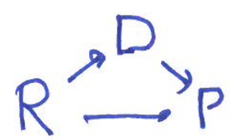
SPANS FOR 2-CATEGORIES AND COMMA OBJECTS



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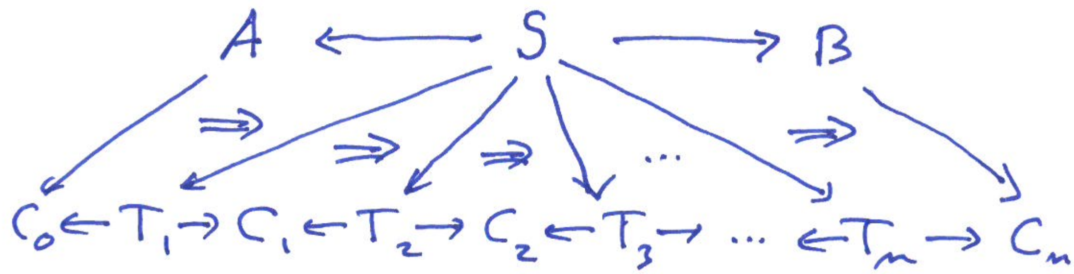
CALAIS 2008

SPANS FOR A 2-CATEGORY $\underline{\underline{A}}$

CONSTRUCT AN OPLAX DOUBLE CATEGORY (C.F. LEINSTER'S FC-CATEGORIES) $\mathbb{S}\underline{\underline{A}}$

- SAME OBJECTS AS $\underline{\underline{A}}$
- HORIZONTAL ARROWS - SPANS OF $\underline{\underline{A}}$
- VERTICAL ARROWS - THE ARROWS OF $\underline{\underline{A}}$
- THE CELLS



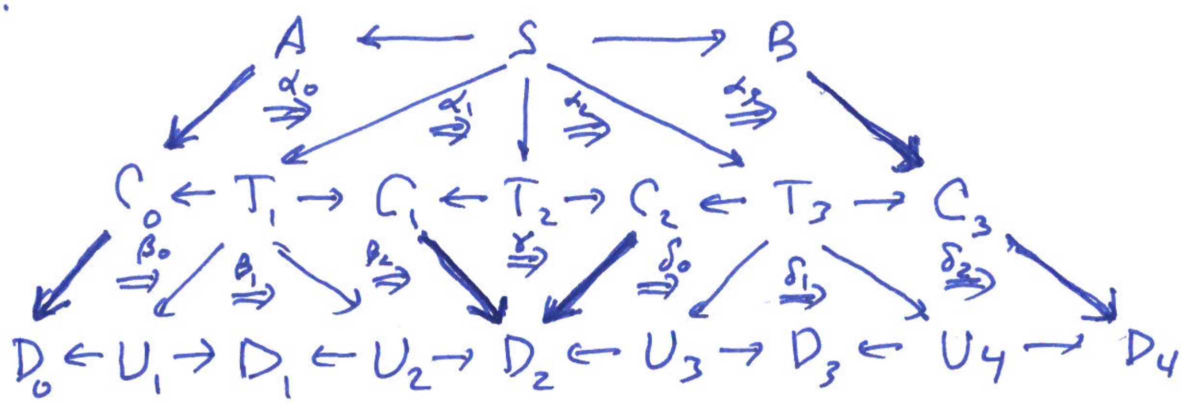
$(n \geq 0)$

- IDENTITIES

$$\begin{array}{ccccc} A & \leftarrow & S & \rightarrow & B \\ \| & \stackrel{1}{\Rightarrow} & \| & \stackrel{1}{\Rightarrow} & \| \\ A & \leftarrow & S & \rightarrow & B \end{array}$$

COMPOSITION (VERTICAL)

E.X.

