Section 4.5 Dimension

4.5.6. Problem Restatement: Find (a) a basis, and (b) state the dimension of the subspace
\[ \{ \begin{bmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{bmatrix} ; a, b, c \in \mathbb{R} \}. \]

Final Answer: (a) \{ \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix} \} \text{ is a basis of the } \mathbb{R}^3 \text{ subspace.}

(b) The dimension of the subspace is 2 since a basis for it has two vectors in it.

Work: (a) \{ \begin{bmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{bmatrix} ; a, b, c \in \mathbb{R} \} = \text{Span}\{ \begin{bmatrix} 3 \\ 6 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 5 \\ 3 \end{bmatrix} \}. \text{ The first and third vectors are multiples of each other, and the first and second vectors are linearly independent of each other so we can remove the third to get a basis (that is, a linearly independent set of spanning vectors) of the subspace.}

(b) No work required.

4.5.12. Problem Restatement: Determine the dimension of
\[ W = \text{Span}\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix} \}. \]

Final Answer: \( \text{dim}(W) = 3. \)

Work: \( \text{dim}(W) = \text{dim}(\text{Col} \begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix}) = \text{dim}(\text{Col} \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & 1 & 5 & 3 \end{bmatrix}). \)

Since there are 3 pivot columns, \( \text{dim}(W) = 3. \)

4.5.14. Problem Restatement: Determine the dimensions of \( \text{Nul}(A) \) and \( \text{Col}(A) \) for
\[
A = \begin{bmatrix}
1 & 3 & -4 & 2 & -1 & 6 \\
0 & 0 & 1 & -3 & 7 & 0 \\
0 & 0 & 0 & 1 & 4 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

**Final Answer:** \( \dim(\text{Col}(A)) = \text{Rank}(A) = 3 \) and \( \dim(\text{Nul}(A)) = 6 - \text{Rank}(A) = 6 - 3 = 3 \).

**Work:** \( A \) is in echelon form and we count 3 pivot columns in \( A \).

### 4.5.24. Problem Restatement:
Let \( B \) be the basis of \( P_2 \) consisting of the first three Laguerre polynomials, \( \{1, 1 - t, 2 - 4t + t^2\} \), and let \( p(t) = 7 - 8t + 3t^2 \). Find the coordinate vector of \( p \) relative to \( B \).

**Final Answer:** \([p(t)]_B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}\).

**Work:** Several equivalent methods to solve this problem are possible. One of these follows.

Let \( S = \{1, t, t^2\} \) be the standard basis of \( P_2 \). \([p(t)]_S = P_B^{-1}(p(t)) = P_S^{-1}(P_B \begin{bmatrix} 7 \\ -8 \\ 3 \end{bmatrix}) =

(P_B^{-1}P_S)\begin{bmatrix} 7 \\ -8 \\ 3 \end{bmatrix} = (P_S^{-1}P_B)^{-1} \begin{bmatrix} 7 \\ -8 \\ 3 \end{bmatrix} \). Therefore, if \( b = \begin{bmatrix} 7 \\ -8 \\ 3 \end{bmatrix}, \) then \([p(t)]_B\) is the solution of \((P_S^{-1}P_B)x = b\). Now \( P_S^{-1}P_B \) is easy to construct; it is the matrix \(\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 1 \end{bmatrix}\).

Passing to the augmented matrix and row reducing to reduced echelon form gives us

\([P_S^{-1}P_B|b] = \begin{bmatrix} 1 & 1 & 2 & 7 \\ 0 & -1 & -4 & -8 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \). Therefore, \([p(t)]_B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}\).

### 4.5.26. Problem Restatement:
Let \( H \) be an \( n \)-dimensional subspace of an \( n \)-dimensional vector space \( V \). Show that \( H = V \).

**Final Answer:** Choose a basis \( B = \{b_1, \ldots, b_n\} \) of \( H \), so that \( H = \text{Span}(B) \). By the Basis Theorem, page 257, \( B \) is also a basis of \( V \), so \( V = \text{Span}(B) \). Therefore, \( H = V \), since they are each equal \( \text{Span}(B) \).

**Work:** None required.
Section 4.6 Rank

4.6.2. Problem Restatement: Assume that $A$ is row equivalent to $B$ where

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

Without calculation, list $\text{Rank}(A)$ and $\text{dim}(\text{Nul}(A))$. Then find a basis for $\text{Col}(A)$, $\text{Row}(A)$, and $\text{Nul}(A)$.

