Math 2040: Matrix Theory and Linear Algebra II
Solutions to Assignment 3

Section 5.2 The Characteristic Equation

5.2.2. Problem Restatement: Find the characteristic polynomial and the eigenvalues of

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}.$$

Final Answer: The characteristic polynomial of A is $\lambda^2 - 10\lambda + 16$. The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 8$.

Work: $det(A - \lambda I) = det \begin{pmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{pmatrix} = (5-\lambda)^2 - 9 = 16 - 10\lambda + \lambda^2 = (\lambda - 2)(\lambda - 8)$.

5.2.8. Problem Restatement: Find the characteristic polynomial and the eigenvalues of

$$A = \begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}.$$

Final Answer: The characteristic polynomial of A is $\lambda^2 - 10\lambda + 25$. The eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = 5$.

Work: $det(A - \lambda I) = det \begin{pmatrix} 7-\lambda & -2 \\ 2 & 3-\lambda \end{pmatrix} = (7-\lambda)(3-\lambda) + 4 = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2$.

5.2.14. Problem Restatement: Find the characteristic polynomial and the eigenvalues of

$$A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}.$$

Final Answer: The characteristic polynomial of A is $(1-\lambda)(\lambda - 7)(\lambda + 4)$. (Also, although not requested, the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 7$, and $\lambda_3 = -4$.)

Work: $det(A - \lambda I) = det \begin{pmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{pmatrix}$. Cofactor expansion along the second row gives $det(A - \lambda I) = (1-\lambda) det \begin{pmatrix} 5-\lambda & 3 \\ 6 & -(2+\lambda) \end{pmatrix} = (1-\lambda)(-(5-\lambda)(2+\lambda) - 18) = (1-\lambda)(\lambda^2 - 3\lambda - 28) = (1-\lambda)(\lambda - 7)(\lambda + 4)$.

5.2.20. Problem Restatement: Use the properties of determinants to show that A and A^T have the same characteristic polynomial.

Final Answer: If B is any square matrix then $det(B) = det(B^T)$. Therefore, we have $det(A - \lambda I) = det((A - \lambda I)^T) = det(A^T - \lambda I^T) = det(A^T - \lambda I)$.

Work: None required.

5.2.24. Problem Restatement: Show that if A and B are similar, then $det(A) = det(B)$.

Final Answer: Suppose A and B are similar. Then there is an invertible matrix P such that $A = P^{-1}BP$. Therefore, $det(A) = det(P^{-1}BP) = det(P^{-1})det(B)det(P) = det(B)$, with the final equality justified by $det(P^{-1}) = 1/det(P)$ for an invertible matrix P.

Work: None required.

Section 5.3 Diagonalization

5.3.6. Problem Restatement: Find the eigenvalues of $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix} = PDP^{-1}.$$

Final Answer: A has the eigenvalues $\lambda_1 = 5$ of multiplicity 2 and $\lambda_2 = 4$ of multiplicity 1. A basis of the eigenspace of $\lambda_1 = 5$ is given by the first two columns of P and a basis of the eigenspace of $\lambda_2 = 4$ is given by the last column of P.

Work: None required.

5.3.10. Problem Restatement: Diagonalize $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

Final Answer: $A = PDP^{-1}$ where $D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$.

Work: $det(A - \lambda I) = det \begin{pmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda) - 12 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$.

Therefore, the eigenvalues of A are $\lambda_1 = 5$, and $\lambda_2 = -2$.

$Nul(A - 5I) = Nul \begin{pmatrix} -3 & 3 \\ 4 & -4 \end{pmatrix} = Nul \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = Span\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$.

$Nul(A - (-2)I) = Nul \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} = Nul \begin{pmatrix} 1 & 3/4 \\ 0 & 0 \end{pmatrix} = Span\{\begin{bmatrix} -3/4 \\ 1 \end{bmatrix}\} = Span\{\begin{bmatrix} -3 \\ 4 \end{bmatrix}\}$.

5.3.16. Problem Restatement: Diagonalize $A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$, assuming the eigenvalues are $\lambda = 2, 1$ as given on page 325.

Final Answer: $A = PDP^{-1}$ where $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

Work: $Nul(A-2I) = Nul \begin{pmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{pmatrix} = Nul \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = Span\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \}.$

$Nul(A - I) = Nul \begin{pmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{pmatrix} = Nul \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = Span\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \}.$

5.3.30. Problem Restatement: With $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$, find an invertible matrix $P_2 \neq \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = P$ such that $A = P_2 D P_2^{-1}$ with $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

Final Answer: Any other eigen-basis will do. Thus, we can use any nonzero multiples of the columns of P to construct P_2. This works by the Diagonalization Theorem on page 320. $P_2 = 2P$ will suffice.

Work: None required.

