Mathematical Applications of Category Theory

Applications mathématiques de la théorie des catégories

Robert Pare

Kan extensions for double categories

Western

CMS
SMC

KAN EXTENSIONS FOR DOUBLE CATEGORIES

MARCO GRANDIS
&
ROBERT PARÉ

CAT

(WEAK) DOUBLE CATEGORY

- OBJECTS : SMALL CATEGORIES
- HORIZONTAL MORPHS : FUNCTORS
- VERTICAL MORPHS : PROFUNCTORS

$$\frac{\dfrac{P : \underline{A} \longrightarrow\!\!\!\!\!\!\bullet\;\; \underline{B}}{P : \underline{A}^{op} \times \underline{B} \longrightarrow \underline{SET}}}{\overline{P} : \underline{A} \longrightarrow \left(\underline{SET}^{\underline{B}}\right)^{op}}$$

$$\frac{x \in P(A, B)}{A \xrightarrow[P]{x} B}$$

- CELLS :

$$\begin{tikzcd} \underline{A} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "t"] & \underline{C} \arrow[d, "Q"] \\ \underline{B} \arrow[r, "G"'] & \underline{D} \end{tikzcd}$$

NATURAL TRANSFORMATION

$$t : P \longrightarrow Q(F-, G-)$$

$$\begin{tikzcd} A \arrow[d, "x"', "P"] \\ B \end{tikzcd} \longmapsto \begin{tikzcd} FA \arrow[d, "tx"', "Q"] \\ GB \end{tikzcd}$$

- HORIZONTAL COMPOSITION : OBVIOUS
- VERTICAL COMPOSITION : "MATRIX MULT"

$$\frac{A \xrightarrow[\overline{P} \otimes P]{y \otimes x} E}{\left[A \xrightarrow[P]{x} B \xrightarrow[\overline{P}]{y} E\right]_B}$$

A COEND $\longrightarrow$ WEAK.

EXAMPLE

$$F : \underline{A} \to \underline{B} \quad \longmapsto \quad F_* : \underline{A} \xrightarrow{\bullet} \underline{B}$$

$$F_*(A, B) = \underline{B}(FA, B)$$

$$\begin{tikzcd}
\underline{A} \arrow[r, "F"] \arrow[d, "F_*"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & \underline{B} \arrow[d, "I_{\underline{B}}", "\bullet"' marking] \\
\underline{B} \arrow[r, "1_{\underline{B}}"'] & \underline{B}
\end{tikzcd}$$

$$\beta : F_*(A, B) \longrightarrow I_{\underline{B}}(FA, B)$$

$$\underline{B}(FA, B) \xrightarrow{1} \underline{B}(FA, B)$$

$$\begin{tikzcd}
\underline{A} \arrow[r, "1_{\underline{A}}"] \arrow[d, "I_{\underline{A}}"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & \underline{A} \arrow[d, "F_*", "\bullet"' marking] \\
\underline{A} \arrow[r, "F"'] & \underline{B}
\end{tikzcd}$$

$$\alpha : I_{\underline{A}}(A, A') \longrightarrow F_*(A, FA')$$

$$\underline{A}(A, A') \xrightarrow{\text{“}F\text{”}} \underline{B}(FA, FA')$$

ALSO $\quad F^* : \underline{B} \xrightarrow{\bullet} \underline{A} \qquad F^*(B, A) = \underline{B}(B, FA)$

$\mathbb{C}$AT IS THE RECIPIENT FOR DOUBLE CATEGORY HOMS

$$\mathbb{A}(A, -) : \mathbb{A} \longrightarrow \mathbb{C}\mathrm{AT}$$

$\mathbb{A}(A, B)$

OBJECTS $\quad A \xrightarrow{h} B \quad$ (HORIZ.)

ARROWS

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equal] \\
A \arrow[r, "k"'] & B
\end{tikzcd}$$

(CELLS)

HORIZONTALLY FUNCTORIAL BY COMPOSITION.