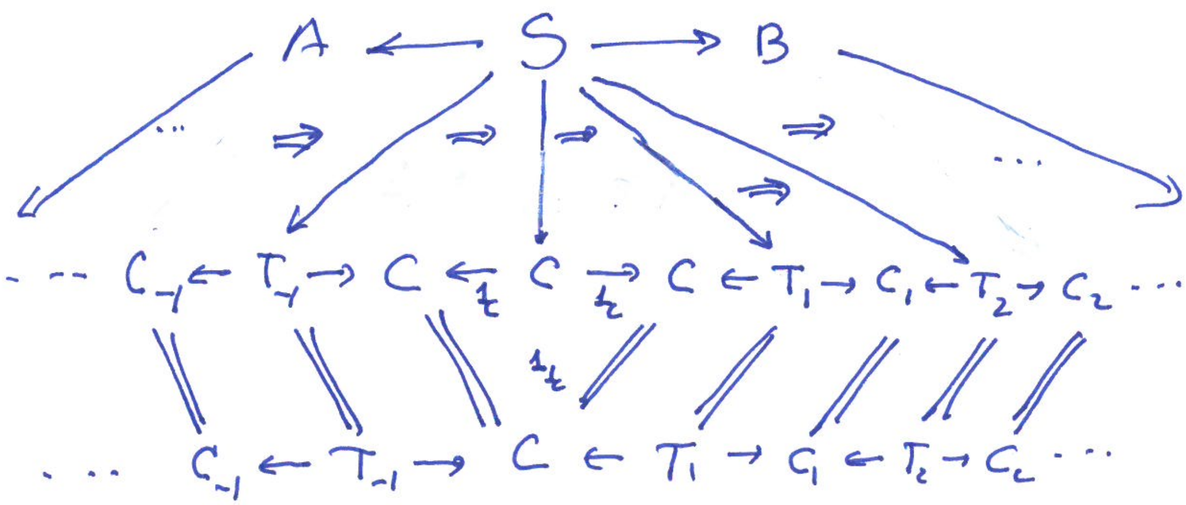


(THE COMPOSITE $(\delta,\gamma,\beta)\cdot\alpha$)

PROP: $\mathcal{S}\underline{\underline{A}}$ IS AN OPLAX DOUBLE CATEGORY.

$\mathcal{S}\underline{\underline{A}}$ IS NOT NORMAL – IDENTITIES NOT REP.

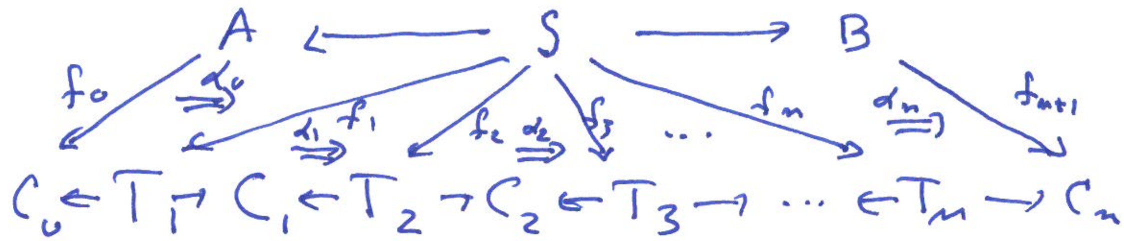
I.E. COMPOSING

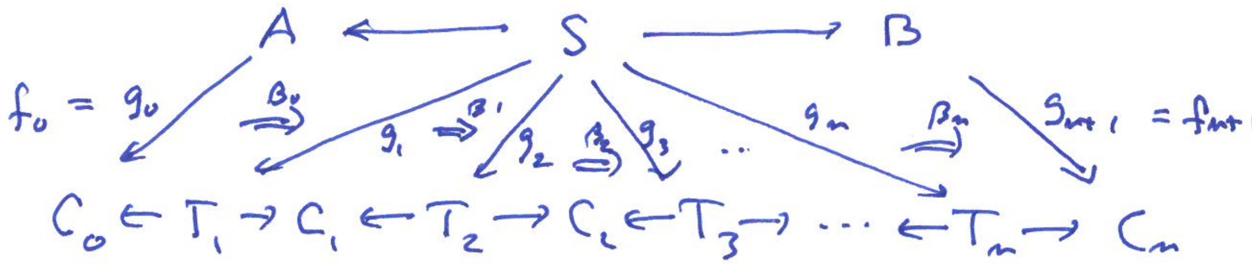


IS NOT A BIJECTION.