Final Answer:

$\text{Rank}(A) = 3$ and $\text{dim}(\text{Nul}(A)) = 2$.

A basis of $\text{Col}(A)$ is given by the first, third, and fifth columns of $A$.
A basis of $\text{Row}(A)$ is given by the first three rows of $A$ (or of $B$).

A basis of $\text{Nul}(A)$ is \{ $\begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ \} (see work below).

Work: $A \sim B \sim \begin{bmatrix} 1 & -3 & 0 & 5 & 0 \\ 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, so if $x$ is a solution of the homogeneous system $Ax = 0$, we have $x_2$ and $x_4$ free, giving us

$x_1 = 3x_2 - 5x_4$,

$x_2 = x_1 + 0x_4$,

$x_3 = 0x_2 + (3/2)x_4$,

$x_4 = 0x_2 + x_4$,

$x_5 = 0x_2 + 0x_4$.

4.6.6. Problem Restatement: If a $6 \times 3$ matrix $A$ has rank 3, find $\text{dim}(\text{Nul}(A))$, $\text{dim}(\text{Row}(A))$, and $\text{Rank}(A^T)$.

Final Answer: $A : \mathbb{R}^3 \rightarrow \mathbb{R}^6$, $A$ having 6 rows and 3 columns. $\text{dim}(\text{Nul}(A)) = 3 - \text{Rank}(A) = 3 - 3 = 0$, by the Rank Theorem. $\text{dim}(\text{Row}(A)) = \text{Rank}(A)$, so $\text{dim}(\text{Row}(A)) = 3$, and $\text{Rank}(A^T) = \text{Rank}(A)$ so $\text{Rank}(A^T) = 3$.

Work: None required.
4.6.10. Problem Restatement: If the null space of a 7 \times 6 matrix \( A \) is 5-dimensional, what is the dimension of the column space of \( A \)?

**Final Answer:** The dimension of the column space of \( A \) is \( \text{Rank}(A) \) by definition. Then, by the Rank Theorem, we have \( \dim(\text{Col}(A)) = 6 - \dim(\text{Nul}(A)) = 6 - 5 = 1 \).

**Work:** None required.

4.6.16. Problem Restatement: If \( A \) is a 6 \times 4 matrix, what is the smallest possible dimension of \( \text{Nul}(A) \)?

**Final Answer:** Zero is the smallest possible dimension of the null space of a 6 \times 4 matrix \( A \).

**Work:**

\( A : \mathbb{R}^4 \to \mathbb{R}^6 \), \( A \) having 6 rows and 4 columns. \( 0 \leq \dim(\text{Nul}(A)) \), but if \( \{ e_1, e_2, e_3, e_4, e_5, e_6 \} \) is the standard basis of \( \mathbb{R}^6 \) and \( A = [e_1 e_2 e_3 e_4] \), say, then \( A \) is one to one and, therefore, \( \text{Nul}(A) = \{0\} \), so \( \dim(\text{Nul}(A)) = 0 \). Therefore, if \( A \) is a 6 \times 4 matrix, the smallest possible dimension of \( \text{Nul}(A) \) is 0.

4.6.30. Problem Restatement: Suppose \( A \) is \( m \times n \) and \( b \) is in \( \mathbb{R}^m \). What has to be true about the two numbers \( \text{Rank}([A \ b]) \) and \( \text{Rank}(A) \) in order for the equation \( Ax = b \) to be consistent?

**Final Answer:** It must be that \( \text{Rank}([A \ b]) = \text{Rank}(A) \).

**Work:** In order for \( Ax = b \) to be consistent, the last column of the augmented matrix \([A \ b]\) cannot be a pivot column. Therefore, in order for \( Ax = b \) to be consistent, \( \text{Rank}([A \ b]) \) must equal the number of pivot columns of \( A \).

Section 4.7 Change of Basis

4.7.2. Problem Restatement: Let \( B = \{b_1, b_2\} \) and \( C = \{c_1, c_2\} \) be bases for a vector space \( V \), and suppose \( b_1 = -c_1 + 4c_2 \) and \( b_2 = 5c_1 - 3c_2 \).

a. Find the change-of-coordinate matrix from \( B \) to \( C \).

b. Find \([x]_C \) for \( x = 5b_1 + 3b_2 \).

