Math 2040: Matrix Theory and Linear Algebra II
Solutions to Assignment 4

5.5 Complex Eigenvalues

5.5.4. Problem Restatement: Find the eigenvalues and a basis of the eigenspace in $\mathbf{C}^2$ of $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$.

Final Answer: The complex eigenvalues are $\lambda = 4+i$ and $\overline{\lambda} = 4-i$. A basis of the eigenspace corresponding to $\lambda = 4+i$ is $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i\}$, and a basis of the eigenspace corresponding to $\overline{\lambda} = 4-i$ is $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} i\}$.

Work: The characteristic polynomial is $det(A-\lambda I) = det \begin{bmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} = (5-\lambda)(3-\lambda) + 2 = \lambda^2 - 8\lambda + 17$. The roots of the characteristic polynomial are the complex eigenvalues $\lambda = (8+\sqrt{64-68})/2 = 4+i$ and $\overline{\lambda} = 4-i$. $A - \lambda I = \begin{bmatrix} 5-(4+i) & -2 \\ 1 & 3-(4+i) \end{bmatrix} = \begin{bmatrix} -1-i & -2 \\ 1 & -1-i \end{bmatrix}$. If $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is a complex eigenvector corresponding to $\lambda = 4+i$ then $\begin{bmatrix} -1-i & -2 \\ 1 & -1-i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $(-1-i)z_1 - 2z_2 = 0$ and $z_1 + (-1-i)z_2 = 0$. From the second equation, we get $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_2 \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = z_2(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i)$ with z_2 a free variable. Therefore, a basis of the eigenspace corresponding to $\lambda = 4+i$ is $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i\}$. To get a basis of the eigenspace corresponding to $\overline{\lambda} = 4-i$, we take the complex conjugate, giving us $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} i\}$ (we can do this because, if v is a complex eigenvector corresponding to λ then $\overline{v}$ is a complex eigenvector corresponding to $\overline{\lambda}$).

Check: We will test $Av = (4+i)v$ for $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$. The left hand side is $Av = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} i = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} i$ and the right hand side is $(4+i)v = (4+i)(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i) = (\begin{bmatrix} 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}) + (\begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix})i = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} i$.

The computed left and right hand sides show $Av = (4+i)v$ holds. Thus, the check is successful, verifying that v is indeed a complex eigenvector corresponding to $\lambda = 4+i$. Since complex conjugation preserves eigenvectors (that is, $\overline{v}$ is an eigenvector corresponding to $\overline{\lambda} = 4-i$), we do not need to check the conjugate eigenvector.

5.5.8. Problem Restatement: List the eigenvalues of $A = \begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix}$. The transformation $x \mapsto Ax$ A is the composition of a rotation and a scaling. Find the angle of rotation θ, where $-\pi \le \theta \le \pi$, and find the scaling factor r .

Final Answer: The eigenvalues are $\lambda = \sqrt{3} - 3i$ and $\overline{\lambda} = \sqrt{3} + 3i$. The scaling factor is $r = \sqrt{3+9} = 2\sqrt{3}$, and $\theta = -\pi/3$ since $\cos\theta = \sqrt{3}/r = 1/2$ and $\sin\theta = -3/r = -\sqrt{3}/2$.

Work: None required.

5.5.16. Problem Restatement: Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = PCP^{-1}$, where $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$. Use the information found in exercise 5.5.4.

Final Answer: With $\lambda = 1+i$, we get $C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. (Alternatively, with $\lambda = 1-i$, we get $C = \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.)

Work: We refer to the work in the solution to 5.5.4. $\lambda = 4+i$, so we set $C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$. We use the eigenvector corresponding to $\overline{\lambda} = 4-i$ to construct P. Therefore, $P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.

Checks: $AP = \begin{bmatrix} 3 & -5 \\ 4 & -1 \end{bmatrix} = PC$. $det(P) = 1 \neq 0$, so P is invertible. This gives us $A = PCP^{-1}$, as required.

5.5.22. Problem Restatement: Let A be a complex (or real) $n \times n$ matrix, and let $x \in \mathbf{C}^n$ be an eigenvector corresponding to an eigenvalue in $\mathbf{C}$. Show that for every nonzero complex scalar μ, the vector μx is an eigenvector of A.

Final Answer: Let μ be any nonzero complex scalar, and let $y = \mu x$. $y \neq 0$ because $\mu \neq 0$ and $x \neq 0$. Let λ be the complex eigenvalue corresponding to x. We have $Ay = A(\mu x) = \mu Ax = \mu(\lambda x) = \lambda(\mu x) = \lambda y$. Therefore, y is an eigenvector of A corresponding to the eigenvalue λ.

Work: None required.

6.1 Inner Product, Length, and Orthogonality

6.1.10. Problem Restatement: Find a unit vector in the direction of $v = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$.

