monoid in $(\mathbf{Oper}, \boxtimes, \mathbb{C})$, unless we are (both willing and) able to "tweak" it further than we already have;

2. the failure of $\mathbb{C}^{n\times n} \xrightarrow{(\ )^T} \mathbb{C}^{n\times n}$ to be a linear complete contraction means that we cannot even model the involution of this C^*-algebra—*i.e.*,

$$\overline{\mathbb{C}^{n\times n}} \xrightarrow{\quad (\ )^H = \overline{(\ )}^T = \overline{(\ )^T} \quad} \mathbb{C}^{n\times n}$$

   —as an arrow in **Oper**, unless we are (both willing and) able to find a different operator space structure on $\overline{\mathbb{C}^{n\times n}}$ than the obvious one (—that of $\mathbb{C}^{n\times n}$).

Remark 2.8

Pisier [7] claims, in effect, that $\mathbf{Mon}(\mathbf{Oper}, \boxtimes, \mathbb{C})$ itself and not the range of the functor

$$\mathbf{Mon}(\mathbf{Oper}, \boxtimes, \mathbb{C}) \longrightarrow \mathbf{Mon}(\mathbf{Oper}, \hat{\otimes}, \mathbb{C}) \longrightarrow \mathbf{Mon}(\mathbf{Ban}, \hat{\otimes}, \mathbb{C})$$

should be regarded as the correct category of *abstract operator algebras*.

In particular, he writes that two concrete operator algebras should be considered equivalent only if they have the same induced operator space structure, in addition to being isomorphic as algebras.

3 Involutive monoidal categories

As noted above, the algebraic theory of involutive monoids can be modelled in any symmetric monoidal category; in particular, it can be modelled in $(\mathbf{Cat}, \times, \imath)$. In this manner one obtains the small and strict version of what we call an involutive monoidal category.

Definition 3.1

An *involutive monoidal category* is a monoidal category $(\mathcal{K}; \otimes, i)$ together with a (covariant!) functor $\mathcal{K} \xrightarrow{\overline{(\ )}} \mathcal{K}$ and natural isomorphisms

$$\overline{x} \otimes \overline{y} \xrightarrow{\quad \psi_{x,y} \quad} \overline{y \otimes x}$$

$$\overline{\overline{x}} \xrightarrow{\quad \varepsilon_x \quad} x$$

satisfying the following coherence conditions:

$$
\begin{array}{ccc}
(\overline{x} \otimes \overline{y}) \otimes \overline{z} & \xrightarrow{\alpha_{\overline{x},\overline{y},\overline{z}}} & \overline{x} \otimes (\overline{y} \otimes \overline{z}) \\
{\scriptstyle \psi_{x,y} \otimes \mathrm{id}_{\overline{z}}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{\overline{x}} \otimes \psi_{y,z}} \\
\overline{y \otimes x} \otimes \overline{z} & & \overline{x} \otimes \overline{z \otimes y} \\
{\scriptstyle \psi_{y \otimes x,z}} \downarrow & & \downarrow {\scriptstyle \psi_{x,z \otimes y}} \\
\overline{x \otimes (y \otimes z)} & \xleftarrow[\overline{\alpha_{z,y,x}}]{} & \overline{(z \otimes y) \otimes x}
\end{array}
\qquad
\begin{array}{ccc}
\overline{\overline{x}} \otimes \overline{\overline{y}} & \xrightarrow{\psi_{\overline{x},\overline{y}}} & \overline{\overline{y} \otimes \overline{x}} \\
{\scriptstyle \varepsilon_x \otimes \varepsilon_y} \downarrow & & \downarrow {\scriptstyle \overline{\psi_{y,x}}} \\
x \otimes y & \xleftarrow[\varepsilon_{x \otimes y}]{} & \overline{\overline{x \otimes y}}
\end{array}
$$

and $\overline{\overline{\overline{x}}} \xrightarrow{\varepsilon_{\overline{x}} = \overline{\varepsilon_x}} \overline{x}$.