5.3.32. Problem Restatement: Construct a non-diagonal 2×2 matrix that is diagonalizable but not invertible.

Final Answer: Many examples are possible. One is $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, since $A = PDP^{-1}$ where $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Work: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible then it must have linearly dependent columns. Therefore, the non-invertibility of A implies $A = \begin{bmatrix} a & \alpha a \\ c & \alpha c \end{bmatrix}$ for some scalar α. We require one of αa or c to be non zero, in order for A to be non-diagonal. The advantage of allowing c to be zero is that A will be $\begin{bmatrix} a & \alpha a \\ 0 & 0 \end{bmatrix}$, and this triangularity implies the eigenvalues are $\lambda_1 = a$ and $\lambda_2 = 0$. Thus, setting $a = 1$ and $\lambda = 1$, say, gives us $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, non-diagonal and not invertible. Still, we need to see this A is diagonalizable. A basis of the eigenspace corresponding to $\lambda_1 = 1$ is $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$, and a basis of the eigenspace corresponding to $\lambda_2 = 0$ is $\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$. Therefore, A is diagonalizable as required.

Section 5.4 Eigenvectors and Linear Transformations

5.4.2. Problem Restatement: Let $D = \{d_1, d_2\}$ and $B = \{b_1, b_2\}$ be bases for vector spaces V and W, respectively. Let $T : V \to W$ be a linear transformation such that $T(d_1) = 2b_1 - 3b_2$ and $T(d_2) = -4b_1 + 5b_2$. Find the matrix of T relative to D and B.

Final Answer: $\begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$.

Work: Let $[a_1\ a_2]$ be the 2×2 matrix of T relative to D and B. We have $P_D : \mathbf{R}^2 \to V$, and $P_B : \mathbf{R}^2 \to W$. $a_1 = P_B^{-1} T P_D e_1 = P_B^{-1} T(d_1) = P_B^{-1}(2b_1 - 3b_2) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Similarly, $a_2 = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$.

5.4.4. Problem Restatement: Let $B = \{b_1, b_2, b_3\}$ be a basis for a vector space V and $T : V \to \mathbf{R}^2$ be a linear transformation with the property that $T(x_1b_1 + x_2b_1 + x_2b_1) = \begin{bmatrix} 2x_1 - 4x_2 + 5x_3 \\ -x_2 + 3x_3 \end{bmatrix}$. Find the matrix for T relative to B and the standard basis of $\mathbf{R}^2$.

Final Answer: $\begin{bmatrix} 2 & -4 & 5 \\ 0 & -1 & 3 \end{bmatrix}$

Work: If S is the standard basis of $\mathbf{R}^2$ then $P_S = I_2$. The first column of the required matrix is $P_S^{-1} T P_B e_1 = I_2 T(b_1) = T(b_1) = T(b_1 + 0b_2 + 0b_3) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Similarly, the second column is $T(b_2) = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$ and the third column is $T(b_3) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

5.4.8. Problem Restatement: Let $B = \{b_1, b_2, b_3\}$ be a basis for a vector space V. Find $T(3b_1 - 4b_2)$ when T is a linear transformation from V to V whose matrix relative to B is $\begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}$.

Final Answer: $T(3b_1 - 4b_2) = 24b_1 - 20b_2 + 11b_3$.

Work: $T(3b_1 - 4b_2) = P_B[T]_B P_B^{-1}(3b_1 - 4b_2) = P_B[T]_B[3b_1 - 4b_2 + 0b_3]_B = P_B[T]_B \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$

$$= P_B[T]_B \begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = P_B \begin{bmatrix} 24 \\ -20 \\ 11 \end{bmatrix} = 24b_1 - 20b_2 + 11b_3.$$

5.4.12. Problem Restatement: Find the B-matrix for the transformation $x \mapsto Ax$ when $A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}$ and $B = \{b_1, b_2\} = \{\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$.

Final Answer: $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

Work: $[A]_B = P_B^{-1} A P_B = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

5.4.14. Problem Restatement: Define $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = Ax$ where $A = \begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix}$. Find a basis B of $\mathbf{R}^2$ such that $[T]_B$ is diagonal.

Final Answer: $B = \{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix}\}$.

Work: Essentially, we are just required to diagonalize A. After examining the eigenvalues and the corresponding eigenspaces of A, we get $P_B^{-1} A P_B = D$, where $D = \begin{bmatrix} 8 & 0 \\ 0 & -2 \end{bmatrix}$ and $P_B = \begin{bmatrix} -1 & 3 \\ 1 & 7 \end{bmatrix}$. We have $A = [T]_S = P_S^{-1} T P_S$ where $S = \{e_1, e_2\}$. Notice $P_S = I_2$, in which case we get $[T]_B = P_B^{-1} T P_B = P_B^{-1} I_2^{-1} T I_2 P_B = P_B^{-1} P_S^{-1} T P_S P_B = P_B^{-1} A P_B = D$.