**Math 2040 Matrix Theory and Linear Algebra II**

# 4 Chapter 4 Lecture Notes. Vector Spaces and Subspaces

## 4.1 Vector Spaces and Subspaces

1. Notation:

   The symbol $\emptyset$ means "the empty set".

   The symbol $\in$ means "is an element of".

   The symbol $\subseteq$ means "is a subset of".

   The symbols $\{x|P(x)\}$ mean "the set of $x$ such that $x$ has the property $P$.

   $\mathbf{R}$ = "the real numbers". Elements of $\mathbf{R}$ are called scalars.

2. The definition of an abstract vector space and examples.

   (a) There are 10 axioms for a vector space, given on page 217 of the text. The first five axioms concern the operation of addition and may be named 1. Closure, 2. Commutivity, 3. Associativity, 4. Unit and 5. Inverse, respectively. These first five axioms are the axioms of addition and alone define the idea of an abelian group. There are some strange and wonderful abelian groups, yet no vector space is strange. This is due to the second set of axioms which guarantee no strange abelian group can be made into a vector space. The axioms 5 through 10 concern scalar multiplication and may be named 6. Closure, 7. Right Distribution, 8. Left Distribution, 9. Associativity and 10. Unit, respectively.

   (b) The idea of a vector space as given above gives our best guess of the objects to study for understanding linear algebra. We will abandon this idea if a better one is found.

   (c) An important consequence of the axioms is the **Cancellation Law**, that $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$ implies $\mathbf{v} = \mathbf{w}$.

   (d) We denote the zero vector by $\mathbf{0}$. This can cause confusion so the student must take care to distinguish the zero vector from the zero scalar. Some practice is gained by reading the next item carefully.

   (e) It follows from the axioms that $0\mathbf{v} = \mathbf{0}$ (here on the left of the equality appears the scalar zero while the vector zero appears on the right), and $-\mathbf{v} = (-1)\mathbf{v}$ for any vector $\mathbf{v}$ and also that $\alpha\mathbf{0} = \mathbf{0}$ for any scalar $\alpha \in \mathbf{R}$. It is also the case that $\mathbf{0} \in V$ is unique; that is, if $\mathbf{v}$ satisfies the property that $\mathbf{v} + \mathbf{u} = \mathbf{u}$ for all vectors $\mathbf{u}$ then $\mathbf{v} = \mathbf{0}$. It is a worthwhile exercise for the student to try to prove these facts.

   (f) $\mathbf{R}^n$ is a vector space.

   (g) $\mathbf{P}_n$ = "the space of real polynomials of degree n or less" is a vector space.

(h) $C[0,1]$ = "the space of continuous real valued functions on the unit interval" is a vector space.

(i) Let $V$ be $\mathbf{R}$ with $\frac{1}{2}$ removed. Compute $v+w$ in $V$ by computing $v+w-2vw$ in $\mathbf{R}$ (e.g. $2+3=-7$ in $V$). We will show in class that $V$ with this addition is an abelian group which cannot be made into a vector space.

3. Subspaces.

(a) A subspace of a vector space $V$ is a subset $W$ which is a vector space under the inherited operations from $V$. Thus, $W \subseteq V$ is a subspace iff $\mathbf{0} \in W$ and $W$ nonempty and is closed under the operations of addition of vectors and multiplication of vectors by scalars.

(b) The trivial subspace of a vector space $V$ is $\{\mathbf{0}\} \subseteq V$. Another example of a subspace of $V$ is $V$ itself.

(c) Subspace Spanned By Vectors: Let $V$ be a vector space and let $\mathcal{S}$ be a set of in $V$. The set $Span(\mathcal{S})$ of all finite linear combinations of the vectors taken from $\mathcal{S}$ is a subspace of $V$. This subspace is called the subspace spanned by $\mathcal{S}$.

## 4.2 Linear Transformation, Null Space, and Column Space

1. Linear transformations.

(a) A linear transformation $T : V_1 \longrightarrow V_2$ between two vector spaces is a function preserving all of the algebra; that is, $T(\alpha\mathbf{v}+\beta\mathbf{u}) = \alpha T(\mathbf{v}) + \beta T(\mathbf{u})$ for all scalars $\alpha, \beta \in \mathbf{R}$ and vectors $\mathbf{v}, \mathbf{u} \in V_1$.

(b) $V_1$ is called the **domain** of $T$ and $V_2$ is called the **codomain** of $T$.

(c) Formal differentiation of polynomials $\frac{d}{dx} : \mathbf{P}_n \longrightarrow \mathbf{P}_{n-1}$ and formal integration $\int : \mathbf{P}_n \longrightarrow \mathbf{P}_{n+1}$ (with 0 as the constant of integration) are examples of linear transformations.