VERTICALLY, $v : B \to C$ GIVES A PROFUNCTOR

$$\mathbb{A}(A,B) \xrightarrow[\bullet]{\mathbb{A}(A,v)} \mathbb{A}(A,C) \qquad \alpha \in \mathbb{A}(A,v)(h,k) \qquad \begin{tikzcd} A \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v", "\bullet"'] \\ A \arrow[r, "k"'] & C \end{tikzcd}$$

IT IS ONLY LAX FUNCTORIAL IN v

$$\begin{tikzcd} \mathbb{A}(A,B) \arrow[r, equal] \arrow[d, "\mathbb{A}(A,v)"'] & \mathbb{A}(A,B) \arrow[dd, "\mathbb{A}(A,w\cdot v)"] \\ \mathbb{A}(A,C) \arrow[d, "\mathbb{A}(A,w)"'] \arrow[r, phantom, "\Rightarrow"] & \\ \mathbb{A}(A,D) \arrow[r, equal] & \mathbb{A}(A,D) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v"] \\ A \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\beta"] & C \arrow[d, "w"] \\ A \arrow[r] & D \end{tikzcd} \Longrightarrow \begin{tikzcd} A \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\beta\cdot\alpha"] & B \arrow[d, "w\cdot v"] \\ A \arrow[r] & D \end{tikzcd}$$

BUT NORMAL – PRESERVES IDENTITIES.

RIGHT KAN EXTENSIONS SHOULD BE PRESERVED BY REPRESENTABLES.

SO WE LOOK AT LAX NORMAL FUNCTORS.

Why Kan Ext? – $\mathbb{C}$at is base for 2-dim cat thy (lots of empirical evid.)

Other 2 dim cats → Construct for it → Like Set, Ab

Need to know completeness & cocompleteness ⟹ Kan.

DOUBLE LIMITS

$S : \mathbb{I} \to \mathbb{A}$ A MORPHISM OF DOUBLE CATS

A <u>DOUBLE CONE</u>

$$\begin{tikzcd} & SI \arrow[d, "Su"] \\ A \arrow[ur, "\gamma I"] \arrow[dr, "\gamma \bar{I}"'] \arrow[r, phantom, "\gamma u"] & S\bar{I} \\ & \end{tikzcd} \qquad \begin{tikzcd} I \arrow[d, "u"] \\ \bar{I} \end{tikzcd}$$

IS
- HORIZONTALLY NATURAL
- VERTICALLY FUNCTORIAL

$\underleftarrow{\mathrm{LIM}}\, S$ IS A UNIVERSAL DOUBLE CONE:

$$\begin{tikzcd} & \underleftarrow{\mathrm{LIM}}\, S \arrow[r, "\lambda I"] \arrow[dr, "\lambda \bar{I}"'] \arrow[r, phantom, "\lambda u", shift right=4] & SI \arrow[d, "Su"] \\ \exists! & & S\bar{I} \\ A \arrow[uur, dashed] \arrow[urr, "\gamma I"] \arrow[urr, "\gamma I'"', bend right] \arrow[urr, phantom, "\gamma u", shift right=3] & & \end{tikzcd}$$

WHICH MUST FURTHER SATISFY A 2 DIM PROPERTY

FOR EVERY CONE OF CELLS

$$\begin{tikzcd} A \arrow[r, "\gamma"] \arrow[d, "w"'] \arrow[dr, phantom, "\alpha"] & S \arrow[d, equal] \\ \bar{A} \arrow[r, "\bar{\gamma}"'] & S \end{tikzcd} \qquad \exists! \begin{tikzcd} A \arrow[r] \arrow[d, "w"'] \arrow[dr, phantom, "\beta"] & \underleftarrow{\mathrm{LIM}}\, S \arrow[d, equal] \\ \bar{A} \arrow[r] & \underleftarrow{\mathrm{LIM}}\, S \end{tikzcd} \qquad \text{S/T} \quad \lambda\beta = \alpha$$

EXAMPLES

(1) $\mathbb{I} = \left\{ \begin{tikzcd} \cdot \arrow[r] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \right\}$, $\mathbb{A}$ A 2-CATEGORY

