Math 2040 Matrix Theory and Linear Algebra II
sample midterm covering material to the end of section 5.1

Instructions: Answer all questions. For each question part, parenthesized key words tell you the idea being tested. (Students should work very hard on these questions themselves before any in class solutions are given.)

1. **General Theory. (Value = 40%)**

   (a) (Definitions.) Give the definition of a subspace of a vector space.

   (b) (Axiom consequences.) Prove the cancellation law for addition in a vector space; that $\mathbf{u}+\mathbf{v}=\mathbf{u}+\mathbf{w}$ implies $\mathbf{v}=\mathbf{w}$ for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

   (c) (Linearity.) Let $T_1 : V_1 \longrightarrow V_2$ and $T_2 : V_1 \longrightarrow V_2$ be two linear transformations between vector spaces V_1 and V_2. Let $T_3 : V_1 \longrightarrow V_2$ be defined by $T_3(\mathbf{v}) = 2T_1(\mathbf{v}) - T_2(\mathbf{v})$ for each $\mathbf{v} \in V_1$. Prove T_3 is linear.

   (d) (Building examples.) Give an example of a linear transformation $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ such that its null space is two dimensional. (Hint: Think about the Rank Theorem.)

   (e) (Abstract vector spaces.) Let $V = \{\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \mid r_1+r_2 = 1 \text{ and } r_1, r_2 \in \mathbf{R}\}$. For any $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in V$, $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in V$, and $\alpha \in \mathbf{R}$, suppose $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in V is defined to be $\begin{bmatrix} x_1+y_1-1 \\ x_2+y_2 \end{bmatrix}$ and $\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in V is defined to be $\begin{bmatrix} \alpha x_1 + (1-\alpha) \\ \alpha x_2 \end{bmatrix}$. It turns out that V is a vector space with this addition and scalar multiplication defined on it. Answer the following four questions about the vector space V.

      i. What is $0 \in V$?

      ii. What is the additive inverse of $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ in V?

      iii. What is $5 \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ in V?

      iv. V is a subset of $\mathbf{R}^2$, there is a $0 \in V$, and the given addition and multiplication by scalars are closed operations on V. However, V a not subspace of $\mathbf{R}^2$ with this vector space structure defined on V. Why?

2. **Representations. (Value = 40%)**
Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis of a vector space V. Let

$\mathbf{c}_1 = \mathbf{b}_1 + \mathbf{b}_2$,

$\mathbf{c}_2 = \mathbf{b}_2 + \mathbf{b}_3$, and

$\mathbf{c}_3 = \mathbf{b}_1 + \mathbf{b}_3$.

Let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$.

(a) (Coordinates.) If $[x]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$, then what is x?

(b) (Basis and isomorphism.) Prove $\mathcal{C}$ is a basis of V by considering $\{[\mathbf{c}_1]_{\mathcal{B}}, [\mathbf{c}_2]_{\mathcal{B}}, [\mathbf{c}_3]_{\mathcal{B}}\}$ in $\mathbf{R}^3$. What about $P_{\mathcal{B}}$ allows this consideration?

(c) (Change of coordinates.) Calculate the change-of-coordinate matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ represented vectors to $\mathcal{C}$ represented vectors. (Hint: Think about $P_{\mathcal{B}\leftarrow\mathcal{C}}$ instead and its relationship to $P_{\mathcal{C}\leftarrow\mathcal{B}}$.)

(d) (Linearity.) Let $T : V \longrightarrow V$ be a linear transformation and $[T]_{\mathcal{B}}$ be the representation of T relative to the basis $\mathcal{B}$. Assume, the first column of $[T]_{\mathcal{B}}$ is $\begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}$. Also, suppose $T(2\mathbf{b}_2) = 2\mathbf{b}_1 + 4\mathbf{b}_2 + 2\mathbf{b}_3$ and $[T]_{\mathcal{B}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

   i. What is the second column of $[T]_{\mathcal{B}}$?

   ii. What is the third column of $[T]_{\mathcal{B}}$?

(e) (Diagram chasing.) Let T be as in the last part d) above. For 7 test points, write a matrix equation expressing $[T]_{\mathcal{C}}$ in terms of the symbols $P_{\mathcal{C}\leftarrow\mathcal{B}}$, $[T]_{\mathcal{B}}$, and $P_{\mathcal{B}\leftarrow\mathcal{C}}$. For 1 test point, use this equation to find the matrix $[T]_{C}$.

3. **Eigenvalues and Eigenvectors. (Value = 20%)**

Let A be the matrix $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

(a) (Eigenvalues.) Show $\lambda = 2$ is an eigenvalue of A (do not use the characteristic polynomial).

(b) (Eigenvalues.) Show $\lambda = 1$ is not an eigenvalue of A (again, do not use the characteristic polynomial).

(c) (Eigenvectors.) Is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ an eigenvector of A? If so, then what is its corresponding eigenvalue and, if not, then why not?

(d) (Eigenvectors.) Is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ an eigenvector of A? If so, then what is its corresponding eigenvalue and, if not, then why not?

(e) (Eigenspace.) Find a basis of the eigenspace corresponding to $\lambda = 2$

End Midterm.