A fuller discussion of involutive monoidal categories will be provided in a subsequent paper, [3].

Let us briefly sketch a few non-trivial examples.

Examples 3.2

1. $(\mathbf{Cat}, \times, 1, (\ )^{\mathsf{op}})$ is an involutive monoidal category.

2. $(\mathbf{Pos}, +_{lex}, 0, (\ )^{\mathsf{op}})$ forms a (not at all symmetric) involutive monoidal category, where $+_{lex}$ denotes the "lexicographic sum" of two posets. [Every element of x is less than every element of y in $x +_{lex} y$.]

3. Let **Flo** denote the full subcategory of **Pos** determined by the finite linearly ordered sets. Clearly **Flo** is closed under $+_{lex}$, 0, and $(\ )^{\mathsf{op}}$, so $(\mathbf{Flo}, +_{lex}, 0, (\ )^{\mathsf{op}})$ again forms an involutive monoidal category.

   This example is noteworthy for the fact that, although $x^{\mathsf{op}} \cong x$ for every object x, there is no natural isomorphism $(\ )^{\mathsf{op}} \cong \mathrm{Id}$.

   [This is just another way of looking at the celebrated fact that $(\mathbf{Flo}, +_{lex}, 0)$ is not a symmetric monoidal category despite the fact that $x +_{lex} y \cong y +_{lex} x$ for every pair of objects x and y.]

4. If R is an involutive ring (*i.e.*, an involutive monoid in the symmetric monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$), then the category ${}_R\mathbf{Mod}_R$ of (two-sided) R-modules form an involutive monoidal category with respect to the usual tensor product, and the involution defined below.

   Given an R-module A (with left and right R-actions denoted by $\triangleright$ and $\triangleleft$ respectively), $\overline{A}$ has the same underlying abelian group as A, but with left and right actions defined by

$$
\begin{array}{rcl}
\rho \mathbin{\overline{\triangleright}} \alpha & = & \alpha \triangleleft \overline{\rho} \\
\alpha \mathbin{\overline{\triangleleft}} \rho & = & \overline{\rho} \triangleright \alpha
\end{array}
$$

   [Of course, this is only part of a more general structure on the bicategory of rings, bimodules and bimodule homomorphisms, but we won't go into that here.]

5. In particular, $\mathbb{C}$ equipped with conjugation is an involutive ring, and so $(\mathbf{Vec}, \otimes, \mathbb{C}, \overline{(\ )})$ forms an involutive monoidal category.

   Moreover, any norm on a complex vector space x is also a norm on $\overline{x}$. Thus the involutive structure of **Vec** extends to **Ban**; and, in particular, $(\mathbf{Ban}, \hat{\otimes}, \mathbb{C}, \overline{(\ )})$ is an involutive monoidal category.

   Here the isomorphism $\overline{x} \hat{\otimes} \overline{y} \xrightarrow{\psi_{x,y}} \overline{y \hat{\otimes} x}$ is the same as the composite

$$\overline{x} \hat{\otimes} \overline{y} \xrightarrow{\chi_{\overline{x},\overline{y}}} \overline{y} \hat{\otimes} \overline{x} \xrightarrow{\sim} \overline{y \hat{\otimes} x}$$

   used (implicitly) in Definition 1.1 above.

One can, of course, also define *involutive monoidal functors* and *involutive monoidal natural transformations*; but in the present paper we shall consider only a special case of these. Definitions of the more general concepts will appear in a subsequent paper, [3].

Definitions 3.3

Let $(\mathcal{K}, \otimes, i, \overline{(\ )})$ be an involutive monoidal category.