$S : \mathbb{I} \to \mathbb{A}$ IS A DIAGRAM
$$\begin{tikzcd} A \arrow[r] & C \arrow[d, equal] \\ B \arrow[r] & C \end{tikzcd}$$

A DOUBLE CONE IS
$$\begin{tikzcd} & A \arrow[r] & C \arrow[d] \\ X \arrow[ur] \arrow[urr] \arrow[drr] \arrow[dr] & \arrow[r, phantom, "\Downarrow"] & {} \\ & B \arrow[r] & C \end{tikzcd}$$

I.E.
$$\begin{tikzcd} & A \arrow[dr] & \\ X \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow"] & & C \\ & B \arrow[ur] & \end{tikzcd}$$

$\underset{\Leftarrow}{\mathrm{LIM}}\, S$ IS COMMA OBJECT

THE 2-DIM PROPERTY MAKES IT A 2 LIMIT.

(2) $\mathbb{I} = \mathbf{2} = \left(\begin{tikzcd} \cdot \arrow[d] \\ \cdot \end{tikzcd} \right)$, $\mathbb{A}$ ARBITRARY DOUBLE CAT.

$S : \mathbf{2} \to \mathbb{A}$ IS A VERTICAL ARROW
$$\begin{tikzcd} A \arrow[d, "w"] \\ \bar{A} \end{tikzcd}$$

$\underset{\Leftarrow}{\mathrm{LIM}}\, S$ IS THE <u>TABULATOR</u>
$$\begin{tikzcd} & A \arrow[d, "w"] \\ T \arrow[ur, "a"] \arrow[dr, "\bar{a}"'] \arrow[r, phantom, "\tau"] & \bar{A} \end{tikzcd}$$

(IN $\mathbb{C}$AT IT IS THE DISCRETE BIFIBRATION ASSOCIATED TO $P : \underline{A}^{op} \times \underline{B} \longrightarrow \underline{\mathrm{SET}}$.)

FUNCTORIALITY OF $\underleftarrow{\text{LIM}}$

<u>DOUB</u> IS CARTESIAN CLOSED

GIVES HORIZONTAL AND VERTICAL TRANSFORMATIONS

(CONES ARE HORIZONTAL TRANSFORMATIONS)

AS WELL AS DOUBLE CELLS BETWEEN THEM.

HAVE $\Delta : \mathbb{A} \longrightarrow \mathbb{A}^{\mathbb{I}}$ AND MIGHT WANT

$\underleftarrow{\text{LIM}} : \mathbb{A}^{\mathbb{I}} \longrightarrow \mathbb{A}$ RIGHT ADJOINT TO Δ.

GIVEN $S, \bar{S} : \mathbb{I} \longrightarrow \mathbb{A}$ AND VERT $s : S \multimap \bar{S}$

WE WOULD LIKE $\underleftarrow{\text{LIM}}\, s : \underleftarrow{\text{LIM}}\, S \multimap \underleftarrow{\text{LIM}}\, \bar{S}$.

THERE'S NOTHING IN UNIV PROPERTY OF $\underleftarrow{\text{LIM}}$

WHICH GIVES A VERTICAL ARROW.

TO GET $\underleftarrow{\text{LIM}} : \mathbb{A}^{\mathbb{I}} \to \mathbb{A}$ IT IS NOT ENOUGH

TO REQUIRE THAT EACH $S : \mathbb{I} \to \mathbb{A}$ HAVE A LIM.

ALSO NEED LIMITS OF VERTICAL TRANSFS.

$$\begin{tikzcd}
\underset{\Leftarrow}{\mathrm{LIM}}\, S \arrow[r, "p_I"] \arrow[d, "\underset{\Leftarrow}{\mathrm{LIM}}\, s"'] \arrow[dr, phantom, "\pi_I"] & SI \arrow[d, "sI"] & \text{DOUBLE CONE} \\
\underset{\Leftarrow}{\mathrm{LIM}}\, \bar{S} \arrow[r, "\bar{p}_I"'] & \bar{S}I & \text{DOUBLE CONE}
\end{tikzcd}$$

- HORIZONTALLY NATURAL (FOR EACH $I \to I'$)
- VERTICALLY COMPATIBLE (FOR EACH $I \to \bar{I}$)

- UNIVERSAL AMONG ALL SUCH
  (ALREADY 2 DIMENSIONAL – NO FURTHER COND.)

