Quillen model categories without equalisers or coequalisers

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Abstract

Quillen defined a *model category* to be a category with finite limits and colimits carrying a certain extra structure. In this paper, we show that only finite products and coproducts (in addition to the certain extra structure alluded to above) are really necessary to construct the homotopy category. This leads to the interesting observation that the homotopy category construction could feasibly be iterated.

1 Introduction

This paper concerns the definition of Quillen model category and the most basic fact which follows from said definition. In particular, we show that this same result can be obtained with slightly weaker hypotheses; *i.e.*, that the definition of Quillen model category can be weakened "at no extra cost".

The bone of contention is the number—or, more accurately, class—of limits and colimits which the category is required to possess. We recall that a category $\mathcal{K}$ has arbitrary (finite) limits if and only if it has equalisers and arbitrary (finite) products, [6, p.113]. Dually, $\mathcal{K}$ has arbitrary (finite) colimits if and only if it has coequalisers and arbitrary (finite) coproducts.

The original motivation for this result lay in the author's attempt to apply the theory of Quillen model structures to categories arising in the study of linear logic, [2, 3]; such categories frequently possess only products and coproducts. Nevertheless, we consider the result to be of independent interest, and conclude the article by discussing some potential applications within the more traditional demesne of topology and geometry.

2 Background

The most basic fact about Quillen model categories, alluded to above, is this: given a Quillen model category $\mathcal{K}$, one can define a category-theoretic *congruence* [6, p.52] $\sim$ on a certain full subcategory of $\mathcal{K}$, denoted $\mathcal{K}_{cf}$, such that

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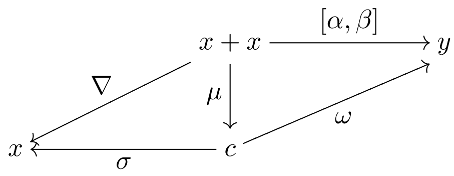
the resultant quotient category $\mathcal{K}_{cf}/\!\sim$ is equivalent to the *category of fractions* obtained by inverting the weak equivalences in $\mathcal{K}$.

We will not repeat the definition of Quillen model structure—which can be found, for example, in [5, 1.1.3]. But we will quickly review the definition(s) of the homotopy relation, $\sim$. For the remainder of the paper, $\mathcal{K}$ will denote a category with finite products and finite coproducts equipped with a Quillen model structure.

Definitions 2.1

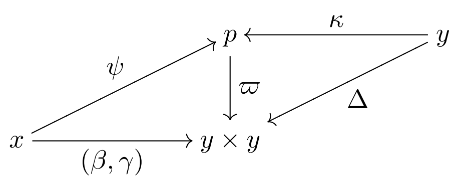
Two arrows $x \underset{\beta}{\overset{\alpha}{\rightrightarrows}} y$ in $\mathcal{K}$ are called

1. *left-homotopic*, denoted $\alpha \overset{\ell}{\sim} \beta$, if there exists a diagram



with μ a cofibration and σ a weak equivalence.

2. *right-homotopic*, denoted $\alpha \overset{r}{\sim} \beta$, if there exists a diagram



with ϖ a fibration and κ a weak equivalence.

The following statements are well-known, and their proofs do not use the existence of limits and colimits other than products and coproducts. [Indeed, the first two are almost tautologous. But the third is quite interesting, as we shall see below.]

Lemmata 2.2

1. The relation $\overset{\ell}{\sim}$ is reflexive, symmetric and satisfies left-congruity—*i.e.*, $\alpha \overset{\ell}{\sim} \beta \Rightarrow \theta\alpha \overset{\ell}{\sim} \theta\beta$, whenever this makes sense.
2. The relation $\overset{r}{\sim}$ is reflexive, symmetric and satisfies right-congruity—*i.e.*, $\alpha \overset{r}{\sim} \beta \Rightarrow \alpha\theta \overset{r}{\sim} \beta\theta$, whenever this makes sense.
3. The restriction of $\overset{\ell}{\sim}$ to $\mathcal{K}_{cf}$ coincides with that of $\overset{r}{\sim}$.

What remains to show is that the common restriction of $\overset{\ell}{\sim}$ and $\overset{r}{\sim}$ to $\mathcal{K}_{cf}$, henceforth denoted $\sim$,[1] is transitive.

It is worth noting at this stage that the usual proof of the transitivity of $\sim$ (as found in [7, Lemma 4]), requires the existence of pushouts but not of pullbacks. Of course, there is a dual proof of the same fact which requires pullbacks but not pushouts. It would seem quite odd if the existence of pushouts-or-pullbacks were necessary as well as sufficient.

3 Transitivity of $\sim$

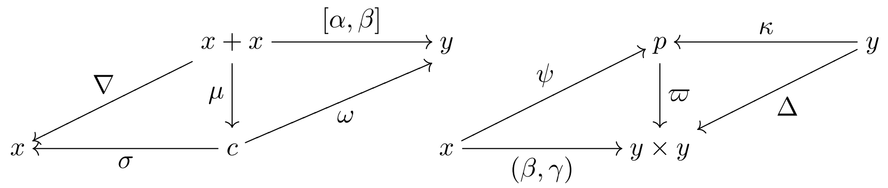
The following lemma, although stated in somewhat more general terms, essentially amounts to the transitivity of $\sim$. Its proof is, in fact, an adaptation of the usual proof of that $\overset{\ell}{\sim}$ and $\overset{r}{\sim}$ coincide on $\mathcal{K}_{cf}$.

Lemma 3.1

Let α, β and γ be parallel arrows $x \longrightarrow y$, and suppose $\alpha \overset{\ell}{\sim} \beta \overset{r}{\sim} \gamma$. Then y fibrant implies $\alpha \overset{\ell}{\sim} \gamma$ dually, x cofibrant implies $\alpha \overset{r}{\sim} \gamma$.