1. An *involutive monoid* in $(\mathcal{K}, \otimes, i, \overline{(\ )})$ is a monoid (m, μ, η) in $(\mathcal{K}, \otimes, i)$ together with an arrow $\overline{m} \xrightarrow{\nu} m$ such that the diagrams

$$\begin{array}{ccccc} \overline{m} \otimes \overline{m} & & \xrightarrow{\quad\psi\quad} & & \overline{m \otimes m} \\ {\scriptstyle \nu \otimes \nu}\downarrow & & & & \downarrow{\scriptstyle \overline{\mu}} \\ m \otimes m & \xrightarrow[\mu]{} & m & \xleftarrow[\nu]{} & \overline{m} \end{array}$$

$$\overline{\overline{m}} \xrightarrow{\overline{\nu}} \overline{m} \xrightarrow{\nu} m, \qquad \nu \circ \overline{\nu} = \varepsilon$$

   commute.

2. An *involutive monoid homomorphism* $(m, \mu, \eta, \nu) \xrightarrow{\vartheta} (n, \mu, \eta, \nu)$ is a monoid homomorphism $(m, \mu, \eta) \xrightarrow{\vartheta} (n, \mu, \eta)$ which respects involution, in the sense that the diagram

$$\begin{array}{ccc} \overline{m} & \xrightarrow{\nu} & m \\ {\scriptstyle \overline{\vartheta}}\downarrow & & \downarrow{\scriptstyle \vartheta} \\ \overline{n} & \xrightarrow[\nu]{} & n \end{array}$$

   commutes.

Examples 3.4

1. Involutive monoids in $(\mathbf{Cat}, \times, 1, (\ )^{\mathsf{op}})$ are [the strict (and small) version of] what are sometimes called $*$-*monoidal categories*—see, for instance, [AbrBluPan].

2. Involutive monoids in $(\mathbf{Ban}, \hat{\otimes}, \mathbb{C}, \overline{(\ )})$ are precisely Banach $*$-algebras.

4 Quantum conjugation

Theorem 4.1

Let $\mathbf{x}$ be an operator space, and let the functions $x^{n\times n} \xrightarrow{\|-\|_{(\mathbf{x}^{\mathsf{op}},n)}} [0,\infty)$ be defined by

$$\|\alpha\|_{(\mathbf{x}^{\mathsf{op}},n)} := \left\|\alpha^T\right\|_{(\mathbf{x},n)}.$$

Then x together with the $\|-\|_{(\mathbf{x}^{\mathsf{op}},n)}$s do indeed form an operator space, denoted $\mathbf{x}^{\mathsf{op}}$.

Moreover, if $\mathbf{x} \xrightarrow{\vartheta} \mathbf{y}$ is a linear complete contraction, then it is also a linear complete contraction $\mathbf{x}^{\mathsf{op}} \xrightarrow{\vartheta} \mathbf{y}^{\mathsf{op}}$.

Thus there is a covariant involution of **Oper** defined by the mappings $\mathbf{x} \mapsto \widetilde{\mathbf{x}}$ and $\vartheta \mapsto \vartheta$. [For convenience sake, we shall define $\vartheta^{\mathsf{op}} := \vartheta$, so that we can refer to this functor as $(\ )^{\mathsf{op}}$.]

Proof

It is self-evident that the functions $\overline{x}^{n\times n} \xrightarrow{\|-\|_{(\mathbf{x}^{\mathsf{op}},n)}} [0,\infty)$ are Cauchy-complete norms, so it remains to check that the two compatibility axioms hold.

Suppose that $\zeta \in \mathbb{C}^{n\times m}$, $\alpha \in \overline{x}^{n\times n}$ and $\omega \in \mathbb{C}^{m\times n}$; then

$$\begin{aligned}
\|\zeta\cdot\alpha\cdot\omega\|_{(\mathbf{x}^{\mathsf{op}},m)} &= \left\|(\zeta\cdot\alpha\cdot\omega)^T\right\|_{(\mathbf{x},m)} \\
&= \left\|\omega^T\cdot\alpha^T\cdot\zeta^T\right\|_{(\mathbf{x},m)} \\
&\le \left\|\omega^T\right\|_\infty \cdot \left\|\alpha^T\right\|_{(\mathbf{x},n)} \cdot \left\|\zeta^T\right\|_\infty \\
&= \|\omega\|_\infty \cdot \|\alpha\|_{(\mathbf{x}^{\mathsf{op}},n)} \cdot \|\zeta\|_\infty \\
&= \|\zeta\|_\infty \cdot \|\alpha\|_{(\mathbf{x}^{\mathsf{op}},n)} \cdot \|\omega\|_\infty .
\end{aligned}$$