E.G. GIVEN VERTICAL $v : A \to B$ AND $w : C \to D$

$$\begin{tikzcd}
A \arrow[d, "v"'] \arrow[dr, phantom, "\pi_1"] & A \times C \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "v \times w"] \arrow[dr, phantom, "\pi_2"] & C \arrow[d, "w"] \\
B & B \times D \arrow[l, "q_1"] \arrow[r, "q_2"'] & D
\end{tikzcd}$$

IN $\mathbb{C}$AT : GIVEN $P : \underline{A} \to \underline{B}$, $Q : \underline{C} \to \underline{D}$

$P \times Q : \underline{A} \times \underline{C} \to \underline{B} \times \underline{D}$ IS GIVEN BY

$$(P \times Q)((A, C), (B, D)) = P(A, B) \times Q(C, D) .$$

<u>PROPOSITION</u> $\mathbb{C}\mathrm{AT}$ HAS ALL DOUBLE LIMITS OF DOUBLE DIAGRAMS AND VERTICAL TRANSFs.

GIVEN DIAGRAMS $S, \bar{S}, \bar{\bar{S}} : \mathbb{I} \to \mathbb{A}$ AND VERTICAL TRANSFORMATIONS $S \xrightarrow{s} \bar{S} \xrightarrow{\bar{s}} \bar{\bar{S}}$, WE HAVE CELLS

$$\begin{tikzcd}
\underset{\Leftarrow}{\mathrm{LIM}}\, S \arrow[r] \arrow[d, "\underset{\Leftarrow}{\mathrm{LIM}}\, s"'] \arrow[dr, phantom, "\pi_I"] & SI = SI \arrow[d, "sI"] \arrow[dd, bend left, "(\bar{s} \cdot s) I"] \\
\underset{\Leftarrow}{\mathrm{LIM}}\, \bar{S} \arrow[r] \arrow[d, "\underset{\Leftarrow}{\mathrm{LIM}}\, \bar{s}"'] \arrow[dr, phantom, "\bar{\pi}_I"] & \bar{S} I \arrow[d, "\bar{s} I"] \\
\underset{\Leftarrow}{\mathrm{LIM}}\, \bar{\bar{S}} \arrow[r] & \bar{\bar{S}} I = \bar{\bar{S}} I
\end{tikzcd}$$

WHICH GIVE

$$\begin{tikzcd}
\underset{\Leftarrow}{\mathrm{LIM}}\, S = \underset{\Leftarrow}{\mathrm{LIM}}\, S \arrow[d, "\underset{\Leftarrow}{\mathrm{LIM}}\, s"'] \arrow[dd, bend left, "\underset{\Leftarrow}{\mathrm{LIM}}\, (\bar{s} \cdot s)"] \arrow[dd, phantom, "\Rightarrow"] \\
\underset{\Leftarrow}{\mathrm{LIM}}\, \bar{S} \arrow[d, "\underset{\Leftarrow}{\mathrm{LIM}}\, \bar{s}"'] \\
\underset{\Leftarrow}{\mathrm{LIM}}\, \bar{\bar{S}} = \underset{\Leftarrow}{\mathrm{LIM}}\, \bar{\bar{S}}
\end{tikzcd}$$

MAKING $\underleftarrow{\mathrm{LIM}} : \mathbb{A}^{\mathbb{I}} \longrightarrow \mathbb{A}$ INTO A LAX MORPHISM OF DOUBLE CATEGORIES.

IS IT STRONG? NOT FOR $\mathbb{C}\mathrm{AT}$!

BUT IT IS <u>NORMAL</u> (PRESERVES IDENTITIES).

THIS IS THE CONTENT OF THE 2 DIMENSIONAL UNIVERSAL PROPERTY OF $\underleftarrow{\mathrm{LIM}}$.

NOTE: RIGHT ADJOINTS ARE USUALLY LAX.