Final Answer: A unit vector in the direction of v is $u = v/||v|| = \begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}$.

Work: $||v|| = \sqrt{(-6)^2 + 4^2 + (-3)^2} = \sqrt{36 + 16 + 9} = \sqrt{61}$.

6.1.14. Problem Restatement: Find the distance between $u = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and $z = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$.

Final Answer: The distance between u and z is $||u - z|| = 2\sqrt{17}$.

Work: $||u - z|| = \sqrt{(0 - (-4))^2 + (-5 - (-1))^2 + (2 - 8)^2} = \sqrt{16 + 16 + 36} = \sqrt{68} = 2\sqrt{17}$.

6.1.18. Problem Restatement: Determine if $y = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}$ and $z = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$ are orthogonal.

Final Answer: y and z are not orthogonal, since $y \cdot z \neq 0$.

Work: $y \cdot z = -3(1) + 7(-8) + 4(15) + 0(-7) = 1 \neq 0$.

6.1.24. Problem Restatement: Verify the *parallelogram law* for vectors u and v in $\mathbf{R}^n$:
$||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2$.

Final Answer: See work below.

Work: Using the properties of dot product, we have

$$||u + v||^2 + ||u - v||^2 = (u + v) \cdot (u + v) + (u - v) \cdot (u - v)$$
$$= u \cdot u + u \cdot v + v \cdot u + v \cdot v + u \cdot u + u \cdot (-v) + (-v) \cdot u + (-v) \cdot (-v)$$
$$= u \cdot u + u \cdot v + v \cdot u + v \cdot v + u \cdot u - u \cdot v - v \cdot u + v \cdot v$$
$$= 2(u \cdot u) + 2(v \cdot v)$$
$$= 2||u||^2 + 2||v||^2.$$

6.2 Orthogonal Sets

6.2.2. Problem Restatement: Determine if $\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}\}$ is an orthogonal set of vectors.

Final Answer: Yes, this is an orthogonal set of vectors.

Work: $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 1(0) + (-2)1 + 1(2) = 0 - 2 + 2 = 0,$

$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} = 1(-5) + (-2)(-2) + 1(1) = -5 + 4 + 1 = 0,$ and

$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} = 0(-5) + 1(-2) + 2(1) = 0 - 2 + 2 = 0.$

6.2.6. Problem Restatement: Determine if $\{\begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix}\}$ is an orthogonal set of vectors.

Final Answer: The second and third vectors in the set are not orthogonal, so the set is not an orthogonal set of vectors.

Work: $\begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix} = (-4)3 + 1(3) + (-3)5 + 8(-1) = -12 + 3 - 15 - 8 - 32 \neq 0.$

6.2.10. Problem Restatement: Show that $\{u_1, u_2, u_3\}$ is an orthogonal basis of $\mathbf{R}^3$ and express x as a linear combinations of the u's where $u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, and $x = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

Final Answer: $u_i \cdot u_j = 0$ for $i \neq j$, all $i = 1, 2, 3$ and $j = 1, 2, 3$, as shown in the work below, so $\{u_1, u_2, u_3\}$ is a linearly independent set of three nonzero orthogonal vectors in $\mathbf{R}^3$.

Therefore, $\{u_1, u_2, u_3\}$ is an orthogonal basis of $\mathbf{R}^3$. Also, in the work below we establish $x = \frac{4}{3}u_1 + \frac{1}{3}u_2 + \frac{1}{3}u_3$.

Work: $u_1 \cdot u_2 = 3(2)+(-3)(2)+0(-1) = 0$, $u_1 \cdot u_3 = 3(1)+(-3)1+0(4) = 0$, and $u_2 \cdot u_3 = 2(1)+2(1)+(-1)4 = 0$. $x = \frac{x \cdot u_1}{u_1 \cdot u_1}u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}u_2 + \frac{x \cdot u_3}{u_3 \cdot u_3}u_3 = \frac{3(5)+(-3)(-3)+0(1)}{3^2+(-3)^2+0^2}u_1 + \frac{2(5)+2(-3)+(-1)(1)}{2^2+2^2+(-1)^2}u_2 + \frac{1(5)+1(-3)+4(1)}{1^2+1^2+4^2}u_3 = \frac{24}{18}u_1 + \frac{3}{9}u_2 + \frac{6}{18}u_3 = \frac{4}{3}u_1 + \frac{1}{3}u_2 + \frac{1}{3}u_3$.

6.2.14. Problem Restatement: Let $y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write y as the sum of two orthogonal vectors, one in $Span\{u\}$ and one orthogonal to u.