Now, suppose that $\alpha \in x^{n\times n}$ and $\beta \in x^{m\times m}$; then

$$\begin{aligned}
\left\|\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right\|_{(\mathbf{x}^{\mathsf{op}},n+m)} &= \left\|\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}^T\right\|_{(\mathbf{x},n+m)} \\
&= \left\|\begin{pmatrix}\alpha^T & 0\\ 0 & \beta^T\end{pmatrix}\right\|_{(\mathbf{x},n+m)} \\
&\le \max\left\{\left\|\alpha^T\right\|_{(\mathbf{x},n)}, \left\|\beta^T\right\|_{(\mathbf{x},m)}\right\} \\
&= \max\left\{\|\alpha\|_{(\mathbf{x}^{\mathsf{op}},n)}, \|\beta\|_{(\mathbf{x}^{\mathsf{op}},m)}\right\}.
\end{aligned}$$

Further, if $x \xrightarrow{\vartheta} y$ is an arbitrary linear transformation, then $x^{n\times n} \xrightarrow{\vartheta^{n\times n}} y^{n\times n}$ clearly satisfies

$$\vartheta^{n\times n}\left(\alpha^T\right) = \vartheta^{n\times n}(\alpha)^T.$$

Therefore, if $\mathbf{x} \xrightarrow{\vartheta} \mathbf{y}$ is a linear complete contraction, then

$$\left\|\vartheta^{n\times n}(\alpha)\right\|_{(\mathbf{x}^{\mathsf{op}},n)} = \left\|\vartheta^{n\times n}(\alpha)^T\right\|_{(\mathbf{x},n)}$$

$$
\begin{aligned}
&= \left\|\vartheta^{n\times n}\left(\alpha^{T}\right)\right\|_{(\mathbf{x},n)} \\
&\leq \left\|\alpha^{T}\right\|_{(\mathbf{y},n)} \\
&= \|\alpha\|_{(\mathbf{y}^{\mathsf{op}},n)} .
\end{aligned}
$$

Q.E.D.

Definition 4.2

We write $\widetilde{(\ )}$ for the composite $\mathbf{Oper} \xrightarrow{\overline{(\ )}} \mathbf{Oper} \xrightarrow{(\ )^{\mathsf{op}}} \mathbf{Oper}$.

Example 4.3

If $A = (a;\mu,\eta,\nu)$ is a C^*-algebra, then $\widetilde{\mathbf{a}} \xrightarrow{\nu} \mathbf{a}$ is a linear complete isometry—*i.e.*, an invertible map in **Oper**.

To see this, let $\alpha = (\alpha_{jk}) \in \overline{a}^{n\times n}$; then

$$
\nu^{n\times n}(\alpha) = \begin{pmatrix} \alpha_{11}^* & \cdots & \alpha_{1n}^* \\ \vdots & \ddots & \vdots \\ \alpha_{n1}^* & \cdots & \alpha_{nn}^* \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{n1} \\ \vdots & \ddots & \vdots \\ \alpha_{1n} & \cdots & \alpha_{nn} \end{pmatrix}^* = \left(\alpha^T\right)^*
$$

and therefore $\|\nu^{n\times n}(\alpha)\|_{(\mathbf{a},n)} = \left\|(\alpha^T)^*\right\|_{(\mathbf{a},n)} = \left\|\alpha^T\right\|_{(\overline{\mathbf{a}},n)} = \|\alpha\|_{(\widetilde{\mathbf{a}},n)}$.

In particular, $\widetilde{\mathbb{C}^{n\times n}} \xrightarrow{(\ )^H} \mathbb{C}^{n\times n}$ is a linear complete contraction; hence the identity function $\widetilde{\mathbb{C}^{n\times n}} \longrightarrow \overline{\mathbb{C}^{n\times n}}$ is not a map in **Oper**, for $n > 1$.