CONSEQUENTLY WE CAN ONLY EXPECT OUR RIGHT KAN EXTENSIONS TO BE LAX NORMAL.

FURTHERMORE WE SHOULD TAKE KAN EXTENSIONS OF LAX NORMAL MORPHISMS.

SO OUR CONTEXT IS THE 2-CATEGORY OF DOUBLE CATEGORIES, LAX NORMAL MORPHISMS, AND HORIZONTAL TRANSFORMATIONS.

<u>DEFINITION</u>: GIVEN DOUBLE CATEGORIES $\mathbb{I}, \mathbb{J}, \mathbb{A}$ AND LAX NORMAL FUNCTORS $\Phi : \mathbb{I} \to \mathbb{J}$, $S : \mathbb{I} \longrightarrow \mathbb{A}$, THE <u>RIGHT KAN EXTENSION OF S ALONG Φ</u>, $\mathrm{Ran}_\Phi S : \mathbb{J} \to \mathbb{A}$, IS A LAX NORMAL FUNCTOR WITH A HORIZONTAL TRANSFORMATION $\lambda : \mathrm{Ran}_\Phi S \circ \Phi \longrightarrow S$ WHICH IS THE UNIVERSAL SUCH: FOR ANY $G : \mathbb{J} \to \mathbb{A}$ AND $\mu : G \circ \Phi \to S$ THERE EXISTS A UNIQUE $\bar{\mu} : G \to \mathrm{Ran}_\Phi S$ SUCH THAT $\lambda(\bar{\mu}\Phi) = \mu$

$$
\begin{tikzcd}
\mathbb{I} \arrow[r, "\Phi"] \arrow[dr, "S"'] & \mathbb{J} \arrow[r, equal] \arrow[d, "\mathrm{Ran}_\Phi S"] \arrow[dl, phantom, "\overset{\lambda}{\Leftarrow}"] & \mathbb{J} \arrow[d, "G"] \arrow[dl, phantom, "\overset{\bar{\mu}}{\Leftarrow}"] \\
& \mathbb{A} \arrow[r, equal] & \mathbb{A}
\end{tikzcd}
$$

IT IS <u>POINTWISE</u> (STREET) IF FOR EVERY $\Psi : \mathbb{K} \longrightarrow \mathbb{I}$, THE EXTERIOR DIAGRAM IN

$$
\begin{tikzcd}
\Psi \downarrow \Phi \arrow[rr, "Q"] \arrow[d, "P"'] & & \mathbb{K} \arrow[d, "\Psi"] \\
\mathbb{J} \arrow[rr, "\Phi"'] \arrow[rru, phantom, "\Swarrow"] \arrow[dr, "S"'] & & \mathbb{I} \arrow[dl, "\mathrm{RAN}_{\Phi} S"] \arrow[ll, phantom, shift right=4ex, "\Swarrow"] \\
& \mathbb{A} &
\end{tikzcd}
$$

IS A KAN EXTENSION DIAGRAM, WHERE THE TOP SQUARE IS THE COMMA OBJECT.

<u>IN THE PRESENT CONTEXT WE REQUIRE THIS CONDITION ONLY FOR Ψ THAT ARE STRONG.</u>

EXAMPLES

1.

$$\begin{tikzcd} \mathbb{I} \arrow[r] \arrow[dr, "S"'] & \mathbb{1} \arrow[d, "\mathrm{RAN}\,S"] \arrow[dl, phantom, "\overset{\lambda}{\Leftarrow}"] \\ & \mathbb{A} \end{tikzcd}$$

GIVES $\underleftarrow{\mathrm{LIM}}\, S$

BUT ONLY 1-DIM. UNIV. PROPERTY

- POINTWISE CONDITION WITH $\mathbb{I} = (\mathbb{2} \to \mathbb{1})$ GIVES 2-DIM. PROPERTY (EQUIVALENT)
- IMPLIES THE FULL POINTWISE COND.