Final Answer: $y = \begin{bmatrix} \frac{14}{5} \\ \frac{2}{5} \end{bmatrix} + \begin{bmatrix} -\frac{4}{5} \\ \frac{28}{5} \end{bmatrix}$. $\begin{bmatrix} \frac{14}{5} \\ \frac{2}{5} \end{bmatrix} \in Span\{u\}$ and $\begin{bmatrix} -\frac{4}{5} \\ \frac{28}{5} \end{bmatrix}$ is orthogonal to u.

Work: $y = \hat{y} + z$ where the orthogonal projection onto $Span\{u\}$ is $\hat{y} = \frac{y \cdot u}{u \cdot u}u = \frac{14+6}{49+1}u = \frac{20}{50}u = \frac{2}{5}u = \begin{bmatrix} \frac{14}{5} \\ \frac{2}{5} \end{bmatrix}$ and the vector orthogonal to u is $z = y - \hat{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} \frac{14}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ \frac{28}{5} \end{bmatrix}$.

6.3 Orthogonal Projections

6.3.2. Problem Restatement: $u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$, $u_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$, and $v = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix}$. Assuming $\{u_1, u_2, u_3, u_4\}$ is an orthogonal basis of $\mathbf{R}^4$, write v as the sum of two vectors, one in $Span\{u_1\}$ and one in $Span\{u_2, u_3, u_4\}$.

Final Answer: $v = x + y$ where $x = 2u_1 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix}$ in $Span\{u_1\}$, and $y = v - x = \begin{bmatrix} 2 \\ 1 \\ -5 \\ 1 \end{bmatrix}$ in $Span\{u_2, u_3, u_4\}$.

Work: $Span\{u_1\}$ and $Span\{u_2, u_3, u_4\}$ are orthogonal complements of each other, so if x is the orthogonal projection of v onto $Span\{u_1\}$ then $y = v - x$ is the orthogonal projection of v onto $Span\{u_2, u_3, u_4\}$. $\frac{u_1 \cdot v}{u_1 \cdot u_1} = \frac{1(4)+2(5)+1(-3)+1(3)}{1^2+2^2+1^2+1^2} = \frac{14}{7} = 2$. Therefore, $x = 2u$ and $y = v - 2u$.

6.3.6. Problem Restatement: $y = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$, $u_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$, and $u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Verify that $\{u_1, u_2\}$ is an orthogonal set, and then find the orthogonal projection of y onto $Span\{u_1, u_2\}$.

Final Answer: $u_1 \cdot u_2 = (-4)0 + (-1)1 + 1(1) = 0$, so $\{u_1, u_2\}$ is an orthogonal set. The orthogonal projection of y onto $Span\{u_1, u_2\}$ is $\hat{y} = -\frac{3}{2}u_1 + \frac{5}{2}u_2$. Since $-\frac{3}{2}u_1 + \frac{5}{2}u_2 = y$, we get $\hat{y} = y$; that is $y \in Span\{u_1, u_2\}$.

Work: $\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1}u_1 + \frac{y \cdot u_1}{u_2 \cdot u_2}u_2 = \frac{6(-4)+4(-1)+1(1)}{(-4)^2+(-1)^2+1^2}u_1 + \frac{6(0)+4(1)+1(1)}{0^2+1^2+1^2}u_2 = -\frac{27}{18}u_1 + \frac{5}{2}u_2 = -\frac{3}{2}u_1 + \frac{5}{2}u_2$.

6.3.12. Problem Restatement: Find the closest point to $y = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$ in the subspace W spanned by $v_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$.

Final Answer: the closest point to $y =$ in the subspace W spanned by $v_1 =$ and v_2 is

$$\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1}v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2}v_2 = 3v_1 + v_2 = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}.$$

Check: If $z = y - \hat{y} = 4\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, then it is easy to see z is orthogonal to $v_1 =$ and v_2.

Work:

$v_1 \cdot v_2 = (1)(-4) + (-2)(1) + (-1)(0) + (2)(3) = 0$, so $v_1 \perp v_2$.

$y \cdot v_1 = (3)(1) + (-1)(-2) + (1)(-1) + (13)(2) = 30$.
$y \cdot v_2 = (3)(-4) + (-1)(1) + (1)(0) + (13)(3) = 26$.
$v_1 \cdot v_1 = (1)^2 + (-2)^2 + (-1)^2 + (2)^2 = 10$.
$v_2 \cdot v_2 = (-4)^2 + (1)^2 + (0)^2 + (3)^2 = 26$.
$\frac{y \cdot v_1}{v_1 \cdot v_1} = 3$, and $\frac{y \cdot v_2}{v_2 \cdot v_2} = 1$.