**Final Answer:**

a. \( P_{C \to B} = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix} \), b. \([x]_C = \begin{bmatrix} 10 \\ 11 \end{bmatrix} \).

**Work:**

a. \( P_{C \to B} = P_C^{-1}P_B = [P_C^{-1}P_B e_1 \ P_C^{-1}P_B e_2] = [P_C^{-1}b_1 \ P_C^{-1}b_2] = [[b_1]_C \ [b_2]_C] \\
\quad = [\begin{bmatrix} -c_1 + 4c_2 \end{bmatrix}_C \ [5c_1 - 3c_2]_C] = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}. \)

b. \([x]_C = [5b_1 + 3b_2]_C = 5[b_1]_C + 3[b_2]_C = P_{C \to B} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 5 \begin{bmatrix} -1 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}. \)
4.7.6. Problem Restatement: Let $D = \{d_1, d_2, d_3\}$ and $F = \{f_1, f_2, f_3\}$ be bases for a vector space $V$, and suppose $f_1 = 2d_1 - d_2 + d_3$, $f_2 = 3d_2 + d_3$, and $f_3 = -3d_1 + 2d_3$.

a. Find the change-of-coordinate matrix from $F$ to $D$.

b. Find $[x]_D$ for $x = f_1 - 2f_2 + 2f_3$.

Final Answer: a. $P_{D-F} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$. b. $[x]_D = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$.

Work: See 4.7.2. above for methodology, replacing $B$ by $F$ and $C$ by $D$ throughout and adjusting for dimension.

4.7.8. Problem Restatement: Let $B = \left\{ \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be bases for $\mathbb{R}^2$. Find the change-of-coordinated matrix from $B$ to $C$ and the change-of-coordinated matrix from $C$ to $B$.

Final Answer: $P_{C-B} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$ and $P_{B-C} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$.

Work: $P_{C-B} = P_{C}^{-1}P_{B} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 \\ 8 & -5 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ -4/3 & 1/3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 8 & -5 \end{bmatrix}$

$= \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$. $P_{B-C} = P_{C}^{-1} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$. Other calculations will lead to the same answers.

4.7.14. Problem Restatement: In $P_2$, find the change-of-coordinates matrix from the basis $B = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ to the standard basis. Then write $t^2$ as a linear combination of the polynomials in $B$.

Final Answer: $P_{S-B} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$. $t^2 = 3(1 - 3t^2) - 2(2 + t - 5t^2) + (1 + 2t)$.

Work: The standard basis is $S = \{1, t, t^2\}$.

$P_{S-B} = P_{S}^{-1}P_{B} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. $[t^2]_B = P_{B}^{-1}(t^2) = P_{S}^{-1}P_{B}P_{S}^{-1}(t^2) = P_{B}^{-1}P_{S} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$= (P_{S}^{-1}P_{B})^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = P_{S-B}^{-1}e_3$. 

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Therefore, \([t^2]_B\) is the solution of the equation \(P_{S-B}x = e_3\). Passing to the augmented matrix and row reducing, gives us

\[
[P_{S-B} \mid e_3] = \begin{bmatrix}
1 & 2 & 1 & | & 0 \\
0 & 1 & 2 & | & 0 \\
-3 & -5 & 0 & | & 1
\end{bmatrix} \sim \ldots \sim \begin{bmatrix}
1 & 0 & 0 & | & 3 \\
0 & 1 & 0 & | & -2 \\
0 & 0 & 1 & | & 1
\end{bmatrix}.
\]

4.7.x. Problem Restatement: Calculate the change of coordinate matrix \(P_{C-B}\) where 
\(B = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}\) and \(C = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}\).
You may assume \(B\) and \(C\) are each basis of \(P_2\).