Theorem 4.4

There exists a natural isomorphism

$$
\mathbf{x}^{\mathsf{op}} \boxtimes \mathbf{y}^{\mathsf{op}} \xrightarrow{\psi_{\mathbf{x},\mathbf{y}}} (\mathbf{y} \boxtimes \mathbf{x})^{\mathsf{op}}
$$

which, together with the identity map $(\mathbf{x}^{\mathsf{op}})^{\mathsf{op}} \longrightarrow \mathbf{x}$, satisfies all the coherence conditions listed in Definition 3.1.

Thus $(\mathbf{Oper}, \boxtimes, \mathbb{C}, (\ )^{\mathsf{op}})$ is an involutive monoidal category. Similarly, $(\mathbf{Oper}, \boxtimes, \mathbb{C}, \widetilde{(\ )})$ is also an involutive monoidal category.

Proof

The crucial observation is that, given $\alpha \in x^{n\times n}$ and $\beta \in y^{n\times n}$, $(\alpha \odot \beta)^T$ is almost, but not quite, the same as $\beta^T \odot \alpha^T$.

More precisely, we have

$$
(\alpha \odot \beta)^T_{jk} = (\alpha \odot \beta)_{kj} = \sum_{l=1}^{n} \alpha_{kl} \otimes \beta_{lj}
$$

and

$$
(\beta^T \odot \alpha^T)_{jk} = \sum_{l=1}^{n} \beta^T_{jl} \otimes \alpha^T_{lk} = \sum_{l=1}^{n} \beta_{lj} \otimes \alpha_{kl}
$$

—thus

$$\chi_{x,y}\left((\alpha\odot\beta)^T_{jk}\right) = (\beta^T\odot\alpha^T)_{jk}$$

for all $j,k\leq n$—or, more simply put,

$$\left(\chi^{n\times n}_{x,y}(\alpha\odot\beta)\right)^T = \chi^{n\times n}_{x,y}\left((\alpha\odot\beta)^T\right) = \beta^T\odot\alpha^T.$$

Now it follows that

$$\begin{aligned}\left\|\chi^{n\times n}_{x,y}(\alpha\odot\beta)\right\|_{(\mathbf{y}\boxtimes\mathbf{x}^{\mathsf{op}},n)} &= \left\|\left(\chi^{n\times n}_{x,y}(\alpha\odot\beta)\right)^T\right\|_{(\mathbf{y}\boxtimes\mathbf{x},n)}\\ &= \left\|\beta^T\odot\alpha^T\right\|_{(\mathbf{y}\boxtimes\mathbf{x},n)}\\ &\leq \left\|\beta^T\right\|_{(\mathbf{y},n)}\cdot\left\|\alpha^T\right\|_{(\mathbf{x},n)}\\ &= \|\alpha\|_{(\mathbf{x}^{\mathsf{op}},n)}\cdot\|\beta\|_{(\mathbf{y}^{\mathsf{op}},n)}.\end{aligned}$$

Hence the composite

$$x\times y\xrightarrow{\otimes}x\otimes y\xrightarrow{\chi_{x,y}}y\otimes x\rightarrowtail\mathcal{V}(\mathbf{y}\boxtimes\mathbf{x}) = \mathcal{V}\left((\mathbf{y}\boxtimes\mathbf{x})^{\mathsf{op}}\right)$$

is a completely contractive bilinear map; by the universal property of $\boxtimes$, this establishes the existence of a linear complete contraction

$$\mathbf{x}^{\mathsf{op}}\boxtimes\mathbf{y}^{\mathsf{op}}\xrightarrow{\psi_{\mathbf{x},\mathbf{y}}}(\mathbf{y}\boxtimes\mathbf{x})^{\mathsf{op}}$$

as desired.

Naturality and the coherence conditions are absolutely trivial, and the second of them implies that $\psi_{\mathbf{x},\mathbf{y}}$ is invertible, with inverse $\psi^{\mathsf{op}}_{\mathbf{x}^{\mathsf{op}},\mathbf{y}^{\mathsf{op}}}$.