2. A VERTICAL TRANSFORMATION $s : S \twoheadrightarrow \bar{S}$ GIVES $\tilde{S}$

$$\begin{tikzcd} \mathbb{2} \times \mathbb{I} \arrow[r, "P_1"] \arrow[dr, "\tilde{S}"'] & \mathbb{2} \arrow[d, "\mathrm{RAN}_{P_1} \tilde{S}"] \\ & \mathbb{A} \end{tikzcd}$$

PT WISE KAN GIVES $\underleftarrow{\mathrm{LIM}}\, s$

- POINTWISE WITH $\mathbb{1} \overset{0}{\underset{1}{\rightrightarrows}} \mathbb{2}$ FIX THE DOMAIN AND CODOMAIN OF $\underleftarrow{\mathrm{LIM}}\, S$.

NOTE: ASSUME $\mathbb{A}$ IS HORIZONTALLY INVARIANT

$$\begin{tikzcd} A \arrow[r, "\cong"] \arrow[d, "\forall\, v"'] \arrow[dr, phantom, "\cong"] & A' \arrow[d, dashed, "\exists!"] \\ B \arrow[r, "\cong"'] & B' \end{tikzcd}$$

COMPANIONS & CONJOINTS

<u>DEFINITION</u> $f : A \to B$ HORIZONTAL MORPHISM IN $\mathbb{A}$. IF THERE IS A VERTICAL MORPH $f_* : A \to B$ AND CELLS

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "f_*"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

SUCH THAT $\beta\alpha = id_f$ & $\beta \cdot \alpha = 1_{f_*}$ WE SAY THAT f_* IS A (THE) <u>COMPANION</u> OF f.

DUALLY f^* IS THE <u>CONJOINT</u> (VERT ADJ) OF f IF THERE ARE

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[r, equal] \arrow[d, "f^*"] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\
A \arrow[r, equal] & A \arrow[r, "f"'] & B
\end{tikzcd}
$$

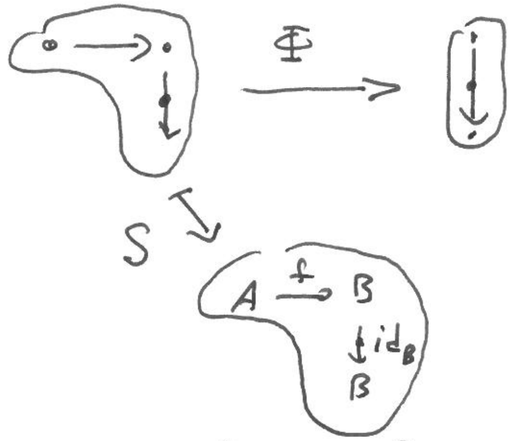
S.T. $\beta\alpha = id_f$

$\alpha \cdot \beta = 1_{f^*}$.

IN CAT THE PROFUNCTORS F_* AND F^* ARE COMPANION AND CONJOINT OF THE FUNCTOR $F : \underline{A} \longrightarrow \underline{B}$.

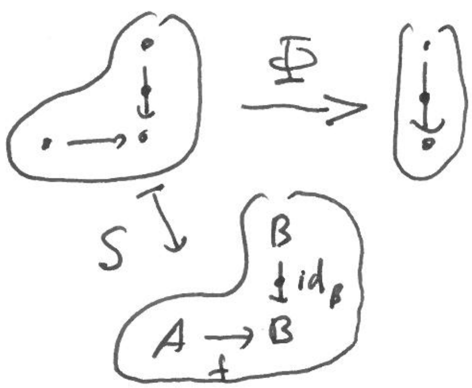
EXAMPLES (CONTINUED)

3.
$$\begin{tikzcd} \mathbb{I} \arrow[r, "\Phi"] \arrow[d, "S"'] & \mathbb{J} \\ \mathbb{A} & \end{tikzcd}$$



POINTWISE $\mathrm{RAN}_{\Phi} S$ IS f_* (COMPANION)

4.



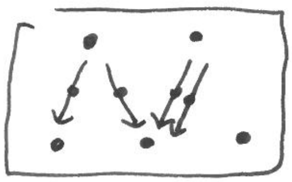
GIVES f^* (CONJOINT) AS $\mathrm{RAN}_{\Phi} S$

NOTE 1: HORIZONTAL INVARIANCE IS EQUIVALENT TO THE EXISTENCE OF COMPANIONS AND CONJOINTS FOR ISOMORPHISMS.