Final Answer: \(P_{C-B} = \begin{bmatrix}
23 & 67 & -1 \\
-14 & -41 & 0 \\
6 & 18 & 1
\end{bmatrix}\)

Work: \(P_{C-B} = P_C^{-1}P_B = P_C^{-1}P_S^{-1}P_B = (P_S^{-1}P_C)^{-1}(P_S^{-1}P_B) = P_{S-C}^{-1}P_{S-B}\). Therefore, \(P_{C-B}\) is the solution of the equation \(P_{S-C}X = P_{S-B}\). The matrices \(P_{S-C}\) and \(P_{S-B}\) are easy to formulate (see 4.7.14. above). They are \(P_{S-B} = \begin{bmatrix}
1 & 3 & 0 \\
-2 & -5 & 2 \\
1 & 4 & 3
\end{bmatrix}\) and

\(P_{S-C} = \begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & 2 \\
-3 & -5 & 0
\end{bmatrix}\). Passing to the augmented matrix and row reducing allows us to solve for \(X = P_{C-B}\).

\[
[P_{S-C} \mid P_{S-B}] = \begin{bmatrix}
1 & 2 & 1 & | & 1 & 3 & 0 \\
0 & 1 & 2 & | & -2 & -5 & 2 \\
-3 & -5 & 0 & | & 1 & 4 & 3
\end{bmatrix}
\sim \ldots \sim \begin{bmatrix}
1 & 0 & 0 & | & 23 & 67 & -1 \\
0 & 1 & 0 & | & -14 & -41 & 0 \\
0 & 0 & 1 & | & 6 & 18 & 1
\end{bmatrix} = [I_3 \mid P_{C-B}].
\]
Section 5.1 Eigenvectors and Eigenvalues

5.1.8. Problem Restatement: Is $\lambda = 3$ an eigenvalue of $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find one corresponding eigenvector.

Final Answer: $\lambda = 3$ is an eigenvalue of $A$. A corresponding eigenvector is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Work: $A - \lambda I = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$.

5.1.10. Problem Restatement: Let $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$. Find a basis of the eigenspace of $A$ corresponding to the eigenvalue $\lambda = 4$.

Final Answer: $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$ or, equivalently, $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Work: $A - \lambda I = \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}$.

5.1.16. Problem Restatement: Let $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Find a basis of the eigenspace of $A$ corresponding to the eigenvalue $\lambda = 4$.

Final Answer: $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$.

Work: $A - \lambda I = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

5.1.24. Problem Restatement: Construct an example of a $2 \times 2$ matrix with only one distinct eigenvalue.
**Final Answer:** By Theorem 1 of section 5.1, we can take any $2 \times 2$ triangular matrix with two equal diagonal elements. One example, therefore, is the $2 \times 2$ zero matrix. Another example is the $2 \times 2$ identity matrix $I_2$. Many other examples are possible.

**Work:** None required.

**5.1.26. Problem Restatement:** Show that if $A^2$ is the zero matrix, then the only eigenvalue of $A$ is 0.

**Final Answer:** Suppose $A^2 = 0$. We must show 0 is an eigenvalue of $A$, and we must show that if $\lambda$ is an eigenvalue of $A$ then $\lambda = 0$.

First, we show 0 is an eigenvalue of $A$. To do this, in $\mathbb{R}^n$, choose any vector $v \neq 0$ (tacitly, we are assuming $A$ is an $n \times n$ matrix with $n > 0$). Let $Av = b$. There are two cases to consider; $b = 0$ and $b \neq 0$. In the first case, if $b = 0$ then $b = 0v$, and this gives us $Av = 0v$ with $v \neq 0$, allowing us to conclude 0 is an eigenvalue of $A$. In the other case, we assume $b \neq 0$. Then because $A^2 = 0$, we get $0 = Ab$ upon multiplying on the left both sides of $Av = b$ by $A$. But $0b = 0$, so we get $0b = Ab$ with $b \neq 0$, allowing us to conclude again that 0 is an eigenvalue of $A$. Therefore, in both cases ($b = 0$ and $b \neq 0$), 0 is shown to be an eigenvalue of $A$. This concludes the first part of the proof.

Next, we show that if $\lambda$ is an eigenvalue of $A$ then $\lambda = 0$. To do this, assume $\lambda$ is an eigenvalue of $A$. Therefore, there is a vector $v \neq 0$ such that $Av = \lambda v$. Then $A^2v = A(\lambda v) = \lambda Av = \lambda \lambda v = \lambda^2 v$. But then $0 = \lambda^2 v$ because $A^2v = 0v = 0$. Thus, $0 = \lambda^2$ because $v \neq 0$. Finally, because $\lambda$ is a scalar, $0 = \lambda^2$ implies $0 = \lambda$ as required, and this completes the second part of the proof.

**Work:** None required.