THE 3-DIMENSIONAL STRUCTURE OF SPANS

AUGMENT $\mathbb{S}\underline{\underline{A}}$ to $\mathcal{S}$PAN $\underline{\underline{A}}$ BY ADDING 3-CELLS:

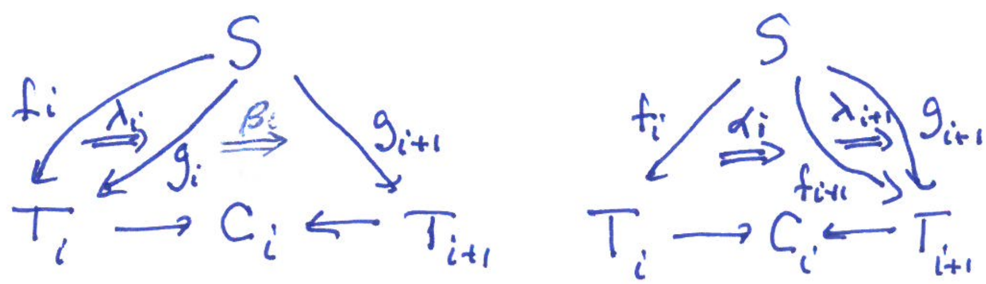




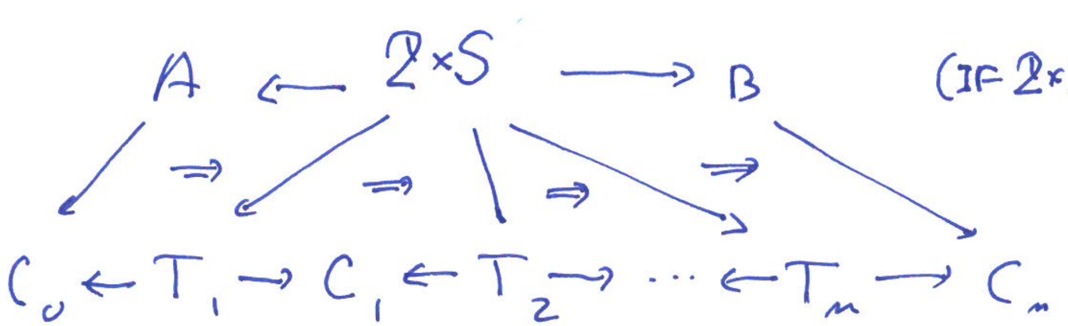
(SAME BOUNDARY)

A 3-CELL IS $\langle \lambda_i : f_i \to g_i \rangle_{i=1,\ldots,n}$

SUCH THAT



I.E. A CELL



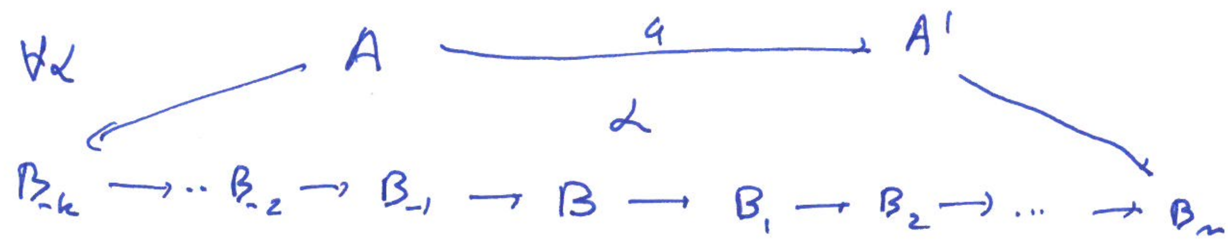
(IF 2×S EXISTS)

THE OPLAX DOUBLE CATEGORY $\mathbb{S}\mathrm{PAN}\,\underline{\underline{A}}$

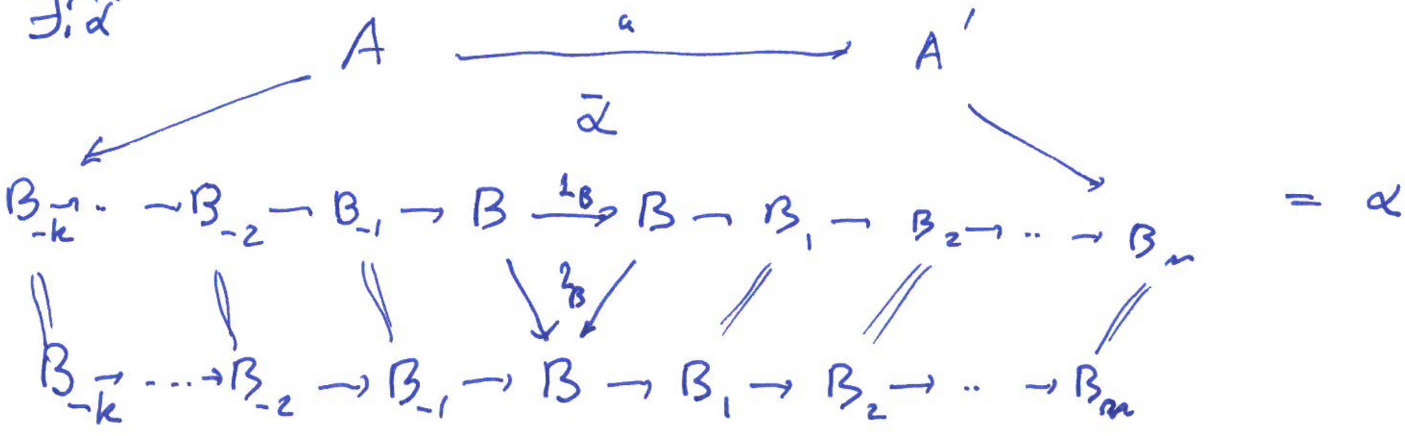
COLLAPSE THE 3-CELLS BY TAKING π_0, CONNECTED COMPONENTS. THIS GIVES THE OPLAX DOUBLE CATEGORY $\mathbb{S}\mathrm{PAN}\,\underline{\underline{A}}$ WITH THE RIGHT PROPERTIES.