(d) A linear transformation $T : V_1 \longrightarrow V_2$ is **onto** if for each $\mathbf{u} \in V_2$ there is at least one $\mathbf{v} \in V_1$ such that $\mathbf{u} = T(\mathbf{v})$. Formal differentiation is an onto linear transformation.

(e) When $T : V_1 \longrightarrow V_2$ and $v \in V_1$, we write $v \mapsto T(v)$.

(f) An $m \times n$ matrix transformation $A$ is onto iff the number of pivot columns is $m$.

(g) A linear transformation $T : V_1 \longrightarrow V_2$ is **one to one** if for each $\mathbf{u} \in V_2$ there is at most one $\mathbf{v} \in V_1$ such that $\mathbf{u} = T(\mathbf{v})$. Formal differentiation is not one to one.

(h) An $m \times n$ matrix transformation $A$ is one to one iff the number of pivot columns is $n$.

(i) A linear transformation $T : V_1 \longrightarrow V_2$ is an **isomorphism** if it is one to one and onto. When $T$ is an isomorphism, we say $V_1$ and $V_2$ are isomorphic vector spaces and we write $V_1 \simeq V_2$.

(j) If $T : V_1 \longrightarrow V_2$ is an isomorphism then for any $\mathbf{u} \in V_2$ there is exactly one $\mathbf{v} \in V_1$ such that $\mathbf{u} = T(\mathbf{v})$. Thus, we can define a function $T^{-1} : V_2 \longrightarrow V_1$ by taking $T^{-1}(\mathbf{u})$ to be the unique $\mathbf{v} \in V_1$ such that $\mathbf{u} = T(\mathbf{v})$. $T^{-1}$ is linear whenever $T$ is linear and $T^{-1}$ is an isomorphism of vector spaces whenever $T$ is an isomorphism.

(k) The identity linear transformation $1_V : V \longrightarrow V$ associated with any vector space is defined by $1_V(\mathbf{v}) = \mathbf{v}$. $1_V$ is an isomorphism.

(l) If $T_1 : V_1 \longrightarrow V_2$ and $T_2 : V_2 \longrightarrow V_3$ are two linear transformations then the **composition (or composite)** of $T_2$ with $T_1$ is $(T_2 \circ T_1) : V_1 \longrightarrow V_3$ defined by $(T_2 \circ T_1)(\mathbf{v}) = T_2(T_1(\mathbf{v}))$ for each $\mathbf{v} \in V_1$.

(m) If $T_1$ and $T_2$ are both linear (both one to one, both onto or both isomorphisms) then so is the composition $T_2 \circ T_1$.

(n) If $T : V_1 \longrightarrow V_2$ is a linear transformation then $T = T \circ 1_{V_1}$ and $T = 1_{V_2} \circ T$.

(o) If $T : V_1 \longrightarrow V_2$ is an isomorphism then $1_{V_1} = T^{-1} \circ T$ and $1_{V_2} = T \circ T^{-1}$.

2. Kernel or Null space.

   (a) The Kernel or Null Space of a linear transformation $T : V_1 \longrightarrow V_2$ is $Nul(T) = \{\mathbf{u} | \mathbf{0} = T(\mathbf{u})\}$.

   (b) $Nul(T)$ is a subspace of $V_1$.

   (c) When $T$ is a matrix transformation, row reducing the augmented matrix $[T|\mathbf{0}]$ to reduced echelon form produces a vector equation describing $Nul(T)$. We call this equation the kernel equation of the matrix. It provides us with a basis of $Nul(T)$ (see "basis" below).

3. Range, Image and Column Space.

   (a) The **image or range** of a linear transformation $T : V_1 \longrightarrow V_2$ is $Image(T)$, the set of all vectors $\mathbf{u} \in V_2$ such that $\mathbf{u} = T(\mathbf{v})$ for some vector $\mathbf{v} \in V_1$.

   (b) The image of a linear transformation is a subspace of the codomain.

   (c) When $T$ is a matrix transformation, its image is called the column space of $T$, denoted $Col(T)$, and it is the subspace spanned by the columns of the matrix. $Col(T) \subseteq V_2$.

   (d) To find a basis of the column space of a matrix, row reduce it to an echelon form. This identifies the pivot columns and these pivot columns (of the original matrix, not the columns of the echelon form) provide a basis of the column space (see "basis" below).

### 4.3 Linearly Independence; Basis

1. Linear independence.

   (a) A subset $\mathcal{B}$ of a vector space is **linearly independent** if any finite linear combination $\alpha_1\mathbf{b_1} + ... + \alpha_n\mathbf{b_n} = \mathbf{0}$ of elements $\mathbf{b}_1, .., \mathbf{b}_n \in \mathcal{B}$ implies $\alpha_1 = 0$,..., $\alpha_n = 0$.