The composite

$$\overline{\mathbf{x}}^{\mathsf{op}}\boxtimes\overline{\mathbf{y}}^{\mathsf{op}}\longrightarrow(\overline{\mathbf{y}}\boxtimes\overline{\mathbf{x}})^{\mathsf{op}}\longrightarrow\overline{\mathbf{y}\boxtimes\mathbf{x}}^{\mathsf{op}}$$

defines a natural isomorphism $\widetilde{\mathbf{x}}\boxtimes\widetilde{\mathbf{y}}\longrightarrow\widetilde{\mathbf{y}\boxtimes\mathbf{x}}$ which, by abuse of language, we also denote $\psi_{\mathbf{x},\mathbf{y}}$. Q.E.D.

Theorem 4.5

Every C^*-algebra can be regarded as an involutive monoid in $(\mathbf{Oper},\boxtimes,\mathbb{C},\widetilde{(\ )})$.

Proof

Let A be a C^*-algebra. Then $\mathbf{a}\times\mathbf{a}\stackrel{\cdot}{\longrightarrow}\mathbf{a}$ is a completely contractive bilinear transformation; hence we obtain a monoid structure on $\mathbf{a}$ in $(\mathbf{Oper},\boxtimes,\mathbb{C})$. Moreover, we have already shown that $\widetilde{\mathbf{a}}\xrightarrow{(\ )^*}\mathbf{a}$ is a completely contractive map.

So it remains to show that the appropriate diagrams commute: but this is an elementary diagram chase. Q.E.D.

It is not true, however, that every involutive monoid in $(\mathbf{Oper}, \boxtimes, \mathbb{C})$ is a C^*-algebra.

Example 4.6

Let A denote the concrete operator algebra consisting of those (2×2)-matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$$

and let $\mathbf{a}$ denote its underlying operator space.

It is plain that A is commutative, but not of the form $C(X)$ for any compact Hausdorff space X; hence, it can not underlie any C^*-algebra.

But, although the identity function $\widetilde{\mathbb{C}^{2\times 2}} \longrightarrow \overline{\mathbb{C}^{2\times 2}}$ is not a linear complete contraction (see above), its restriction to $\mathbf{a}$ is.

[By Smith's Lemma [2], it suffices to show that $\widetilde{\mathbf{a}}^{2\times 2} \xrightarrow{\text{id}} \overline{\mathbf{a}}^{2\times 2}$ is a linear contraction—*i.e.*, that

$$\left\| \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha & 0 & \gamma \\ \zeta & \vartheta & \psi & \omega \\ 0 & \zeta & 0 & \psi \end{pmatrix} \right\|_{(\overline{\mathbf{a}},2)} \leq \left\| \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha & 0 & \gamma \\ \zeta & \vartheta & \psi & \omega \\ 0 & \zeta & 0 & \psi \end{pmatrix} \right\|_{(\widetilde{\mathbf{a}},2)} = \left\| \begin{pmatrix} \alpha & 0 & \gamma & 0 \\ \beta & \alpha & \delta & \gamma \\ \zeta & 0 & \psi & 0 \\ \vartheta & \zeta & \omega & \psi \end{pmatrix} \right\|_{(\overline{\mathbf{a}},2)} .$$

But since

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha & 0 & \gamma \\ \zeta & \vartheta & \psi & \omega \\ 0 & \zeta & 0 & \psi \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha & 0 & \gamma & 0 \\ \beta & \alpha & \delta & \gamma \\ \zeta & 0 & \psi & 0 \\ \vartheta & \zeta & \omega & \psi \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

and $\left\| \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \right\|_\infty = 1$, it follows from axiom (a) of Definition 2.1.1 that

$$\left\| \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha & 0 & \gamma \\ \zeta & \vartheta & \psi & \omega \\ 0 & \zeta & 0 & \psi \end{pmatrix} \right\|_{(\overline{\mathbf{a}},2)} \leq 1 \cdot \left\| \begin{pmatrix} \alpha & 0 & \gamma & 0 \\ \beta & \alpha & \delta & \gamma \\ \zeta & 0 & \psi & 0 \\ \vartheta & \zeta & \omega & \psi \end{pmatrix} \right\|_{(\overline{\mathbf{a}},2)} \cdot 1$$

as desired.]