AN OPLAX DOUBLE CATEGORY IS <u>NORMAL</u> IF FOR EVERY OBJECT B THERE IS AN ARROW $1_B : B \to B$ AND A CELL 2_B S.T.

$\forall \alpha$

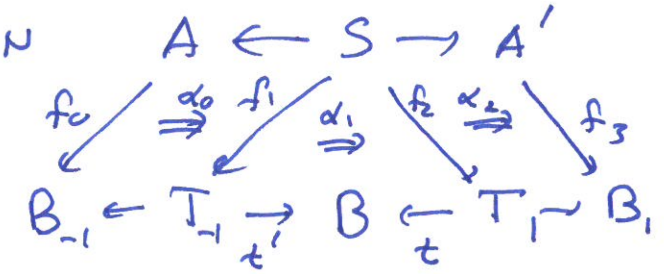


$\exists!\, \bar{\alpha}$



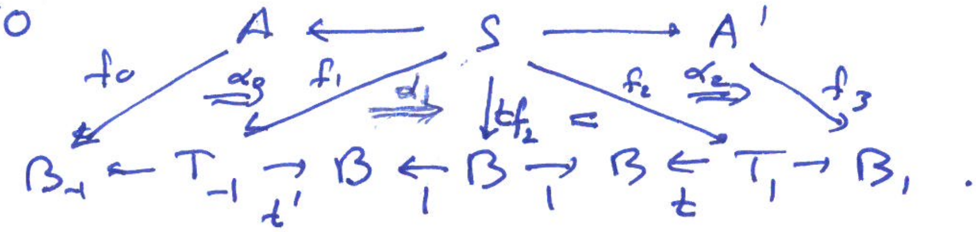
PROP: SPAN $\underline{\underline{A}}$ IS NORMAL.

PROOF: GIVEN

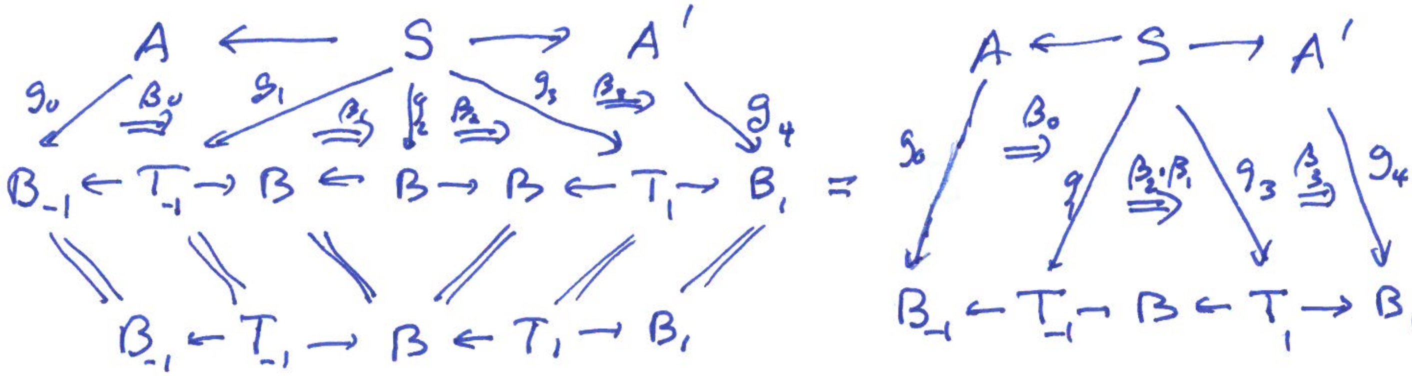


(1)