NOTE 2: C.F. BROWN & MOSA FOR CONNECTIONS WITH COMPANIONS AND CONJOINTS.

THEOREM: SUPPOSE $\mathbb{A}$ HAS DOUBLE LIMITS (OF DIAGRAMS AND VERTICAL TRANSFORMATIONS) AND COMPANIONS AND CONJOINTS, THEN IT HAS ALL POINTWISE RT KAN EXTENSIONS OF THE FORM

$$\begin{tikzcd} \mathbb{I} \arrow[r, "\Phi"] \arrow[dr, "S"'] & \mathbb{D} \arrow[d, "\mathrm{Ran}_\Phi S"] \\ & \mathbb{A} \end{tikzcd}$$

Ex:  $\overset{S}{\longmapsto}$

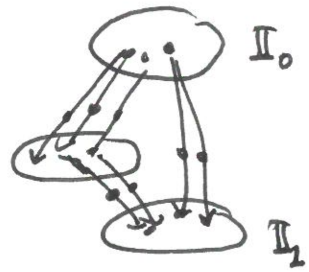
$$\begin{tikzcd} A_1 \arrow[d, "v_1"'] \arrow[dr, "v_2"] & A_2 \arrow[d, shift right, "v_3"'] \arrow[d, shift left, "v_4"] & \\ B_1 & B_2 & B_3 \end{tikzcd}$$

$$\begin{tikzcd} A_1 \times A_2 \arrow[r, "\Delta"] \arrow[d, dashed, "\bar{\Delta}^* \cdot w \cdot \Delta_*"'] & A_1 \times A_1 \times A_2 \times A_2 \arrow[d, "v_1 \times v_2 \times v_3 \times v_4 = w"] \\ B_1 \times B_2 \times B_3 \arrow[r, "\bar{\Delta}"'] & B_1 \times B_2 \times B_2 \times B_2 \end{tikzcd}$$

CONSIDER $\mathbb{I} \xrightarrow{\Phi} \mathbb{B} = $

$$\begin{tikzcd} 0 \arrow[d] \arrow[dd, "\bullet"] \\ 1 \arrow[d] \\ 2 \end{tikzcd}$$

$\mathbb{I}$ LOOKS LIKE



FOR $S : \mathbb{I} \to \mathbb{A}$, $\mathrm{RAN}_{\Phi} S$, IF IT EXISTS WILL BE

$$\begin{tikzcd}[column sep=large]
\underset{I_0 \in \mathbb{I}_0}{\overleftarrow{\mathrm{LIM}}}\, SI_0 \arrow[r, "p_0 I_0"] \arrow[d, "\bullet"] \arrow[ddd, bend right=60, "\underset{v \in \mathbb{I}_{02}}{\overleftarrow{\mathrm{LIM}}}\, Sv"'] & SI_0 \arrow[ddd, "Sv"] & I_0 \arrow[ddd, "v"] \\
\underset{I_1 \in \mathbb{I}_1}{\overleftarrow{\mathrm{LIM}}}\, SI_1 \arrow[from=1-1, "\underset{v_0 \in \mathbb{I}_{01}}{\overleftarrow{\mathrm{LIM}}}\, Sv_0"'] \arrow[dd, "\underset{v_1 \in \mathbb{I}_{12}}{\overleftarrow{\mathrm{LIM}}}\, Sv_1"'] \arrow[r, phantom, "\overset{?}{\Rightarrow}"] & \pi_v & \forall \\
& & \\
\underset{I_2 \in \mathbb{I}_2}{\overleftarrow{\mathrm{LIM}}}\, SI_2 \arrow[r, "p_2 I_2"'] & SI_2 & I_2
\end{tikzcd}$$

NEED : $\forall v : I_0 \to\!\!\!\bullet\, I_2$ THERE IS A CELL

$$\begin{tikzcd}
I_0' \arrow[r] \arrow[d, "v_0"'] & I_0 \arrow[dd, "v"] \\
I_1' \arrow[d, "v_1"'] \arrow[r, phantom, "\alpha"] & \\
I_2' \arrow[r] & I_2
\end{tikzcd}$$