This entails that the map

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \mapsto \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ 0 & \overline{\alpha} \end{pmatrix} .$$

—which is plainly an involution on A—defines a linear complete contraction $\widetilde{\mathbf{a}} \xrightarrow{\nu} \mathbf{a}$, which in turn makes $\mathbf{a}$ into an involutive monoid in $(\mathbf{Oper}, \boxtimes, \mathbb{C}, \widetilde{(\ )})$.

5 Conclusions and Future Work

Obviously, the original problem of finding a diagrammatic definition of C^*-algebras remains open.

But we feel that even the partial result contained in this paper has profound implications for the project of C.J. Mulvey *et al.* to capture the essence of non-commutative topology in terms of involutive quantales.

An involutive quantale is, of course, an involutive monoid in the trivially-involutive monoidal category $(\mathbf{Sup}, \hat{\otimes}, 2, \mathrm{Id})$. But, as indicated in the preamble to Definitions 3.3, involutive monoids are (in the usual way) merely a special case of *involutive monoidal functors.*

Therefore, an involutive monoidal functor

$$(\mathbf{Ban}, \hat{\otimes}, e, \overline{(\ )}) \xrightarrow{(M,\mu,\eta,\nu)} (\mathbf{Sup}, \hat{\otimes}, 2, \mathrm{Id})$$

determines a functor

$$\mathbf{B^*Alg} := \mathbf{IM}(\mathbf{Ban}, \hat{\otimes}, \mathbb{C}, \overline{(\ )}) \longrightarrow \mathbf{IM}(\mathbf{Sup}, \hat{\otimes}, 2, \mathrm{Id}) =: \mathbf{IQ}$$

by composition:

$$\begin{array}{ccc} \mathbf{1} & \cdots\cdots\rightarrow & (\mathbf{Ban}, \hat{\otimes}, \mathbb{C}, \overline{(\ )}) \\ & \searrow & \Big\downarrow (M,\mu,\eta,\nu) \\ & & (\mathbf{Sup}, \hat{\otimes}, 2, \mathrm{Id}) \end{array}$$

—and it appears that the functor $\mathbf{C^*Alg} \xrightarrow{Max} \mathbf{IQ}$ arises as the composite of such a functor with the forgetful functor $\mathbf{C^*Alg} \longrightarrow \mathbf{B^*Alg}$.

But as pointed out in [6], for instance, the functor *Max* is far from ideal.

As a result of Theorem 4.5, we are presented with potential alternatives: if we were to find an involutive monoidal functor

$$(\mathbf{Oper}, \boxtimes, C, \widetilde{(\ )}) \xrightarrow{(M,\mu,\eta,\nu)} (\mathbf{Sup}, \hat{\otimes}, 2, \mathrm{Id})$$

which did not factor through the forgetful involutive monoidal functor $(\mathbf{Oper}, \boxtimes, \mathbb{C}, \widetilde{(\ )}) \longrightarrow (\mathbf{Ban}, \hat{\otimes}, \mathbb{C}, \overline{(\ )})$ then the corresponding functor

$$\mathbf{IOA} := \mathbf{IM}(\mathbf{Oper}, \boxtimes, \mathbb{C}, \widetilde{(\ )}) \longrightarrow \mathbf{IM}(\mathbf{Sup}, \hat{\otimes}, 2, \mathrm{Id}) =: \mathbf{IQ}$$

would not necessarily factor through $\mathbf{B^*Alg}$; therefore its composite with the "less forgetful" functor $\mathbf{C^*Alg} \longrightarrow \mathbf{IOA}$ could retain more information about a C^*-algebra, and therefore (potentially) have better properties than *Max*.

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