LIFTS TO



GIVEN ANOTHER LIFTING



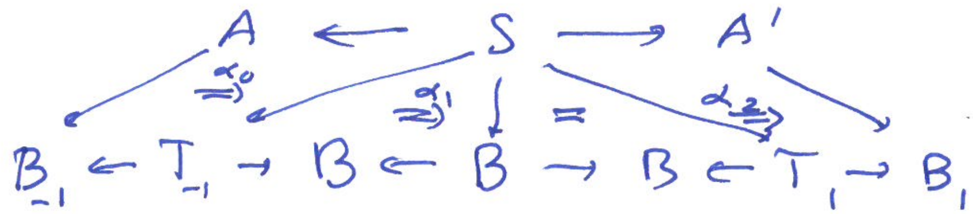
IF THIS IS EQUAL TO (1) THEN $f_0 = g_0$, $f_3 = g_4$

AND THERE IS A ZIG-ZAG SEQUENCE

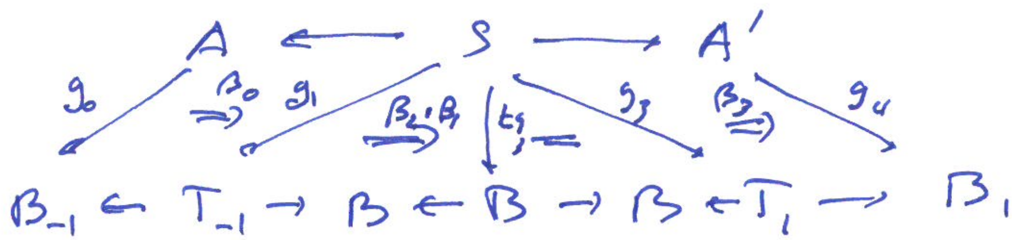
$\langle \lambda_{1i}, \lambda_{2i} \rangle$ RELATING f_1 TO g_1, f_2 TO g_3

α_0 TO β_0, α_1 TO $\beta_2 \cdot \beta_1$, α_2 TO β_3.

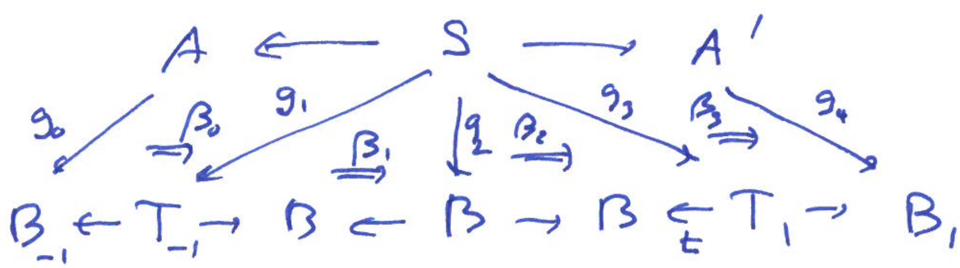
THEN $\langle \lambda_{1i}, t\lambda_{2i}, \lambda_{2i} \rangle$ RELATES



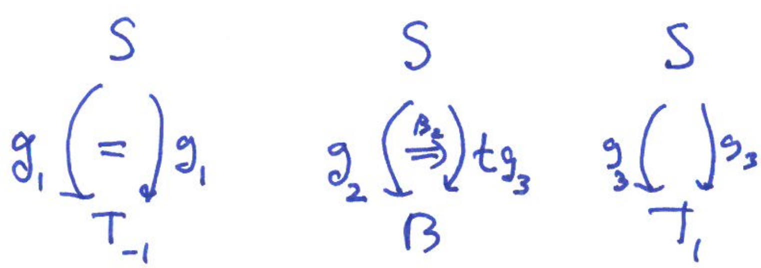
TO



WHICH IS RELATED TO



BY



∎

NORMALITY IS IMPORTANT !

PROP: EVERY VERTICAL ARROW $f: A \to B$ IN $\mathbb{S}\text{PAN}\,\underline{\underline{A}}$ HAS A **COMPANION** $f_* = (A \xleftarrow{1} A \xrightarrow{f} B)$ AND A **CONJOINT** $f^* = (B \xleftarrow{f} A \xrightarrow{1} A)$.