Proof

Suppose that y is fibrant and that the relations $\alpha \overset{\ell}{\sim} \beta \overset{r}{\sim} \gamma$ are witnessed as follows:



with μ a cofibration, ϖ a fibration, and σ, κ weak equivalences.

Let ϖ_0, ϖ_1 denote the two components of ϖ so that $\varpi = (\varpi_0, \varpi_1)$. By a standard argument, y fibrant implies that $y \times y \xrightarrow{\pi_0} y$ and $y \times y \xrightarrow{\pi_1} y$ are fibrations; hence also $\varpi_0 = \pi_0 \varpi$ and $\varpi_1 = \pi_1 \varpi$. Moreover, $\varpi_0 \kappa = \mathrm{id}_y = \varpi_1 \kappa$, so by two-out-of-three, ϖ_0, ϖ_1 are also weak equivalences.

Now we can factor $[\alpha, \gamma]$ through ϖ_1 as follows:

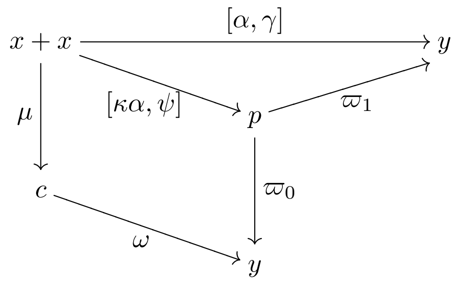
$$[\alpha, \gamma] = [\varpi_1 \kappa \alpha, \varpi_1 \psi] = \varpi_1 [\kappa \alpha, \psi]$$

and moreover,

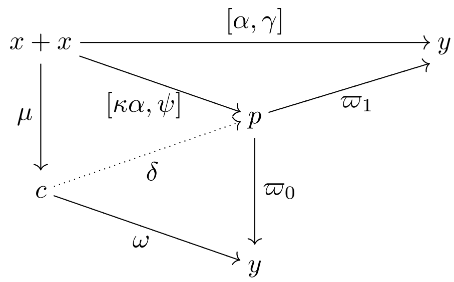
$$\varpi_0 [\kappa \alpha, \psi] = [\varpi_0 \kappa \alpha, \varpi_0 \psi] = [\alpha, \beta] = \omega \mu.$$

[1] In this, my notation is slightly non-standard. More often, one writes $\psi \sim \omega$ to mean that both $\psi \overset{\ell}{\sim} \omega$ and $\psi \overset{r}{\sim} \omega$ hold.

Hence, we have a diagram



with μ a cofibration and ϖ_0 a trivial fibration. Therefore, we can find a diagonal lift



and so the composite $\varpi_1\delta$ witnesses $\alpha \overset{\ell}{\sim} \gamma$. Q.E.D.

Theorem 3.2
The relation $\sim$ is a congruence on $\mathcal{K}_{cf}$; moreover, the quotient category $\mathcal{K}_{cf}/{\sim}$ is equivalent to $\mathcal{K}[\mathcal{W}^{-1}]$.

Proof
We have already established the first statement via a series of lemmata; the second is proven as in [5]. Q.E.D.

4 Iterated Homotopy

It is well known that, for an arbitrary Quillen model category $\mathcal{K}$, the homotopy category $\mathsf{Ho}[\mathcal{K}]$ does not have arbitrary limits and colimits. It does, however, inherit discrete limits and colimits from $\mathcal{K}$. Thus the result of this article shows that, if we can find a Quillen model structure on $\mathcal{H} = \mathsf{Ho}[\mathcal{K}]$, then there is no obstruction to forming a further homotopy category, $\mathsf{Ho}[\mathcal{H}] = \mathsf{Ho}[\mathsf{Ho}[\mathcal{K}]]$.

This observation could be utilised in two opposite ways: one might try to find Quillen model structures on already known homotopy categories—this might prove a useful way of studying individual homotopy invariants; or, perhaps more interestingly, one might attempt to 'subdivide' ordinary homotopy into smaller, and hopefully more tractable, steps. Let us illustrate the latter idea with an example.

Consider the concept of *thin homotopy* which arises in differential geometry, [1].

Questions 4.1

1. Does there exist a Quillen model structure on some category $\mathcal{M}$ such that:
   (a) $\mathcal{M}$ contains the category of finite-dimensional manifolds and smooth maps;
   (b) $\mathcal{M}$ is closed under finite products and coproducts; and
   (c) arrows between finite-dimensional manifolds in $\mathsf{Ho}[\mathcal{M}]$ correspond to thin-homotopy equivalence classes of smooth maps?
2. If so, does there exist a Quillen model structure on $\mathcal{H} = \mathsf{Ho}[\mathcal{M}]$ such that arrows between finite-dimensional manifolds in $\mathsf{Ho}[\mathcal{H}]$ $(= \mathsf{Ho}[\mathsf{Ho}[\mathcal{M}]])$ correspond to (ordinary) homotopy equivalence classes of continuous maps?

At first glance, the ideal candidate for $\mathcal{M}$ would seem to be the category of finite disjoint unions of finite-dimensional manifolds (of possibly differing dimension). This category is indeed closed under finite products and coproducts; but, at second glance, it would seem to lack appropriate cylinder objects. For this reason, it would seem to be necessary to consider a larger category—perhaps something along the lines of diffeological spaces [8], or Frölicher spaces [4].

But, note that, even if we were to choose $\mathcal{M}$ to be closed under arbitrary (finite) limits and colimits—and so avoid using the result of this article with respect to the first question posed above—there is still no guarantee that $\mathcal{H} = \mathsf{Ho}[\mathcal{M}]$ would be similarly closed.

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