   (b) A set of vectors $\{\mathbf{b}_1, ..., \mathbf{b}_p\}$ in $\mathbf{R}^n$ is linearly independent iff the matrix $[\mathbf{b}_1 ... \mathbf{b}_p]$ is a one to one linear transformation.

   (c) A set of vectors $\{\mathbf{b}_1, ..., \mathbf{b}_p\}$ in $\mathbf{R}^n$ is linearly independent iff the matrix $[\mathbf{b}_1 ... \mathbf{b}_p]$ has $p$ pivot columns.

   (d) The standard basis vectors $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ in $\mathbf{R}^n$ is a linearly independent set.

   (e) The set of vectors $\{1, t, t^2, ..., t^n\}$ is linearly independent in $\mathbf{P}_n$.

   (f) If $\mathcal{B}$ is a linearly independent set of vectors then $\mathbf{0} \notin \mathcal{B}$.

   (g) No vector in a linearly independent set of vectors is a linear combination of the other vectors of the set.

   (h) Some vector in a linearly dependent set of more than one vector is a linear combination of the other vectors of the set.

   (i) One to one linear transformations preserve and reflect linear independence.

   (j) In class problem: Decide if $\mathcal{B} = \{1, 2t, -2 + 4t^2\}$ and $\mathcal{C} = \{1, 1 - t, 2 - 4t + t^2\}$ are each linearly independent sets in $P_2$.

2. Basis.

   (a) A **basis** of a vector space $V$ is a set of linearly independent vectors spanning $V$.

   (b) $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ is a basis of $\mathbf{R}^n$. It is called the standard basis of $\mathbf{R}^n$.

   (c) The set of vectors $\{1, t, t^2, ..., t^n\}$ is a basis of $\mathbf{P}_n$. It is the standard basis of $\mathbf{P}_n$.

   (d) One of the most important facts (definitely theorem material) about a basis is this: THEOREM: If $\{\mathbf{b}_1, ..., \mathbf{b}_n\}$ is a basis of a vector space and if $\mathbf{v}$ is any vector in the space then there is exactly one set $\{\alpha_1, ..., \alpha_n\}$ of scalars such that

   $$\mathbf{v} = \alpha_1\mathbf{b}_1 + ... + \alpha_n\mathbf{b}_n.$$

   (e) The above theorem guarantees if $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ is a basis of a vector space $V$ then the linear transformation $P_B : \mathbf{R}^n \longrightarrow V$ defined at a standard basis vector $\mathbf{e}_i \in \mathbf{R}^n$ by $P_B(\mathbf{e}_i) = \mathbf{b}_i$, $i = 1, ..., n$, is an isomorphism of vector spaces.

   (f) If $\{\mathbf{b}_1, ..., \mathbf{b}_n\}$ is a set of vectors in $\mathbf{R}^n$ then it is a basis of $\mathbf{R}^n$ iff the $n \times n$ column matrix $[\mathbf{b}_1 ... \mathbf{b}_n]$ is invertible.

   (g) If any vector not in a basis is added to the basis then the resulting set is not a basis.

(h) If any vector is deleted from a basis then the resulting set is not a basis.

(i) These last two facts imply $n = m$ if $\mathbf{R}^m \simeq \mathbf{R}^n$. This is very important.

(j) The zero vector is never a basis vector.

(k) Isomorphisms of vector spaces preserve and reflect basis.

(l) In class problem: Show $\mathcal{B} = \{1, 2t, -2 + 4t^2\}$ and $\mathcal{C} = \{1, 1 - t, 2 - 4t + t^2\}$ are each basis of $P_2$.

### 4.4 Coordinate Systems

1. Coordinate Systems: Let $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ be a basis of a vector space $V$. Let $\mathbf{x} \in V$.

   (a) From the notes of the last section above there are unique scalars $c_1, ..., c_n \in \mathbf{R}^n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + ... + c_n\mathbf{b}_n$. The vector $[\mathbf{x}]_B \in \mathbf{R}^n$ whose coordinates are $c_1, ..., c_n$, respectively, is called the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

   (b) The assignment of $[\mathbf{x}]_B \in \mathbf{R}^n$ to $\mathbf{x} \in V$ is the inverse $P_B^{-1} : V \longrightarrow \mathbf{R}^n$ to the isomorphism $P_B : \mathbf{R}^n \longrightarrow V$, called the **coordinate mapping** determined by $B$.

   (c) In case $V = \mathbf{R}^n$, $P_B$ is the column matrix $[\mathbf{b}_1...\mathbf{b}_n]$ and it is called the **change-of-cordinate matrix** of $\mathcal{B}$. Thus, the change-of-coordinate equations are $[\mathbf{x}]_B = P_B^{-1}(\mathbf{x})$ and $\mathbf{x} = P_B([\mathbf{x}]_B)$.