"PROOF": THE BINDING CELLS FOR f_* ARE

$$\begin{array}{ccccc} A & \xleftarrow{1} & A & \xrightarrow{f} & B \\ f\downarrow & = & f\downarrow & = & \| \\ B & \xleftarrow[1]{} & B & \xrightarrow[1]{} & B \end{array} \qquad \text{AND} \qquad \begin{array}{ccccc} A & \xleftarrow{1} & A & \xrightarrow{1} & A \\ \| & = & \| & = & \downarrow f \\ A & \xleftarrow[1]{} & A & \xrightarrow[f]{} & B \end{array}$$

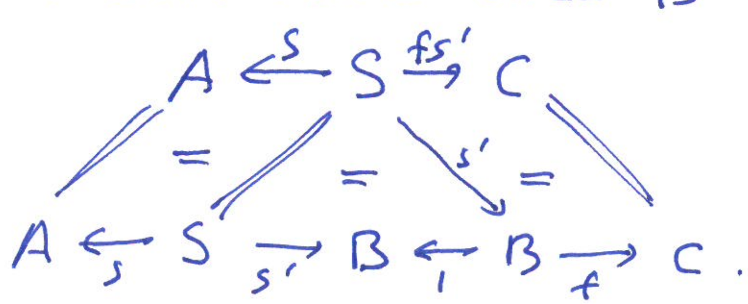
$\square$

PROP: IN $\mathbb{S}\text{PAN}\,\underline{\underline{A}}$ COMPOSITES $f_* S$

$$A \xleftarrow[s]{} S \xrightarrow[s']{} B \xleftarrow[1]{} B \xrightarrow[f]{} C$$

EXIST (STRONGLY REPRESENTABLE) AND ARE EQUAL TO $A \xleftarrow[s]{} S \xrightarrow[fs']{} C$.

"PROOF": THE UNIVERSAL CELL IS



$\square$

DUALITY $\quad \mathrm{SPAN}(\underline{\underline{A}}^{co}) \cong (\mathrm{SPAN}\,\underline{\underline{A}})^{op}$

COR: IN $\mathrm{SPAN}\,\underline{\underline{A}}$ COMPOSITES $S g^*$ EXIST.

PROP: IN $\mathrm{SPAN}\,\underline{\underline{A}}$, $f_* \dashv f^*$.

UNIT

$$\begin{array}{ccccccccc} & & A & \xleftarrow{1} & A & \xrightarrow{1} & A & & \\ & /\!\!/ & = & /\!\!/ & = & \backslash\!\!\backslash & = & \backslash\!\!\backslash & \\ A & \leftarrow & A & \xrightarrow[f]{} & B & \xleftarrow[f]{} & A & \xrightarrow[1]{} & A \end{array}$$

(USES NORMAL)

COUNIT

$$\begin{array}{ccccc} B & \xleftarrow{f} & A & \xrightarrow{f} & B \\ \| & = & \downarrow f & = & \| \\ B & \xleftarrow[1]{} & B & \xrightarrow[1]{} & B \end{array}$$

(USES $B \xleftarrow{f} A \xrightarrow{f} B$
$= B \xleftarrow[f]{} A \xrightarrow[1]{} A \xleftarrow[1]{} A \xrightarrow[f]{} B$
$= f_* f^*$)

PROP: If $f \dashv u$ in $\underline{\underline{A}}$, THEN

$$A \xleftarrow{u} S \xrightarrow{v} B \quad \cong \quad A \xleftarrow{1} A \xrightarrow{vf} B.$$

"PROOF":

$$\begin{array}{ccccc} A & \xleftarrow{u} & S & \xrightarrow{v} & B \\ \| & = & u\downarrow & \overset{v\varepsilon}{\Rightarrow} & \| \\ A & \xleftarrow[1]{} & A & \xrightarrow[vf]{} & B \end{array} \quad \& \quad \begin{array}{ccccc} A & \xleftarrow{1} & A & \xrightarrow{vf} & B \\ \| & \overset{\eta}{\Rightarrow} & \downarrow f & = & \| \\ A & \xleftarrow[u]{} & S & \xrightarrow[v]{} & B \end{array}$$

ARE INVERSE.