UNIQUE UP TO THE EQUIVALENCE RELATION GENERATED BY

$$\alpha(\alpha_1 \bullet \alpha_0) \sim \alpha$$

$$\begin{tikzcd}
I_0'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_0"] & I_0' \arrow[r] \arrow[d] & I_0 \arrow[dd, "\bullet"] \\
I_1'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_1"] & I_1' \arrow[d] \arrow[r, phantom, "\alpha"] & \\
I_2'' \arrow[r] & I_2' \arrow[r] & I_1
\end{tikzcd} \qquad (*)$$

<u>THEOREM</u>: IF $\mathbb{A}$ SATISFIES THE CONDITIONS OF THE PREVIOUS THEOREM AND $\mathbb{I}$ SATISFIES (*) THEN $\mathrm{Ran}_{\Phi} S$ EXISTS.

IF WE APPLY THIS TO THE COMMA CATEGORIES

$$\begin{tikzcd} \mathbb{B} \Downarrow \Phi \arrow[r] \arrow[d] & \mathbb{B} \arrow[d] \\ \mathbb{I} \arrow[r, "\Phi"'] \arrow[ur, phantom, "\swArrow"] & \mathbb{J} \end{tikzcd}$$

WE GET

<u>DEFINITION</u>: Φ SATISFIES THE <u>RIGHT CONDUCHÉ CONDITION</u> IF EVERY CELL

$$\begin{tikzcd} J_0 \arrow[r] \arrow[d] & \Phi I_0 \arrow[dd, "\Phi u"] \\ J_1 \arrow[d] \arrow[r, phantom, "\beta"] & {} \\ J_2 \arrow[r] & \Phi I_2 \end{tikzcd}$$

FACTORS AS

$$\begin{tikzcd} J_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_0"] & \Phi I_0' \arrow[r] \arrow[d, "\Phi u_0"] & \Phi I_0 \arrow[dd, "\Phi u"] \\ J_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_1"] & \Phi I_1' \arrow[d, "\Phi u_1"] \arrow[r, phantom, "\Phi\alpha"] & {} \\ J_2 \arrow[r] & \Phi I_2' \arrow[r] & \Phi I_2 \end{tikzcd}$$

UNIQUELY UP TO THE EQUIVALENCE RELATION GENERATED BY THE EXISTENCE OF A "MORPHISM OF FACTORIZATIONS".

<u>THEOREM</u> IF $\mathbb{A}$ HAS $\underleftarrow{\text{LIM}}$ AND COMPANIONS AND CONJOINTS AND Φ SATISFIES THE RT CONDUCHÉ CONDITION THEN $\text{RAN}_\Phi S$ EXISTS AND IS POINTWISE FOR ALL LAX NORMAL S.

<u>NOTE 1</u>: IF $\mathbb{I}$, $\mathbb{J}$ HORIZONTALLY DISCRETE, I.E. OBTAINED FROM CATEGORIES $\underline{I}$, $\underline{J}$ BY MAKING THE ARROWS VERTICAL, THEN OUR CONDUCHÉ CONDITION REDUCES TO THE USUAL ONE.

ALSO $\quad \text{LAXN}(\mathbb{I}, \mathcal{C}\text{AT}) \simeq \underline{\text{CAT}}/\underline{I} \quad$ AND $\quad \text{RAN}_\Phi = \Pi_\Phi$

<u>NOTE 2</u>: POINTWISE FORCES

$$\text{RAN}_\Phi S(J) = \underleftarrow{\text{LIM}}\left[J{\downarrow}\Phi \longrightarrow \mathbb{I} \xrightarrow{S} \mathbb{A} \right].$$

EVEN IF $\mathbb{I}, \mathbb{J}, \mathbb{A}$ ARE 2-CATEGORIES, $J{\Downarrow}\Phi$ IS A PROPER DOUBLE CATEGORY. THIS REPLACES THE WEIGHTED LIMITS OF ENRICHED CATEGORY THEORY.