   (d) The inverse $P_B : \mathbf{R}^n \longrightarrow V$ of a coordinate mapping defines a **representation** of $V$. Generally it is easier to work with the representing space than with $V$.

   (e) We use the standard basis $\{1, t, t^2, ..., t^n\}$ to represent the vector space $P_n$ as $\mathbf{R}^{n+1}$.

   (f) In general, if $\mathcal{B}$ is a given finite basis of a vector space, then $P_\mathcal{B}$ is easy to compute and $P_\mathcal{B}^{-1}$ is hard to calculate. When $\mathcal{B}$ is a standard basis, both $P_\mathcal{B}$ and $P_\mathcal{B}^{-1}$ are easy to calculate.

   (g) In class problem: Let $\mathcal{B} = \{1, 2t, -2 + 4t^2\}$ and $\mathcal{C} = \{1, 1 - t, 2 - 4t + t^2\}$ in $P_2$. Study the $\mathcal{B}$ and $\mathcal{C}$ representations of quadratic polynomials.

### 4.5 Dimension

1. Dimension.

   (a) Any two basis of a vector space have the same number of elements. In the case of finite basis, this follows from a count of the pivot elements of a matrix transformation constructed from the two basis. In the case of infinite dimensional vector spaces this fact is more difficult to prove and is outside the scope of the course.

   (b) That every vector space has a basis is equivalent to an axiom of mathematics, called The Axiom Of Choice. Consequently, we will accept this fact at face value.

(c) The **dimension** of a vector space is the number of elements in a basis of the vector space. That dimension is a well defined concept follows from parts **(a)** and **(b)** just given.

(d) The dimension of $\mathbf{P}_n$ is $n+1$.

(e) If $A$ is a matrix then the dimension of $Col(A)$ is the number of pivot columns of $A$ and the dimension of $Nul(A)$ is the number of free variables (i.e., the number of columns of $A$ minus the number of pivot columns of $A$).

### 4.6 Rank

1. Rank.
   The **rank** of a matrix is the dimension of its column space. Thus, $Rank(A) = dim(Col(A))$.

2. Rank Theorem.
   If $A : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is a linear transformation then $n = Rank(A) + dim(Nul(A))$.

3. Invertibility.
   An $n \times n$ matrix $A$ is invertible iff $Rank(A) = n$.

### 4.7 Change Of Basis

1. Change-of-coordinates matrix.

   (a) Let $\mathcal{B}$ and $\mathcal{C}$ be two basis of an $n$ dimensional vector space $V$. Both $\mathcal{B}$ and $\mathcal{C}$ contain $n$ elements. For any $\mathbf{x} \in V$ we have $\mathbf{x} = P_B([\mathbf{x}]_B)$ and $[\mathbf{x}]_C = P_C^{-1}(\mathbf{x})$. Composing these gives $[\mathbf{x}]_C = (P_C^{-1} \circ P_B)([\mathbf{x}]_B)$ and this equation describes a linear transformation

   $$(P_C^{-1} \circ P_B) : \mathbf{R}^n \longrightarrow \mathbf{R}^n.$$

   The standard matrix representation $P_{C \leftarrow B}$ of $P_C^{-1} \circ P_B$ is called the **change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$**.

   (b) If $\mathbf{x} \in V$ then $[\mathbf{x}]_{\mathcal{B}} \in \mathbf{R}^n$ is the representation of $\mathbf{x}$ relative to $\mathcal{B}$ and $[\mathbf{x}]_{\mathcal{C}} \in \mathbf{R}^n$ is the representation of $\mathbf{x}$ relative to $\mathcal{C}$. We have

   $$P_{C \leftarrow B}[\mathbf{x}]_{\mathcal{B}} = (P_C^{-1} \circ P_B \circ P_B^{-1})(\mathbf{x}) = (P_C^{-1} \circ 1_V)(\mathbf{x}) = P_C^{-1}(\mathbf{x}) = [\mathbf{x}]_{\mathcal{C}}.$$

   Thus, $P_{C \leftarrow B}$ maps $\mathcal{B}$ based representations to $\mathcal{C}$ based representations.

   (c) Suppose $\mathcal{B}$ and $\mathcal{C}$ are both basis of $\mathbf{R}^n$. Let $B$ be the matrix of columns of $\mathcal{B}$ and let $C$ be the matrix of columns of $\mathcal{C}$. Then $P_B = B$ and $P_C = C$, so $P_{C \leftarrow B} = C^{-1}B$ and consequently $P_{C \leftarrow B}$ may be computed by row reducing the augmented matrix $[C|B]$ to $[I|P_{C \leftarrow B}]$.

2. In class problem: Let $\mathcal{B} = \{1, 2t, -2 + 4t^2\}$ and $\mathcal{C} = \{1, 1 - t, 2 - 4t + t^2\}$ in $P_2$. Find $P_{C \leftarrow B}$.