6.3.14. Problem Restatement: Find the best approximation to z by vectors of the form $c_1v_1 + c_2v_2$, where $z = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}$, and $v_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}$

Final Answer: $\{c_1v_1 + c_2v_2 \mid c_1, c_2 \in \mathbf{R}\} = Span\{v_1, v_2\}$. Therefore, the best approximation to z by vectors of the form $c_1v_1 + c_2v_2$ is $\hat{z} = \frac{z \cdot v_1}{v_1 \cdot v_1}v_1 + \frac{z \cdot v_2}{v_2 \cdot v_2}v_2 = \frac{1}{2}v_1 + 0v_2 = \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}$.

Check: If $u = z - \hat{z} = \begin{bmatrix} 1 \\ 4 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, then it is easy to see $u \perp v_1$ and $u \perp v_2$.

Work:

$v_1 \cdot v_2 = (2)(5) + (0)(-2) + (-1)(4) + (-3)(2) = 0$, so $v_1 \perp v_2$.

$z \cdot v_1 = (2)(2) + (4)(0) + (0)(-1) + (-1)(-3) = 7.$
$z \cdot v_2 = (2)(5) + (4)(-2) + (0)(4) + (-1)(2) = 0.$
$v_1 \cdot v_1 = (2)(2) + (0)(0) + (-1)(-1) + (-3)(-3) = 14.$
$v_2 \cdot v_2 = (5)(5) + (-2)(-2) + (4)(4) + (2)(2) = 34.$
$c_1 = \frac{v_1 \cdot z}{v_1 \cdot v_1} = \frac{1}{2}$, and $c_2 = \frac{y \cdot v_2}{v_2 \cdot v_2} = 0$.

6.7 Inner Product Spaces

6.7.4. Problem Restatement: Let $\mathbf{P}_2$ have inner product defined by evaluation at -1, 0, and 1 (that is, $< p(t), q(t) >= p(-1)q(-1) + p(0)q(0) + p(1)q(1)$). Compute $< p(t), q(t) >$ where $p(t) = 3t - t^2$, and $q(t) = 3 + 2t^2$.

Final Answer: $< p(t), q(t) >= (3(-1) - (-1)^2)(3 + 2(-1)^2) + (3(0) - (0)^2)(3 + 2(0)^2) + (3(1) - (1)^2)(3 + 2(1)^2) = (-4)(5) + (0)(3) + (2)(5) = -10.$

Work: None required.

6.7.6. Problem Restatement: Compute $||p||$ and $||q||$, for p and q in 6.7.4.

Final Answer: $||p|| = 2\sqrt{5}$, and $||q|| = \sqrt{59}$.

Work: $||p||^2 =< p(t), p(t) >= (-4)^2 + 0^2 + 2^2 = 20$, and $||q||^2 =< q(t), q(t) >= 5^2 + 3^2 + 5^2 = 59$.

6.7.8. Problem Restatement: Find the orthogonal projection of q onto $Span\{p\}$, for p and q in 6.7.4.

Final Answer: Find the orthogonal projection of q onto $Span\{p\}$ is
$\hat{q} = \frac{<q,p>}{<p,p>}p = -\frac{10}{20}(3t - t^2) = -\frac{3}{2}t + \frac{1}{2}t^2$.

Work: None required.

6.7.10. Problem Restatement: Let $\mathbf{R}_3$ have inner product given by evaluation at -3, -1, 1, and 3. Let $p_0(t) = 1$, $p_1(t) = t$, and $q(t) = t^2 - 5$ (found in the back of text). Find the best approximation to t^3 by polynomials in $Span\{p_0, p_1, q\}$.

Final Answer: According to the information given in 4.7.9, $\{p_0, p_1, q\}$ is an orthogonal set. Therefore, the best approximation to t^3 by polynomials in $Span\{p_0, p_1, q\}$ is
$\widehat{t^3} = \frac{<t^3,p_0>}{<p_0,p_0>}p_0 + \frac{<t^3,p_1>}{<p_1,p_1>}p_1 + \frac{<t^3,q>}{<q,q>}q = \frac{0}{4}p_0 + \frac{164}{20}p_1 + \frac{0}{64}q = \frac{41}{5}p_1 = \frac{41}{5}t$.

Work:

$$< p_0, p_0 >= 1^2 + 1^2 + 1^2 + 1^2 = 4.$$

$$< p_1, p_1 >= (-3)^2 + (-1)^2 + (1)^2 + (3)^2 = 20.$$
$$< q, q >= (4)^2 + (-4)^2 + (-4)^2 + (4)^2 = 64.$$
$$< t^3, p_0 >= (-3)^3(1) + (-1)^3(1) + (1)^3(1) + (3)^3(1) = 0.$$
$$< t^3, p_1 >= (-3)^3(-3) + (-1)^3(-1) + (1)^3(1) + (3)^3(3) = 164.$$
$$< t^3, q >= (-3)^3(4) + (-1)^3(-4) + (1)^3(-4) + (3)^3(4) = 0.$$