$\square$

<u>QUINTETS</u> $\mathbb{Q}\underline{\underline{A}}$ CELLS

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ h\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow k \\ C & \xrightarrow[g]{} & D \end{array}$$

<u>PROP</u>:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ h\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow k \\ C & \xrightarrow[g]{} & D \end{array} \longmapsto \begin{array}{ccccc} A & \xleftarrow{1} & A & \xrightarrow{f} & B \\ h\downarrow & = & h\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow k \\ C & \xleftarrow[1]{} & C & \xrightarrow[g]{} & D \end{array}$$

DEFINES A PSEUDO MORPHISM

$$\mathbb{Q}\underline{\underline{A}} \xrightarrow{(\ )_*} \mathbb{S}\mathrm{PAN}\, \underline{\underline{A}}$$

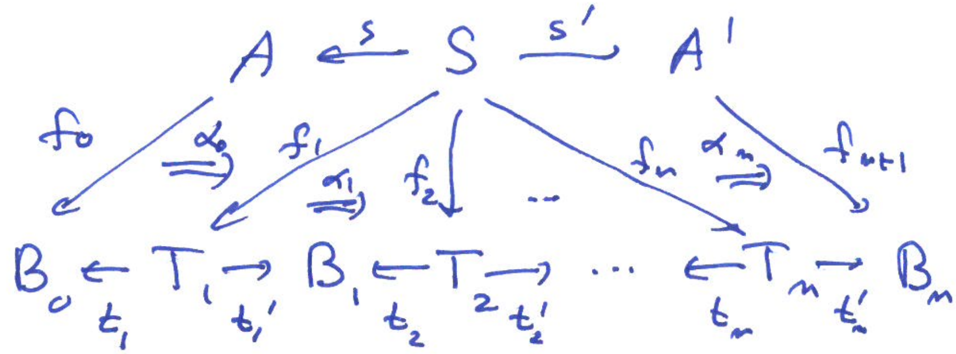
WHICH IS LOCALLY FULL & FAITHFUL.

"<u>PROOF</u>":

$$\begin{array}{ccccc} A & \xleftarrow{1} & A & \xrightarrow{f} & B \\ h\downarrow & \overset{\alpha_0}{\Rightarrow} & l\downarrow & \overset{\alpha_1}{\Rightarrow} & \downarrow k \\ C & \xleftarrow[1]{} & C & \xrightarrow[g]{} & D \end{array} = \begin{array}{ccccc} A & \xleftarrow{1} & A & \xrightarrow{f} & B \\ h\downarrow & = & h\downarrow & \overset{\alpha_1 \cdot \alpha_0}{\Rightarrow} & \downarrow k \\ C & \xleftarrow[1]{} & C & \xrightarrow[g]{} & D \end{array}$$

$\square$

CONSIDER A CELL



IN WHICH $t_i : T_i \longrightarrow B_{i-1}$ IS A FIBRATION.

THEN WE CAN MAKE α_{i-1} INTO AN IDENTITY BY CHANGING f_i AND α_i BUT NOTHING ELSE.

- IF ALL t_i ARE FIBRATIONS WE CAN MAKE ALL THE α_i IDENTITIES EXCEPT α_n.
- IF FURTHERMORE $B_{n-1} \xleftarrow{t_n} T_n \xrightarrow{t_n'} B_n$ IS A BIFIBRATION WE CAN MAKE α_n AN IDENTITY TOO.
- EQUIVALENCE OF CELLS CAN BE REALIZED BY THIS KIND OF CELL.

Comma Objects

Prop: If the comma object

$$\begin{array}{ccc} (s'/t) & \xrightarrow{p_1} & T \\ {\scriptstyle p_0}\downarrow & \Rightarrow & \downarrow{\scriptstyle t} \\ S & \xrightarrow[s']{} & B \end{array}$$

exists, then the composite of

$$A \xleftarrow[s]{} S \xrightarrow[s']{} B \xleftarrow[t]{} T \xrightarrow[t']{} C$$

is strongly represented by

$$A \xleftarrow[sp_0]{} (s'/t) \xrightarrow[t'p_1]{} C .$$

Cor: (Beck condition)

If

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & \Rightarrow & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

is a comma object,

then $f_* h^* \cong k^* g_*$,

$$\begin{array}{ccc} A & \xrightarrow{f_*} & B \\ {\scriptstyle h^*}\uparrow & \overset{\cong}{\Rightarrow} & \uparrow{\scriptstyle k^*} \\ C & \xrightarrow[g_*]{} & D \end{array}$$

Proof:

$$C \xleftarrow{1} C \xrightarrow{g} D \xleftarrow{k} B \xrightarrow{1} B \;\cong\; C \xleftarrow{h} A \xrightarrow{f} B$$

$$\cong\; C \xleftarrow{h} A \xrightarrow{1} A \xleftarrow{1} A \xrightarrow{f} B \qquad \square$$

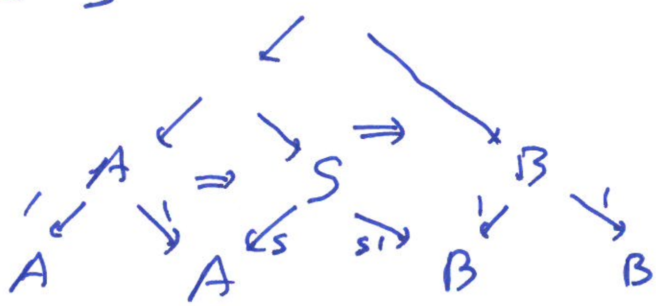
ASSUME $\underline{\underline{A}}$ HAS COMMA OBJECTS.

- $$\begin{array}{ccc} A^2 & \xrightarrow{d_1} & A \\ {\scriptstyle d_0}\downarrow & \Rightarrow & \downarrow{\scriptstyle 1} \\ A & \xrightarrow[1]{} & A \end{array}$$ COMMA OBJECT $\Rightarrow$ $$\begin{array}{ccccc} & & A^2 & & \\ & {\scriptstyle d_0}\swarrow & & \searrow{\scriptstyle d_1} & \\ A & & & & A \end{array} \cong \begin{array}{ccccc} & & A & & \\ & {\scriptstyle 1}\swarrow & & \searrow{\scriptstyle 1} & \\ A & & & & A \end{array}.$$

- $\mathcal{S}$PAN $\underline{\underline{A}}$ "IS" A (PSEUDO) DOUBLE CATEGORY.

- EVERY SPAN S HAS AN ASSOCIATED BIFIB

$I_B S I_A \cong S$



SO WE CAN TAKE $\mathcal{S}$PAN $\underline{\underline{A}}$ TO HAVE BIFIBRATIONS AS HORIZONTAL ARROWS AND AS CELLS <u>EQUIVALENCE CLASSES</u> OF DIAGRAMS

$$\begin{array}{ccccc} A & \leftarrow & S & \rightarrow & B \\ \downarrow & = & \downarrow & = & \downarrow \\ C & \leftarrow & T & \rightarrow & D \end{array}.$$

Spans in $\underline{\underline{\mathrm{Cat}}}$ vs Profunctors

$\mathbb{C}\mathrm{at}$

$$\begin{array}{ccc} \underline{A} & \overset{P}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & \underline{B} \\ F\downarrow & \Downarrow\varphi & \downarrow G \\ \underline{C} & \underset{Q}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & \underline{D} \end{array}$$

$P : \underline{A}^{op} \times \underline{B} \longrightarrow \underline{\mathrm{Set}}$ (profunctor)

$\varphi : P \to Q(F-, G-)$ nat. transf.

$\mathbb{E}\mathrm{l} : \mathbb{C}\mathrm{at} \longrightarrow \mathbb{S}\mathrm{pan}\ \underline{\underline{\mathrm{Cat}}}$

$$\underline{A} \overset{P}{\longrightarrow\!\!\!\!\!\!\!+\;\;} \underline{B} \longmapsto \underline{A} \leftarrow \mathrm{El}(P) \rightarrow \underline{B}$$

$\mathbb{P}\mathrm{O} : \mathbb{S}\mathrm{pan}\ \underline{\underline{\mathrm{Cat}}} \longrightarrow \mathbb{C}\mathrm{at}$

$$\underline{A} \overset{U}{\longleftarrow} \underline{S} \overset{V}{\longrightarrow} \underline{B} \longmapsto \underline{A} \overset{V_* \otimes U^*}{\longrightarrow\!\!\!\!\!\!\!+\;\;} \underline{B}$$

Prop:
- $\mathbb{E}\mathrm{l}$ lax normal,
- $\mathbb{P}\mathrm{O}$ pseudo,
- $\mathbb{P}\mathrm{O}$ is left (upper?) adjoint to $\mathbb{E}\mathrm{l}$. ∎

Compare:

$$\mathrm{Span}(\mathrm{Set}) \underset{\mathrm{Inc}}{\overset{\mathrm{Fact}}{\rightleftarrows}} \mathrm{Rel}(\mathrm{Set}) .$$